Spiteful Bidding in Sealed-Bid Auctions

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Abstract. We study the bidding behavior of spiteful agents who, contrary to the common assumption of self-interest, maximize the weighted difference of their own profit and their competitors' profit. This assumption is motivated by inherent spitefulness, or, for example, by competitive scenarios such as in closed markets where the loss of a competitor will likely result in future gains for oneself. We derive symmetric Bayes Nash equilibria for spiteful agents in 1st-price and 2nd-price sealed-bid auctions. In 1st-price auctions, bidders become "more truthful" the more spiteful they are. Surprisingly, the equilibrium strategy in 2nd-price auctions does not depend on the number of bidders. Based on these equilibria, we compare revenue in both auction types. It turns out that expected revenue in 2nd-price auctions is higher than expected revenue in 1st-price auctions whenever agents have the slightest interest in reducing others' profit as long as they still care for their own profit. In other words, revenue equivalence only holds for auctions in which all agents are either self-interested or completely malicious.

Keywords. Auctions, Externalities, Spite, Revenue

1 Introduction

One of the fundamental assumptions of game theory is that agents are self-interested, *i.e.*, they maximize their own utility without considering the utility of other agents. However, there is some evidence that certain types of behavior in the real world can be better explained by models in which agents have other-regarding preferences. While there are settings where an agent is interested in the well-being of other agents, there are also others where an agent intends to degrade competitors in order to improve his own standing. This is typically the case in competitive scenarios such as in closed markets where the loss of a competitor will likely result in future gains for oneself (*e.g.*, when a competitor is driven out of business) or, more generally, when agents intend to maximize their relative rather than their absolute utility. To give an example, consider the popular Trading Agent Competition (TAC) where an agent's goal clearly

is to accumulate more revenue than his competitors instead of maximizing his revenue.

For another example, consider the German 3G mobile phone spectrum license auction in 2000 which happens to be one of the most revenue generating auctions to date (50.8 billion Euro). One of the bidders (German Telekom) kept raising the price beyond a point at which there were as many remaining bidders as licences. This behavior has been interpreted as an (unsuccessful) attempt to crowd out one of the weaker competitors [1,2]. Nevertheless, it turned out that some of the weaker phone companies were unable to cope with the resulting high amounts of money spent for the licenses. As of today, one of the six winning bidders (Mobilcom) is struggling with bankruptcy (and gave away its license) and another one (Quam) filed for bankruptcy. Clearly, a reduced number of competitors is advantageous for the remaining providers because their share of the market increases.

Sealed-bid auctions are well-understood competitive economic processes where questioning the assumption of self-interest is particularly reasonable. For instance, in 2nd-price auctions, according to the dominant strategy equilibrium, a self-interested bidder is still best off bidding his private value of the good to be sold even when he knows all other bids. However, a competitive agent who *knows* he cannot win might have an incentive to place his bid right below the winning bid in order to minimize the winner's profit. In order to make such behavior rational, we need to incorporate other-regarding preferences in the utility function. For this reason, we formalize a notion of utility where a spiteful agent, *i.e.*, an agent with negative externalities, is interested in minimizing the profit of his competitors as well as maximizing his own profit (Section 3). The tradeoff between both goals is controlled by a parameter α called *spite coefficient*. $\alpha = 0$ yields self-interested agents whereas $\alpha = 1$ defines completely malicious agents whose only goal is to reduce others' profit. Given this new definition of utility, the well-known equilibria for 1st- and 2nd-price auctions do not hold anymore whenever $\alpha > 0$. We derive symmetric Bayes Nash equilibria for spiteful agents in both auction types in Section 4 and 5, respectively. Based on these equilibria, we deduce further results, primarily on auction revenue, in Section 6. The paper concludes with an overview of the obtained results and a brief outlook on future research in Section 7.

2   Related Work

Numerous authors in experimental economics [3,4], game theory [5], social psychology [6,7], and multiagent systems [8] have argued in favor of other-regarding preferences, usually with an emphasis on altruism. Levine introduced a model in which utility is defined as a linear function of both the agent's monetary payoff and his opponents, controlled by a parameter called "altruism coefficient" [3]. This model was used to explain data obtained in ultimatum bargaining and centipede experiments. One surprising outcome of that study was that an overwhelming majority of individuals possess a *negative* altruism coefficient, corre-

sponding to spiteful behavior. He concludes that "one explanation of spite is that it is really 'competitiveness', that is, the desire to outdo opponents." Most papers, including Levine's, also consider elements of fairness in the sense that agents are willing to be more altruistic/spiteful to an opponent who is more altruistic/spiteful towards them. Brainov defines a generic type of antisocial agent by letting $\frac{\partial U_i}{\partial u_j} < 0$ for any $j \neq i$ (using the notation defined in Section 3.1) [8]. A game-theoretic model in which buyers have negative identity-dependent externalities which "can stand for expected profits in future interaction" has been studied by Jehiel et al [9].

We extend our previous work on spitefulness in auctions [10], where we have already given an equilibrium strategy for spiteful agents in 2nd-price auctions with *complete information*, *i.e.*, all private values are known, by also considering 1st-price auctions and switching to the more realistic common prior model.[3] Until quite recently, we were oblivious to the existence of a surprisingly similar paper on spiteful bidding [11] that also shows the inequality of revenue in 1st- and 2nd-price auctions with spiteful bidders (as well as other results on English auctions). Nevertheless, we believe that our approach has its merits because the intuitive form of both equilibria as conditional expectations allows for a considerably simpler proof of revenue inequality. Furthermore, we obtain results for malicious bidders for the first time.

