Facility Location under Economies of Scale in the Case of Uncertain Demand

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Abstract. This paper adresses facility location under uncertain demand. The problem is to determine the optimal location of facilities and allocation of uncertain customer demand to these facilities. The operating costs of the facilities are subject to economies of scale.

The objective is to minimize the expected total costs. Total costs are split into two parts: firstly the costs of investing in a facility as well as maintaining and operating it with strictly diminishing average costs, and secondly linear transportation cost. We formulate the problem as a two-stage stochastic programming model and develop a solution method based on Lagrangian Relaxation. We also present some preliminary computional results.

1 Introduction

Facility location models usually deal with a trade-off between transportation costs from satisfying customer demand and the costs of opening and operating a facilities at certain locations. Traditionally, facility location models treat the facility costs as fixed set-up costs [1, 2, 3, 4]. In real-world problems however, facility costs often decrease in the size of the facility [5, 6, 7].

There is an extensive amount of research literature available in the field of facility location. These problems have already been studied since the 1950's [8, 9]. The importance of economies of scale (see [10] or [11] for a definition) in combination with facility location was recognized at the same time [12]. Economies of scale have been incorporated in facility location models in several ways: A location problem with transportation costs that are concave in the amount shipped and facility costs that are concave in the amount produced can be found in [13].

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A branch-and-bound algorithm, which is a particular case of minimizing separable concave functions over linear polyhedra, is presented there as well. Another variant where the fixed costs of opening a facility include a component that is convex and decreasing in the number of facilities is examined in [14]. A formulation for a multi-product facility location problem with concave production costs can be found in [15]. Another approach is to model the facility costs using a staircase cost function. Solution methods for this approach are presented in [16, 17]. A solution method for facility location under economies of scale with non-convex, non-concave facility costs is developed in [18]. The aforementioned problem formulations all assume deterministic demand. The formulation of a production-transportation problem with stochastic demand and concave production costs is given together with a solution method in [19].

This paper discusses the case of facility location under economies of scale and uncertain demand. The total facility costs curve has a typical S-shape, often found in long-run cost curves: it is concave in the beginning of the production interval and convex towards the end. The cost function exhibits economies of scale over the whole production interval as the average costs per unit produced are strictly decreasing. In combination with linear transportation costs, the resulting objective function is non-linear, non-convex, and non-concave.

We provide the stochastic programming formulation for a facility location problem with a non-linear, non convex, non-concave objective function and uncertain demand in Section 2. Our solution method based on Lagrangian relaxation is described in Section 3. Some preliminary computational results are shown in Section 4. We finish this paper with the conclusions in Section 5.

2 The Mathematical Programming Model

In this section we provide a two-stage stochastic programming formulation for a facility location problem subject to economies of scale and uncertain demand. The first-stage decision is to determine the location of a facility and how much capacity should be installed at this location. After observing the demand, we decide in the second stage upon the allocation of customer demand to the facilities opened in the first stage.

The objective function consists of non-linear facility costs and linear transportation costs. The first stage facility cost function is both non-convex and non-concave and derived from the long-run total curve, whereas the convex second-stage facility cost function is derived from the short-run total cost function. For a more detailed description of how we obtain the costs functions see [20]. We approximate both the first-stage facility cost function and the second-stage facility cost function by a piecewise linear function. Thus, the objective function becomes a piecewise linear, non-convex, non-concave function as well.

We model the non-convex, non-concave first-stage cost function using the standard approach of a special ordered set of type 1 (see e. g.[21]), i. e. using an ordered set of binary variables, one for each breakpoint of the function, that have to sum up to one. In a feasible solution, exactly one of the variables will be equal to one, corresponding to the chosen linepiece in the facility cost function. The first-stage decision is then to choose a linepiece k on the first-stage facility cost function, opening a facility with a designed lower and an upper capacity limit.

Once customer demand is known, the second-stage decision is to allocate demand to the opened facilities. Depending on the realization of demand, the production level is adjusted according to the short-run expansion path, varying the variable input factors. It is possible to either exceed the upper capacity limit installed in the first stage up to a certain limit, e. g. by using overtime hours, or to assign less demand to a facility than capacity was installed for. The total facility costs however, always exceed the costs that would have occured if the right linepiece for the production level had been chosen in the fist stage. We approximate the convex second-stage cost function by a piecewise linear function. The second stage cost function is modelled using a special ordered set of type 2 for each linepiece and each scenario.