There are interesting equivalences of the proposed bidding equilibria to loosely related settings such as charity auctions, auctions with cheating sellers, toeholds, and knockout sales [12,13,14,15,16].

3   Preliminaries

In this section, we define the utility function of rational spiteful agents and the framework of the auction setting for which we study the bidding behavior of these agents.

3.1   Spiteful Agents

A spiteful agent maximizes the weighted difference of his own utility and his competitors' utility. It seems reasonable to take the average or maximum utility when speaking of the competitors' utility. However, since we only consider single-item auctions where all utilities except the winner's are zero, we simply employ the sum of all remaining agents' utilities.

Definition 1. *A* spiteful agent *maximizes the utility given by*

$$U_i = (1 - \alpha_i) \cdot u_i - \alpha_i \cdot \sum_{j \neq i} u_j \quad ,$$

where $\alpha_i \in [0, 1]$ is a parameter called spite coefficient*.*

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[3] We have used the term "antisocial" instead of "spiteful" in previous publications.

In the following, the term "utility" will refer to spiteful utility U_i. We will use the term "profit" to denote conventional utility u_i. Obviously, $\alpha = 0$ yields self-interested agents (whose utility equals their profit) whereas $\alpha = 1$ defines a completely *malicious* agent whose only goal is to minimize the profit of other agents without caring for his own well-being. When $\alpha = \frac{1}{2}$, we say that an agent is *balanced spiteful*.[4]

As mentioned in Section 2, other authors have suggested utility functions with a linear trade-off between self-interest and others' well-being. In contrast to these proposals, our definition differs in that the weight of one's own utility is not normalized to 1, allowing us to capture malicious agents who have no self-interest at all.

3.2   Auction Setting

Except for a preliminary result in Section 6, we assume that bidders are symmetric, in particular they all have the *same spite coefficient α*. Before each auction, private values v_i are drawn independently from a commonly known probability distribution over interval the $[0, 1]$ defined by the *cumulative distribution function (cdf) $F(v)$*. The *cdf* is defined as the probability that a random sample V drawn from the distribution does not exceed v: $F(v) = Pr(V \leq v)$. Its derivative, the *probability density function (pdf)*, is denoted by $f(v)$.

Once the auction starts, each bidder submits a bid based on his private value. The bidder who submitted the highest bid wins the auction. In the 1st-price auction, he pays the amount he bid whereas in the 2nd-price (or Vickrey) auction he pays the amount of the second-highest bid. Extending the notation of Krishna [17], we will denote equilibrium strategies of 1st- and 2nd-price auctions by $b_\alpha^I(v)$ and $b_\alpha^{II}(v)$, respectively. When bidders are self-interested ($\alpha = 0$), there all well-known equilibria for both auction types. The unique Bayes Nash equilibrium strategy for 1st-price auctions is to bid at the expectation of the second-highest private value, conditional on one's own value being the highest, $b_0^I(v) = E[X \mid X < v]$ where X is distributed according to $G(x) = F^{n-1}(x)$ [18,19]. 2nd-price auctions are strategy-proof, *i.e.*, $b_0^{II}(v) = v$ for any distribution of values [18]. Vickrey also first made the observation that expected revenue in both auction types is identical which was later generalized to a whole class of auctions in the Revenue Equivalence Theorem [20,19].

4   1st-Price Auctions

As is common in auction theory, we study symmetric equilibria, that is, equilibria in which all bidders use the same bidding function (mapping from valuations to bids). Symmetric equilibria are considered the most reasonable equilibria, but in principle need not be the only equilibria (see Section 4 for an asymmetric

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[4] In the case of only two balanced spiteful agents, the game at hand becomes a zero-sum game.

equilibrium in an auction with malicious bidders). Furthermore, we guess that the bidding function is strictly increasing and differentiable over $[0, 1]$. These assumptions impose no restriction on the general setting. They are only made to reduce the search space.

Theorem 1. *The following bidding strategy constitutes a Bayes Nash equilibrium for spiteful bidders in 1st-price auctions:*

$$b_{\alpha}^{\mathrm{I}}(v) = E[X \mid X < v] \quad \text{where } X \text{ is drawn from} \quad G_{\alpha}^{\mathrm{I}}(x) = F^{\frac{n-1}{1-\alpha}}(x).$$

Proof. Let $W_i = \big(b_i(v_i) > b_{(1)}(v_{-i})\big)$ be the event that bidder i wins the auction. $b_{(1)}(v_{-i})$ denotes the highest of all bids except i's. We will also write $v_{(1)}$ to denote the highest private value. $\bar{v}_i(b)$ is the inverse function of $b_i(v)$. We will use the short notation $\bar{v}$ for $\bar{v}_{(1)}(b_i(v_i))$ to improve readability. It is important to keep in mind that $\bar{v}$ is a function of $b_i(v_i)$, *e.g.*, when taking the derivative of the expected utility.