Let n be the number of locations in the problem. By K we denote the number of linepieces used to approximate the non-linear first-stage facility cost function, thus resulting in $K+1$ breakpoints. $P_1, \ldots, P_{K+1}$ are the per unit cost, and $F_1, \ldots, F_{K+1}$ the volumes at the breakpoints of the piecewise linear function. We also define an artificial linepiece $k = 0$ with $P_0 = 0$ and $F_0 = 0$, such that the choice of this linepiece means that no facility is opened. The first-stage decision variables are represented by the $n \cdot (K+1)$-dimensional vector y that is made up by all y_{jk}, $j = 1, \ldots, n$, $k = 0, \ldots, K$. If $y_{jk} = 1$, $k \neq 0$, the linepiece between F_k and F_{k+1} is chosen. In order to properly represent the fixed costs of opening a facility, F_2 is chosen small. Thus, P_2 becomes high as the fixed costs of the facility are distributed only over these few units.

The second-stage cost function is approximated by a piecewise linear function, depending on the choice of linepiece k in the first stage. The second-stage cost function consists of B linesegments, thus having $B+1$ breakpoints. We denote the breakpoints of this function by Q_{kb}, $\forall k, b = 1, \ldots, B+1$. We define $Q_{kb_1} = F(k)$ and $Q_{kb_2} = F(k+1)$ such that the linepiece between breakpoints b_1 and b_2 of the second-stage cost function is equal to the linepiece chosen in the first-stage. In addition, let $Q_{kB+1} = (1+\alpha) \cdot F(k+1)$, with $\alpha \geq 0$ being the percentage by which the upper capacity limit from the first-stage decision can be exceeded. The slope of every linepiece is given by u_{kb}, representing the per unit production costs. The total costs at each breakpoint are given by C_{kb}, $\forall k, b = 1, \ldots, B+1$. With Q_{kb_1} and Q_{kb_2}, we get $C_{kb_1} = P_k F_k$, and $C_{kb_2} = P_{k+1} F_{k+1}$. Let μ_{kb}, $\forall k, b = 1, \ldots, B+1$, denote the weight of breakpoint b given linepiece k. Only two of the weights can be non-zero and in this case they must be adjacent. This means that $\{\mu_{k1}, \ldots, \mu_{kB+1}\}$, $\forall k$ is a special ordered set of type 2. The approximated the first-stage and the second-stage facility cost functions are illustrated in Figure 1.

By D_i we denote demand at a given location i. T_{ij} is the per unit transportation cost of satisfying demand at location i from location j. We define the

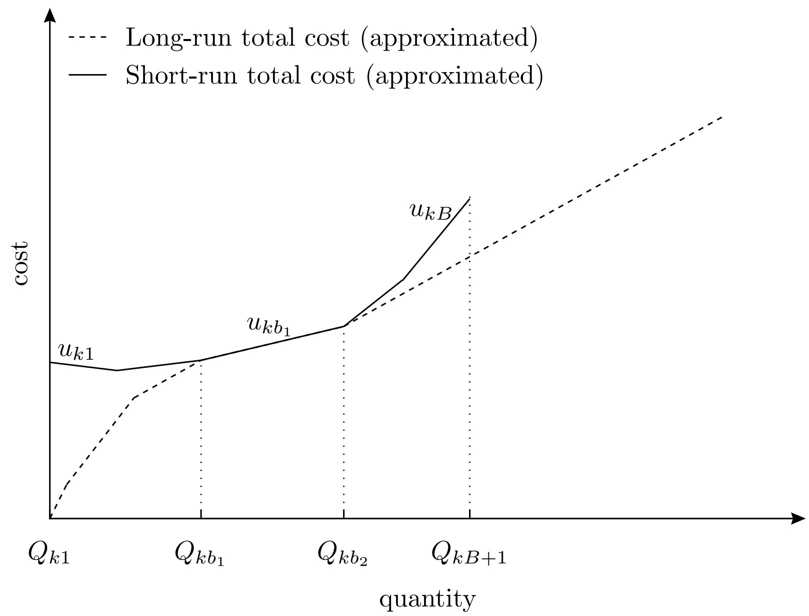

Fig. 1. Approximated first-stage and second-stage facility cost function

parameters L_{ij} as $L_{ij} = 1$ if demand at location i can be satisfied from location j and $L_{ij} = 0$ otherwise.

We use continous decision variables x_{ij} to denote the amount of customer demand at location i that is served from location j. The uncertainty in customer demand is modelled by scenarios. The distribution of demand is discretized using the set of scenarions S. We split the first-stage decision variables y_{jk} and add non-anticipativity constraints (3) to the modell formulation [22].