We will now give the utility of a spiteful agent as defined in Definition 1 for 1st-price auctions. Recall that agent i knows his own private value v_i, but only has probabilistic beliefs about the remaining $n - 1$ private values (and bids).

$$
\begin{aligned}
E\big[U_i(b_i(v_i))\big] = (1 - \alpha) \cdot Pr(W_i) \cdot \Big(v_i - b_i(v_i)\Big) - \\
\alpha \cdot (1 - Pr(W_i)) \cdot \Big(E\big[v_{(1)} \mid \neg W_i\big] - E\big[b_{(1)}(v_{-i}) \mid \neg W_i\big]\Big)
\end{aligned}
\tag{1}
$$

We can ignore ties in this formulation because they are zero probability events in the continuous setting we consider. By definition, the probability that any private value is lower than i's value is given by $F(v_i)$. Since all values are independently distributed, the probability that bidder i has the highest private value is $F^{n-1}(v_i)$. Thus, the probability that i submits the highest *bid* can be expressed by using the inverse bid function.

$$Pr(W_i) = F^{n-1}(\bar{v}) \tag{2}$$

The *cdf* of the highest of $n-1$ private values is $F_{(1)}(v) = F^{n-1}(v)$. The associated *pdf* is $f_{(1)}(v) = (n-1)F^{n-2}(v) \cdot f(v)$. Using standard formulas for the conditional expectation (see Appendix A), this allows us to compute both expectation values on the right-hand side of Equation 1.

$$E\big[v_{(1)} \mid \neg W_i\big] = \frac{1}{1 - F^{n-1}(\bar{v})} \int_{\bar{v}}^{1} t \cdot (n-1)F^{n-2}(t) \cdot f(t) \, dt \tag{3}$$

$$E\big[b_{(1)}(v_{-i}) \mid \neg W_i\big] = \frac{1}{1 - F^{n-1}(\bar{v})} \int_{b_i(v_i)}^{b_i(1)} t \cdot (n-1)F^{n-2}(\bar{v}(t)) \cdot f(\bar{v}(t)) \cdot \bar{v}'(t) \, dt \tag{4}$$

Inserting these expectations in Equation 1 and simplifying the result yields

$$
\begin{aligned}
E\big[U_i(b_i(v_i))\big] &= (1-\alpha)(F^{n-1}(\bar{v})v_i - F^{n-1}(\bar{v})b_i(v_i))- \\
&\quad \alpha(n-1)\Bigg(\int_{\bar{v}}^{1} t \cdot F^{n-2}(t) \cdot f(t)\ dt- \\
&\quad \int_{b_i(v_i)}^{b_i(1)} t \cdot F^{n-2}(\bar{v}(t)) \cdot f(\bar{v}(t)) \cdot \bar{v}'(t)\ dt \Bigg).
\end{aligned}
$$

When taking the derivative with respect to $b_i(v_i)$, both integrals vanish due to the Fundamental Theorem of Calculus and the following observation:

$$
\frac{\partial \int_{\bar{v}(b)}^{1} g(t)\ dt}{\partial b} = \frac{\partial \big(G(1) - G(\bar{v}(b))\big)}{\partial b} = 0 - g\big(\bar{v}(b)\big) \cdot \bar{v}'(b) \tag{5}
$$

In order to obtain the strategy that generates maximum utility we take the derivative and set it to zero.

$$
\begin{aligned}
0 = (1-\alpha)\Big(&(n-1)F^{n-2}(\bar{v}) \cdot f(\bar{v}) \cdot \bar{v}' \cdot v_i- \\
&(n-1)F^{n-2}(\bar{v}) \cdot f(\bar{v}) \cdot \bar{v}' \cdot b_i(v_i) - F^{(n-1)}(\bar{v})\Big)- \\
&\alpha(n-1)\Big(\big(0 - \bar{v} \cdot F^{n-2}(\bar{v}) \cdot f(\bar{v}) \cdot \bar{v}'\big) - \big(0 - b_i(v_i) \cdot F^{n-2}(\bar{v}) \cdot f(\bar{v}) \cdot \bar{v}'\big)\Big)
\end{aligned}
$$

From this point on, we treat v_i as a variable (instead of $b_i(v_i)$) and assume that all bidding strategies are identical, *i.e.*, $\bar{v} = \bar{v}_{(1)}(b_i(v_i)) = v_i$. Using the fact that the derivative of the inverse function is the reciprocal of the original function's derivative $(\bar{v}'(b_i(v_i)) = \frac{1}{b_i'(v_i)})$, we can rearrange terms to obtain the following differential equation:

$$
b(v) = v - \frac{(1-\alpha) \cdot F(v) \cdot b'(v)}{(n-1) \cdot f(v)} \tag{6}
$$

It follows that $b(v) \leq v$ because the fraction on the right-hand side of the previous equation is always non-negative (the bidding function is strictly increasing). Since we assume that there are no negative bids, this yields the boundary condition $b(0) = 0$. The solution of Equation 6 with boundary condition $b(0) = 0$ is

$$
b(v) = \frac{1}{F^{\frac{n-1}{1-\alpha}}(v)} \int_{0}^{v} t \cdot \frac{n-1}{1-\alpha} \cdot F^{\frac{n-1}{1-\alpha}-1}(t) \cdot f(t)\ dt. \tag{7}
$$

Strikingly, the right-hand side of this equation is a conditional expectation (see Appendix A). More precisely, it is the expectation of the highest of $\frac{n-1}{1-\alpha}$ private values below v:

$$
b(v) = E[X \mid X < v] \quad \text{where the } cdf \text{ of } X \text{ is given by} \quad G_\alpha^{\mathrm{I}}(x) = F^{\frac{n-1}{1-\alpha}}(x) \tag{8}
$$

$\square$

In 1st-price auctions, bidders face a tradeoff between the probability of winning and the profit conditional on winning. An intuition behind the equilibrium for spiteful agents is that the more spiteful a bidder is, the less emphasis he puts on his expected profit. Whereas a self-interested bidder bids at the expectation of the highest of $n-1$ private values below his own value, a balanced spiteful agents bids at the expectation of the highest of $2(n-1)$ private values below his value. Interestingly, agents are "least truthful" when they are self-interested. Any level of spite makes them more truthful. Furthermore, parameter α defines a continuum of bidding equilibria between the well-known standard equilibria of 1st-price and 2nd-price auctions. Even though the strategy defined in Equation 8 is not defined for $\alpha = 1$, it can easily be seen from Equation 6, that $b_1^{\text{I}}(v) = v$.

Corollary 1. *The 1st-price auction is (Bayes Nash) incentive-compatible for malicious bidders ($\alpha = 1$).*

This result is perhaps surprising because one might expect that always bidding 1 is an optimal strategy for malicious bidders. The following consideration shows why this is not the case. Assume that all agents are bidding 1. Agent i's expected utility depends on the tie resolution policy. In $n-1$ cases, his expected utility is positive. In the remaining case, if he wins, his utility is zero. By bidding less than 1, he can ensure that his expected utility is always positive.

Curiously, there are other, asymmetric, equilibria for malicious bidders, *e.g.*, a "threat" equilibrium where one bidder always bids 1 and everybody else bids some value below his private value. It is well-known that asymmetric equilibria like this exist in 2nd-price auctions (see [21] for a complete characterization). However, asymmetric equilibria in 2nd-price auctions are (weakly) *dominated* whereas the one given above is not, making it more reasonable.

One way to gain insight in the equilibrium strategy is to instantiate $F(v)$ with the uniform distribution.

Corollary 2. *The following bidding strategy constitutes a Bayes Nash equilibrium for spiteful bidders in 1st-price auctions when private values are uniformly distributed.*

$$b_\alpha^{\text{I}}(v) = \frac{n-1}{n-\alpha} \cdot v$$

Whereas one can get full intuition in the extreme points of the strategy ($\alpha = \{0,1\}$), the fact that the scaling between both endpoints of the equilibrium spectrum is not linear in α, even for a uniform prior, is somewhat surprising. Interestingly, there is a direct correspondence between the equilibrium proposed in Theorem 1 and the equilibrium for 2nd-price auctions when the seller may cheat with probability $1 - \alpha$ (see [16]). Here, cheating means that the seller, with some probability, places a bogus bid right below the highest bid in order to increase his revenue.

5 2nd-Price Auctions

In this section, we derive an equilibrium strategy for spiteful agents in 2nd-price auctions using the same set of assumptions made in Section 4.

Theorem 2. *The following bidding strategy constitutes a Bayes Nash equilibrium for spiteful bidders in 2nd-price auctions.*

$$b_\alpha^{\mathrm{II}}(v) = E[X \mid X > v] \quad \text{where } X \text{ is drawn from} \quad G_\alpha^{\mathrm{II}}(x) = 1 - (1 - F(x))^{\frac{1}{\alpha}}$$

Proof. W_i, $\bar{v}$ and all other notations are defined as in the proof of Theorem 1. The expected utility of spiteful agent i in 2nd-price auctions can be described in the following way. There are two general cases depending on whether bidder i wins or loses. In the former case, the utility is simply v_i minus the expected highest bid (except i's). In the latter case, we can give the expectation of the winner's private value. In order to specify the selling price, we need to distinguish between two subcases: If bidder i submitted the second-highest bid, the selling price is his bid b_i. Otherwise, *i.e.*, if the second-highest of all remaining bids is greater than b_i, we can again give a conditional expectation.

$$
\begin{aligned}
E\big[U_i(b_i(v_i))\big] = (1 - \alpha) \cdot \ & Pr(W_i) \cdot \Big(v_i - E[b_{(1)}(v_{-i}) \mid W_i] \Big) - \\
\alpha \cdot \ & \Big((1 - Pr(W_i)) \cdot E\big[v_{(1)} \mid \neg W_i\big] - \\
& Pr\big((b_i(v_i) < b_{(1)}(v_{-i})) \wedge (b_i(v_i) > b_{(2)}(v_{-i})) \big) \cdot b_i(v_i) - \\
& Pr(b_i(v_i) < b_{(2)}(v_{-i})) \cdot E[b_{(2)}(v_{-i}) \mid b_i < b_{(2)}(v_{-i})] \Big)
\end{aligned}
$$
(9)

The conditional expectation of the remaining highest bid, in case bidder i wins, is as follows (see Appendix A).

$$E\big[b_{(1)}(v_{-i}) \mid W_i\big] = \frac{1}{F^{n-1}(\bar{v})} \int_{b_i(0)}^{b_i(v_i)} t \cdot (n-1) F^{n-2}(\bar{v}(t)) \cdot f(\bar{v}(t)) \cdot \bar{v}'(t) \, dt \quad (10)$$

We have already given a formula for $E\big[v_{(1)} \mid \neg W_i\big]$ in Equation 3. The probability that b_i is the second-highest bid equals the probability that exactly one bid is higher than b_i and $n - 2$ bids are lower than b_i. Depending on who submitted the highest bid, there are $n - 1$ different ways in which this can occur.

$$Pr\big((b_i(v_i) < b_{(1)}(v_{-i})) \wedge (b_i(v_i) > b_{(2)}(v_{-i})) \big) = (n-1) \cdot F^{n-2}(\bar{v}) \cdot (1 - F(\bar{v})) \quad (11)$$