Let the superscript s denote the scenario and let p^s be the probability associated with scenario s. The stochastic programming formulation is then

$$\min \sum_{s=1}^{S} p^s \left[\sum_{i=1}^{n} \sum_{j=1}^{n} T_{ij} x_{ij}^s + \sum_{j=1}^{n} \sum_{k=1}^{K} \sum_{b=1}^{B+1} C_{kb} \mu_{jkb}^s \right] \qquad (1)$$

subject to

$$\sum_{k=0}^{K} y_{jk}^s = 1, \qquad SOS1,\ \forall j, s, \qquad (2)$$

$$y_{jk}^1 = y_{jk}^2 = \cdots = y_{jk}^{S-1} = y_{jk}^S, \qquad \forall j, k, \qquad (3)$$

$$\sum_{j=1}^{n} x_{ij}^s = D_i^s, \qquad \forall i, s, \qquad (4)$$

$$\sum_{i=1}^{n} x_{ij}^{s} = \sum_{k=1}^{K}\sum_{b=1}^{B+1} Q_{kb}\mu_{jkb}^{s}, \qquad \forall j, s, \tag{5}$$

$$x_{ij}^{s} \le L_{ij} D_i^{s}, \qquad \forall i, j, s, \tag{6}$$

$$\sum_{b=1}^{B+1} \mu_{jkb}^{s} = y_{jk}^{s}, \qquad SOS2,\ \forall j, k, s, \tag{7}$$

$$y_{jk}^{s} \in \{0, 1\}, \qquad \forall j, k, s, \tag{8}$$

$$x_{ij}^{s} \ge 0, \qquad \forall i, j, s, \tag{9}$$

$$\mu_{jkb}^{s} \ge 0, \qquad \forall j, k, b, s. \tag{10}$$

The objective (1) is to minimize the expected costs of operating a set of facilites and satisfying customer demand. The restrictions (2) ensure that only one linepiece is chosen for each location and define the special ordered set of type 1. Constraints (4) force all demand at location i to be assigned to one or more facilities. The set of constraints (6) only allow the assignment of demand to locations from where the demand can be satisfied. By contraints (5) we ensure that all demand is allocated to an open facility. These constraints also prohibit allocating demand to locations without facility. Restrictions (7) link the correct second-stage cost function to the first-stage decision and define the special ordered set of type 2. Finally, constraints (9)-(10) are the non-negativity constraints for the x- and μ-variables.

3 Lagrangian Relaxation

The technique of Lagrangian relaxation has in the past been successfully applied to standard facility location problem, both for the capacitated and the uncapacitated case, see e. g. [23, 24, 25].

We relax the demand constraints (4) in the scenario formulation (1)-(10). Let the vector $\lambda = \left(\lambda_1^1, \ldots, \lambda_1^S, \ldots, \lambda_n^1, \ldots, \lambda_n^S\right)$ denote the Langragian multipliers. For a vector λ, the problem is separable in j. We define $LR(\lambda) = \sum_{j=1}^{n} g_j(\lambda) + E\left[\sum_{i=1}^{n} \lambda_i^s D_i^s\right]$ and get the following Lagrangian subproblem $g_j(\lambda)$ for each location j:

$$g_j(\lambda) = \min \sum_{s=1}^{S} p^s \left[\sum_{i=1}^{n} \left(T_{ij} - \lambda_i^s\right) x_{ij}^{s} + \sum_{k=1}^{K}\sum_{b=1}^{B+1} C_{kb}\mu_{jkb}^{s}\right] \tag{11}$$

subject to

$$\sum_{k=0}^{K} y_{jk}^{s} = 1, \qquad SOS1,\ \forall s, \tag{12}$$

$$y_{jk}^{1} = y_{jk}^{2} = \cdots = y_{jk}^{S-1} = y_{jk}^{S}, \qquad \forall k, \tag{13}$$

$$\sum_{i=1}^{n} x_{ij}^{s} = \sum_{k=1}^{K}\sum_{b=1}^{B+1} Q_{kb}\mu_{jkb}^{s}, \qquad \forall s, \tag{14}$$