The *cdf* of the second-highest private value (of $n - 1$ values) can be derived by computing the probability that the second-highest value is less than or equal to a given v. Either all $n - 1$ values are lower than v, or $n - 2$ values are lower and one is greater than v. As above, there are $n - 1$ different possibilities in the latter case. Thus,

$$F_{(2)}(v) = F^{n-1}(v) + (n-1)F^{n-2}(1 - F(v)) = (n-1) \cdot F^{n-2}(v) - (n-2) \cdot F^{n-1}(v).$$

It follows that the *pdf* is $f_{(2)}(v) = (n-1) \cdot (n-2) \cdot (1 - F(v)) \cdot F^{n-3}(v) \cdot f(v)$. Finally, the conditional expectation of the second-highest bid times the probability of this bid being higher than b_i is

$$
\begin{aligned}
Pr(b_i(v_i) &< b_{(2)}(v_{-i})) \cdot E[b_{(2)}(v_{-i}) \mid b_i < b_{(2)}(v_{-i})] = \\
(n-1) \cdot (n-2) \cdot &\int_{b_i(v_i)}^{b_i(1)} t \cdot (1 - F(\bar{v}(t))) \cdot F^{n-3}(\bar{v}(t)) \cdot f(\bar{v}(t)) \cdot \bar{v}'(t) \, dt. \quad (12)
\end{aligned}
$$

Inserting both expectations and the probability of winning (see Equation 2) into Equation 9 yields

$$
\begin{aligned}
E\big[U_i(b_i(v_i))\big] = \quad & (1-\alpha)\cdot\left(F^{n-1}(\bar{v})v_i-\right.\\
& \left.(n-1)\int_{b_i(0)}^{b_i(v_i)} t\cdot F^{n-2}(\bar{v}(t))\cdot f(\bar{v}(t))\cdot \bar{v}'(t)\, dt\right)-\\
\alpha\cdot(n-1)\cdot & \left(\int_{\bar{v}}^{1} t\cdot F^{n-2}(t)\cdot f(t)\, dt-\right.\\
& F^{n-2}(\bar{v})\cdot(1-F(\bar{v}))\cdot b_i(v_i)-(n-2)\\
& \left.\int_{b_i(v_i)}^{b_i(1)} t\cdot(1-F(\bar{v}(t)))F^{n-3}(\bar{v}(t))f(\bar{v}(t))\bar{v}'(t)\, dt\right).
\end{aligned}
$$

As in the previous section, we now take the derivative with respect to $b_i(v_i)$ and set it to zero. All integrals vanish due to the Fundamental Theorem of Calculus and the formula given in Equation 5.

$$
\begin{aligned}
0 = \quad & (1-\alpha)\cdot\Big((n-1)F^{n-2}(\bar{v})\cdot f(\bar{v})\cdot \bar{v}'\cdot v_i-\\
& (n-1)(b_i(v_i)\cdot F^{n-2}(\bar{v})\cdot f(\bar{v})\cdot \bar{v}')\Big)-\\
\alpha\cdot(n-1)\cdot & \Big((0-\bar{v}\cdot F^{n-2}(\bar{v})\cdot f(\bar{v})\cdot \bar{v}')-\\
& ((n-2)\cdot F^{n-3}(\bar{v})\cdot f(\bar{v})\cdot \bar{v}'\cdot b_i(v_i)+F^{n-2}(\bar{v})-\\
& (n-1)\cdot F^{n-2}(\bar{v})\cdot f(\bar{v})\cdot \bar{v}'\cdot b_i(v_i)-F^{n-1}(\bar{v}))-\\
& (n-2)\cdot(0-b_i(v_i)\cdot(1-F(\bar{v}))\cdot F^{n-3}(\bar{v})\cdot f(\bar{v})\cdot \bar{v}'\Big)
\end{aligned}
$$

Using the fact that the derivative of the inverse function is the reciprocal of the original function's derivative ($\bar{v}'(b_i(v_i)) = \frac{1}{b_i'(v_i)}$) and $\bar{v} = v_i$, we can simplify and rearrange terms to obtain the following differential equation.

$$
b(v) = v + \frac{\alpha\cdot(1-F(v))\cdot b'(v))}{f(v)} \tag{13}
$$

It turns out that $b(0) = 0$ does not hold for 2nd-price auctions. However, a boundary condition can easily be obtained by letting $v = 1$. By definition, $F(1) = 1$ which yields $b(1) = 1$. Given this boundary condition, the solution of Equation 13 is

$$
b(v) = \frac{1}{(1-F(v))^{\frac{1}{\alpha}}}\int_{v}^{1} \frac{t\cdot(1-F(t))^{\frac{1}{\alpha}-1}\cdot f(t)}{\alpha}\, dt. \tag{14}
$$

Like in the Theorem 1, the right-hand side of Equation 14 resembles a conditional expectation. In fact, the bidding strategy can be reformulated as the expectation of some random variable X, given that $X > v$.

$$
b(v) = E[X \mid X > v] \quad \text{where the } cdf \text{ of } X \text{ is given by} \quad G_\alpha^{\mathrm{II}}(x) = 1-(1-F(x))^{\frac{1}{\alpha}} \tag{15}
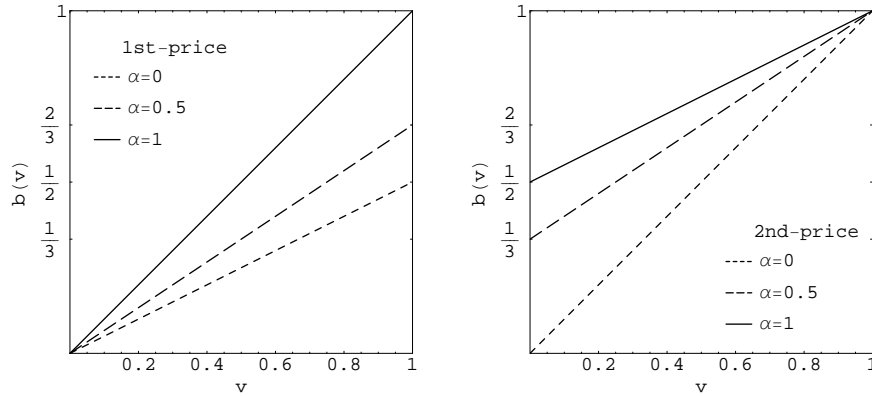$$

It can easily be checked that $G_\alpha^{\mathrm{II}}(x)$ is indeed a valid *cdf* ($G_\alpha^{\mathrm{II}}(0) = 0$, $G_\alpha^{\mathrm{II}}(1) = 1$, and $G_\alpha^{\mathrm{II}}(x)$ is non-decreasing). By inserting this *cdf* in Equation 18, we obtain the equilibrium bidding strategy. The resulting expectation is the expected value of the lowest of $\frac{1}{\alpha}$ values above v. Equation 15 is not defined for $\alpha = 0$, but the correct equilibrium can quickly be obtained from Equation 13. $\square$

Remarkably, the resulting equilibrium strategy is independent of the number of bidders n (though it *does* depend on the prior distribution of private values). For example, a balanced spiteful bidder bids at the expectation of the lowest of two private values above his own value. As in the previous section, we try to get more insight in the equilibrium by instantiating the uniform distribution.

Corollary 3. *The following bidding strategy constitutes a Bayes Nash equilibrium for spiteful bidders in 2nd-price auctions when private value are uniformly distributed.*

$$b_\alpha^{\mathrm{II}}(v) = \frac{v + \alpha}{1 + \alpha}$$

For example, given a uniform prior, the optimal strategy for balanced spiteful agents is $b(v) = \frac{2}{3} \cdot v + \frac{1}{3}$, regardless of the number of bidders. Figure 1 shows equilibrium strategies for both auction types and varying spite coefficients. As in

$b_\alpha^{\mathrm{I}}(v)$ and $b_\alpha^{\mathrm{II}}(v)$ for $n = 2$

Fig. 1. Spiteful equilibrium bidding strategies

the 1st-price auction setting, the surprising equilibrium strategies are those for $0 < \alpha < 1$. There is no linear scaling between both extreme points of the equilibrium spectrum. As we will see in the following section, this leads to important consequences on auction revenue.

6   Consequences

In order to obtain useful results from these equilibria, we compare a key measure in auctions, the seller's revenue. The well-known Revenue Equivalence Theorem, which states that members of a large class of auctions all yield the same revenue, does not hold when agents are spiteful. Figure 2 shows the expected revenue in both auction types when agents are balanced spiteful and private values are uniformly distributed.

It can be shown that the revenue gap visible in the figure exists independently of prior and spite coefficient as long as agents are neither self-interested nor malicious.

Theorem 3. *For the same spite coefficient $0 < \alpha < 1$, the 2nd-price auction yields more expected revenue than the 1st-price auction. When $\alpha \in \{0, 1\}$, expected revenue in both auction types is equal in the symmetric equilibrium.*

Proof. The statement can be deduced from the following three observations:

- *When agents are malicious, expected revenue in both auction types is identical.*
  In the 1st-price auction, truthful bidding is in equilibrium. In the 2nd-price auction, the second-highest bidder bids at the expectation of the highest private value. In both cases, revenue equals the expectation of the highest value.
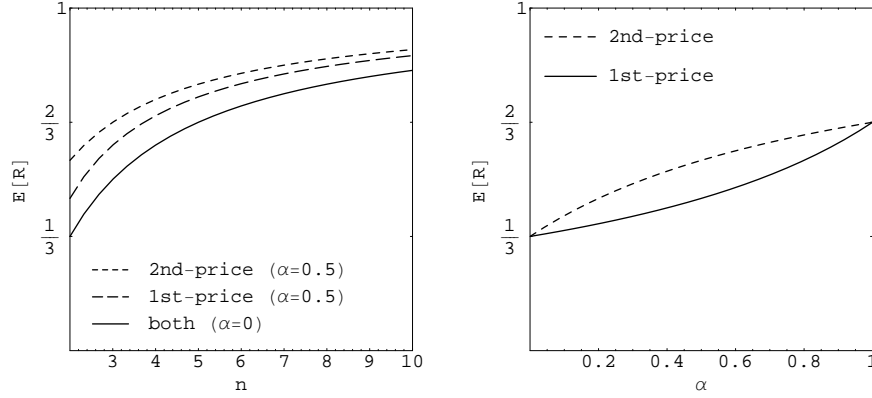- $b_\alpha^I(v)$ *and* $b_\alpha^{II}(v)$ *are strictly increasing in* α.
  In the 1st-price auction, bidders bid at the (conditional) expectation of the highest value of a number of private values that *grows* as α increases. In the 2nd-price auction, bidders bid at the expectation of the lowest value of a number of private values that *shrinks* as α increases. Obviously, both expectations are increasing in α. More formally, G_α^I stochastically dominates G_β^I and G_α^{II} stochastically dominates G_β^{II} for any $\alpha > \beta$.
- $b_\alpha^I(v)$ *is convex in* α. $b_\alpha^{II}(v)$ *is concave in* α.
  Since both equilibria are symmetric, we just need to consider the curvature of expectations distributed according to $G_\alpha^I(X)$ and $G_\alpha^{II}(X)$ for variable α. Bids in 1st-price auctions are the (conditional) expectation of the highest of $\frac{1}{1-\alpha}$ values. The slope of this expectation increases as α rises. In 2nd-price auctions, bids are the expectation of the *lowest* of $\frac{1}{\alpha}$ values. If it were the highest value, the slope would be increasing too. However, since it is the expectation of the lowest value, the slope is strictly decreasing in α.