$$x_{ij}^s \le L_{ij} D_i^s, \quad \forall i, s, \tag{15}$$

$$\sum_{b=1}^{B+1} \mu_{jkb}^s = y_{jk}^s, \quad SOS2,\ \forall k, s, \tag{16}$$

$$y_{jk}^s \in \{0,1\}, \quad \forall k, s, \tag{17}$$

$$x_{ij}^s \ge 0, \quad \forall i, s, \tag{18}$$

$$\mu_{jkb}^s \ge 0, \quad \forall k, b, s. \tag{19}$$

3.1 Solving the Subproblem

The first-stage decision is to choose the designed capacity of the facility to open at a given location j based on the long-run total cost function. This corresponds to choosing a linepiece k of the piecewise linear facility cost function. Once the linepiece k is chosen, the second-stage facility cost function is a piecewise linear and convex function consisting of B linepieces. The slope of these linepieces, u_{kb}, is strictly increasing, with $u_{kb_1} = \frac{P_{k+1}F_{k+1} - P_k F_k}{F_{k+1} - F_k}$ being the slope of the linepiece chosen in the first stage. Choosing a linepiece k for a given location j also takes care of the non-anticipativity constraints (13) as the choice of the linepiece is valid for all scenarios. If we thus consider the problem (11)-(19) for each linepiece $k = 1, \ldots, K$ separately, $g_j(\lambda)$ becomes separable in scenarios. The case $k = 0$ does not have to be calculated, as no facility will be opened and no costs occur. The subproblem $g_{jks}(\lambda)$ for a given location j, linepiece k and scenario s is:

$$g_{jks}(\lambda) = \min \sum_{i=1}^{n} (T_{ij} - \lambda_i^s)\, x_{ij}^s + \sum_{b=1}^{B+1} C_{kb} \mu_{jkb}^s \tag{20}$$

subject to

$$\sum_{i=1}^{n} x_{ij}^s = \sum_{b=1}^{B+1} Q_{kb} \mu_{jkb}^s, \tag{21}$$

$$x_{ij}^s \le L_{ij} D_i^s, \quad \forall i, \tag{22}$$

$$\sum_{b=1}^{B+1} \mu_{jkb}^s = 1, \quad SOS2, \tag{23}$$

$$x_{ij}^s \ge 0, \quad \forall i, \tag{24}$$

$$\mu_{jkb}^s \ge 0, \quad \forall b. \tag{25}$$

The problem (20)-(25) can be easily reformulated as the Lagrangian subproblem described in [18]. They solve a deterministic problem with a piecewise linear, non-convex, non-concave objective function. However, as $g_{jks}(\lambda)$ has a piecewise linear and convex objective function, we can adapt a method for solving continous knapsack problems with a linear objective function [26] . We apply the following procedure to find the solution for $g_{jks}(\lambda)$:

Initialise: Set $g_{jks}(\lambda) = 0$, $b = 1$, and $i_0 = 1$.
Define $q_i^s = T_{ij} - \lambda_i^s + u_{k1}$, $\forall i$, as the extra cost of serving one unit of demand at location i. Sort the locations i in order of increasing q_i^s: $q_1^s \leq \cdots \leq q_n^s$.
Repeat: Until $b > B$,

1. Set $x_{ij}^s = L_{ij} D_i^s$, $i = 1, \ldots, n$, until either
   (a) $x_{ij}^s = L_{ij} D_i^s$, $\forall i$,
   or for the first time for some index (i_b):
   (b) $q_{i_b} > 0$, or
   (c) $\sum_{m=1}^{i_b} x_{mj}^s > Q_{kb+1}$.
   If (a): Set $b = B$ and $i_b = n$. The solution is optimal.
   If (b): Set $x_{mj}^s = 0$, $m = i_b, \ldots, n$ and $b = B$. The solution is optimal.
   If (c): Set $x_{i_b j}^s = Q_{kb+1} - \sum_{m=1}^{i_b - 1} x_{mj}^s$.
2. Calculate $g_{jks}(\lambda) = g_{jks}(\lambda) + \sum_{m=i_{b-1}}^{i_b} q_m^s x_{mj}^s - q_{i_{b-1}}^s \left(Q_{kb} - \sum_{m=1}^{i_{b-1}-1} x_{mj}^s \right)$
3. If $b < B$: update $q_m^s = T_{mj} - \lambda_m^s + u_{kb+1}$, $m = i_1, \ldots, n$. The sequence of locations i is not changed.
4. Set $b = b + 1$.

Output: $g_{jks}(\lambda)$ is the solution to (20)-(25).

Once $g_{jks}(\lambda)$ is calculated for all scenarios, we can calculate the expected costs per location and linepiece $g_{jk}(\lambda) = \sum_{s=1}^{S} p^s g_{jks}(\lambda)$. The solution to subproblem (11)-(19) is then given by

$$g_j(\lambda) = \min_k g_{jk}(\lambda)$$

The computional complexity of this procedure is $O(n \cdot K \cdot S)$.