Let $E[R_\alpha^I]$ and $E[R_\alpha^{II}]$ be the expected revenue in 1st- and 2nd-price auctions, respectively, and consider these as functions of α. So far, we know that both functions are equal for $\alpha \in \{0, 1\}$ and strictly increasing. Furthermore, $E[R_\alpha^I]$ is convex and $E[R_\alpha^{II}]$ is concave. These facts imply that $E[R_\alpha^{II}] > E[R_\alpha^I]$ for any $0 < \alpha < 1$ (see Figure 2). □

Revenue inequalities for other special conditions such as when bidders or the seller are risk-averse have been used to argue in favor of one auction form over the

Left: Expected revenue (interpolated for non-integer n)
Right: Expected revenue for varying α ($n = 2$)

Fig. 2. Expected revenue

other. Hence, it is appropriate to state that Theorem 3 represents an advantage of the 2nd-price auction (from the perspective of a seller), especially because the inequality holds for arbitrarily small $\alpha > 0$. On the other hand, the difference in expected revenue is very small, even for just few bidders. For example, the difference in expected revenue for ten bidders with uniformly distributed private values is less than 2% for any α (see also Figure 3). An interesting aspect of Figure 3 is that revenue seems to be maximal for some α slightly below 0.5.

Theorem 4. *The difference in expected revenue between 2nd-price and 1st-price auctions is maximal for some $\alpha \leq 0.5$ that approaches $\frac{1}{1+\sqrt{2}} \approx 0.4142$ in the limit as n rises, when private values are uniformly distributed.*

Proof. By definition, the revenue difference is the difference of the expectation of the second-highest bid in 2nd-price auctions minus the highest expected bid in 1st-price auctions. Instantiating with the uniform distribution, we obtain

$$E[R_\alpha^{II}] - E[R_\alpha^I] = b_\alpha^{II}\left(E[v_{(2)}]\right) - b_\alpha^I\left(E[v_{(1)}]\right)$$

$$= \frac{\frac{n-1}{n+1} + \alpha}{1 + \alpha} - \frac{n-1}{n-\alpha} \cdot \frac{n}{n+1} = \frac{(1-\alpha) \cdot \alpha}{(\alpha+1) \cdot (n-\alpha)} \overset{[0<\alpha<1]}{>} 0. \tag{16}$$

In order to obtain the maximal revenue difference, we take the derivative of the expression given in Equation 16 with respect to α.

$$0 = \frac{\partial}{\partial \alpha} \cdot \frac{(1-\alpha) \cdot \alpha}{(\alpha+1) \cdot (n-\alpha)} = \frac{\partial}{\partial \alpha} \cdot \frac{\alpha^2 - \alpha}{\alpha - n + \alpha^2 - \alpha \cdot n} =$$

$$-\frac{(\alpha^2 - \alpha)(2 \cdot \alpha - n + 1)}{(\alpha^2 + \alpha - n - \alpha \cdot n)^2} + \frac{2 \cdot \alpha - 1}{\alpha^2 + \alpha - n - \alpha \cdot n}$$

$$\implies \alpha_{max} = \frac{n}{n + \sqrt{2} \cdot \sqrt{n \cdot (n-1)}}$$

When there are only two bidders, $\alpha_{max} = 0.5$. α_{max} is strictly decreasing as n grows.

$$\lim_{n \to \infty} \alpha_{max} = \frac{1}{1 + \sqrt{2}} \approx 0.4142$$
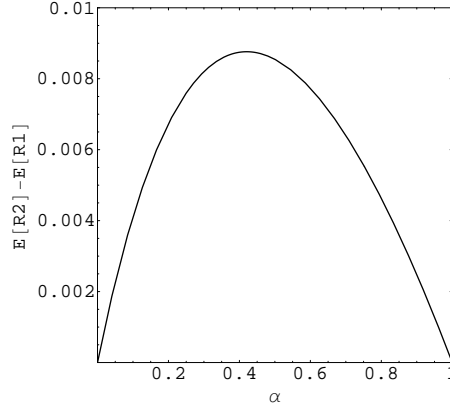
$\square$

Fig. 3. Difference of expected revenue for $n = 20$

An important extension of our setting is one that deals with asymmetries in spitefulness. For example, it would be very desirable to extend the revenue inequality (Theorem 3) to arbitrary profiles of spite coefficients $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ or a general prior from which each α_i is drawn. A first step towards this direction can be made by observing that the equilibrium strategies of self-interested bidders are somewhat "robust" against spiteful bidding.