3.2 The Lagrangian Dual

In order to find the best lower bound on the optimal solution value of the original problem, one has to solve the Lagrangian dual problem (LD):

$$LD = \max_\lambda LR(\lambda).$$

We solve LD by using sub-gradient optimization, see e. g. [25]. The partial derivative of LR is given by

$$\delta_i^s = \frac{\partial LR(\lambda)}{\partial \lambda_i^s} = D_i^s - \sum_{j=1}^{n} x_{ij}^s(\lambda),$$

with $x_{ij}^s(\lambda)$ being the optimal solution of the Lagrangian relaxation given the multipliers λ. Hence, the gradient of LR is given by $\nabla LR(\lambda) = (\delta_1^1, \ldots, \delta_n^S)$. The sub-gradient optimization routine that we apply in this paper is commonly used for facility location problems. A description of this routine can be found in [23].

The procedure relies on finding an upper bound for the problem. We determine the upper bound by finding a feasible solution usign a simple heuristic. Starting out from the solution of $LR(\lambda)$, we assign first as much demand of customer i as possible to the facility j that can serve the demand at lowest transportation costs. If there is still unstatisfied customer demand, we extend the available capacity at the facility that can serve most of the unsatisfied demand. In the case that the expansion of facilities does no longer remove infeasibilities, we open a facility at the location from where we can remove the highest number of unsatisfied demand, assign all possible demand to this location, and install enough capacity to serve at least the expected demand. We repeat this procedure until we have a feasible solution and thus an upper bound on the solution of our problem. Every time we find a new best upper bound we take the locations and solve a linear stochastic transportation problem using Xpress-MP to further improve the solution. In addition we solve a stochastic transportation problem every 100 iterations. See [20] for a detailed description of the heuristic.

4 Computational Results

We tested the solution method with data from the Norwegian meat industry. The problem instance has 435 candidate locations for facilities and 416 demand points, see [18] and [20]. We present here the results for a dataset consistsing of 10 scenarios that assumes normal distributed demand with a standard deviation $\sigma = 0.2\mu$ for each location. In addition, the demand is correlated thus considering only one stochastic variable (total animal population).

First test runs indicated the initial values of the Lagrangian multipliers do not have much influence on the results. The zero tolerance ϵ required by the subgradient optimization routine was set to 0.10^{-20}. The other parameters are given in Table 1. *MaxIt* is the maximum number of iterations after which the routine terminates. By η_0 we denote the initial value of the parameter needed to calculate the step length $t^{(v)} = \eta \frac{UB - LR(\lambda^{(v)})}{||\nabla LR(\lambda^{(v)})||^2}$ at iteration v [27]. If the lower bound is not improved within *IWI* iterations, η is halved. When the upper bound is improved, η is set back to η_0. All calculations are carried out on a computer with two 3GHz Intel P4 Xeon processors and 6GB RAM running with a 2.4.28 Linux kernel. Results are shown in Table 1.

Our best results after 4.25 hours is a lower bound of $232.883 \cdot 10^6$ and an upper bound of $247.256 \cdot 10^6$ (gap 5.81%). The attempt to solve the stochastic problem for the given dataset results in an lower bound of $149.742 \cdot 10^6$ and an upper bound of $340.684 \cdot 10^6$ (gap 56.05%) after running for approx. 4.5 hours. The solution obtained from the expected value problem is infeasible with respect to the stochastic problem. See [20] for more results.

MaxIt	η_0	*IWI*	*K*	*S*	*LB*	*UB*	gap	CPU-time
3000	2.0	150	6	1	$232.791 \cdot 10^6$	$245.496 \cdot 10^6$	5.18%	0:39:59
3000	2.0	100	6	10	$230.869 \cdot 10^6$	$247.256 \cdot 10^6$	6.68%	4:12:53
3000	2.0	75	6	10	$232.883 \cdot 10^6$	$247.256 \cdot 10^6$	5.81%	4:10:36
3000	1.75	125	6	10	$229.403 \cdot 10^6$	$247.256 \cdot 10^6$	7.22%	4:14:59
3000	1.75	25	6	10	$228.160 \cdot 10^6$	$247.384 \cdot 10^6$	7.7%	4:09:36
3000	1.5	150	6	10	$228.724 \cdot 10^6$	$247.256 \cdot 10^6$	7.50%	4:12:06
3000	1.5	125	6	10	$228.893 \cdot 10^6$	$247.256 \cdot 10^6$	7.43%	4:15:45
3000	1.25	100	6	10	$227.968 \cdot 10^6$	$249.872 \cdot 10^6$	8.77%	4:13:24

Table 1. Computional results for K linepieces and S scenarios

5 Conclusions

We presented a general two-stage stochastic programming formulation and a solution method for a facility location problems under economies of scale and uncertain demand.

Applying Lagrangian relaxation to the problem results in an improved lower bound, considerably narrowing the optimality gap. The performance of the approach shown here may be even better if the obtained bounds are incorporated in a branch-and-bound scheme.

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