Proposition 1. *Rational self-interested bidders will stick with their bidding strategy when other agents bid according to the strategies given in Theorem 1 and 2, respectively, and private values are uniformly distributed.*

Proof. The statement for 1st-price auctions follows from a result by Porter et al [16] where it has been proven that bidders in 1st-price auctions will stick with their equilibrium strategy even when other bidders bid constant fractions of their private value larger than $\frac{n-1}{n} \cdot v$ (in the case of a uniform prior). This holds for a certain class of probability distributions, including the uniform distribution. The statement for 2nd-price auctions trivially follows from the fact that bidding truthfully is a dominant strategy for self-interested agents and therefore holds for any given prior. □

The previous proposition relates to a setting in which there are self-interested and spiteful agents participating in the same auction. Self-interested agents are aware of this asymmetry whereas spiteful agents believe that everybody is spiteful.

7   Conclusion

We studied the bidding behavior of spiteful agents who, contrary to the common assumption of self-interest, maximize the weighted difference of their own profit and their competitors' profit. We derived symmetric Bayes Nash equilibria for spiteful agents in 1st-price and 2nd-price sealed-bid auctions. The main results are as follows. In 1st-price auctions, bidders become "more truthful" the more spiteful they are. When bidders are completely malicious, truth-telling is in Nash equilibrium. Surprisingly, the equilibrium strategy in 2nd-price auctions does not depend on the number of bidders. Based on these equilibria, we compared revenue in both auction types. It turned out that revenue equivalence breaks down for this setting. Expected revenue in 2nd-price auctions is higher than revenue in 1st-price auctions whenever the spite coefficient α satisfies $0 < \alpha < 1$. Revenue equivalence only holds for auctions in which all agents are either self-interested ($\alpha = 0$) or malicious ($\alpha = 1$).

There are many open problems left for future work. Most importantly, we intend to extend the revenue inequality (Theorem 3) to settings with asymmetric spitefulness.

Acknowledgements

References

1. Klemperer, P.: How (not) to run auctions: the European 3G telecom auctions. European Economic Review **46** (2002) 829–845
2. Grimm, V., Riedel, F., Wolfstetter, E.: The third generation (UMTS) spectrum auction in Germany. ifo Studien **48** (2002)
3. Levine, D.K.: Modeling altruism and spitefulness in experiments. Review of Economic Dynamics **1** (1998) 593–622
4. Fehr, E., Schmidt, K.: Theories of fairness and reciprocity - Evidence and economic applications. Working Paper (2001)
5. Sobel, J.: Interdependent preferences and reciprocity. Working Paper (2004)
6. Messick, D.M., Sentis, K.P.: Estimating social and nonsocial utility functions from ordinal data. European Journal of Social Psychology **15** (1985) 389–399
7. Loewenstein, G., Thompson, L., Bazerman, M.: Social utility and decision making in interpersonal contexts. Journal of Personality and Social Psychology **57** (1989) 426–441
8. Brainov, S.: The role and the impact of preferences on multiagent interaction. In Jennings, N.R., Lespérance, Y., eds.: Intelligent Agents VI. Volume 1757 of Lecture Notes in Artificial Intelligence (LNAI)., Springer (1999) Revised papers from the 6th International Workshop on Agent Theories, Architectures and Languages (ATAL).
9. Jehiel, P., Moldovanu, B., Stacchetti, E.: How (not) to sell nuclear weapons. American Economic Review **86** (1996) 814–829
10. Brandt, F., Weiß, G.: Antisocial agents and Vickrey auctions. In Meyer, J.J.C., Tambe, M., eds.: Intelligent Agents VIII. Volume 2333 of Lecture Notes in Artificial Intelligence (LNAI)., Springer (2001) 335–347 Revised papers from the 8th International Workshop on Agent Theories, Architectures and Languages (ATAL).
11. Morgan, J., Steiglitz, K., Reis, G.: The spite motive and equilibrium behavior in auctions. Contributions to Economic Analysis & Policy **2** (2003) 1102–1127
12. Deltas, G.: Determining damages from the operation of bidding rings: An analysis of the post-auction 'knockout' sale. Economic Theory **19** (2002) 243–269
13. Bulow, J., Huang, M., Klemperer, P.: Toeholds and takeovers. Journal of Political Economy **107** (1999) 427–454
14. Engelbrecht-Wiggans, R.: Auctions with price-proportional benefits to bidders. Games and Economic Behavior **6** (1994) 339–346
15. Engers, M., McManus, B.: Charity auctions. Working Paper (2002)
16. Porter, R., Shoham, Y.: On cheating in sealed-bid auctions. In: Proceedings of the 4th ACM Conference on Electronic Commerce (ACM-EC), ACM Press (2003) 76–84
17. Krishna, V.: Auction Theory. Academic Press (2002)
18. Vickrey, W.: Counter speculation, auctions, and competitive sealed tenders. Journal of Finance **16** (1961) 8–37
19. Riley, J., Samuelson, W.: Optimal auctions. American Economic Review **71** (1981) 381–392
20. Myerson, R.: Optimal auction design. Mathematics of Operations Research **6** (1981) 58–73
21. Blume, A., Heidhues, P.: All equilibria of the Vickrey auction. Journal of Economic Theory **114** (2004) 170–177

A   Conditional Expectations

Let X be random variable drawn from the interval $[0, 1]$ according to the cumulative distribution function $F(x)$. The expectation of X is $E[X] = \int_0^1 t \cdot f(t) \, dt$. The conditional expectation that X is greater or smaller than some constant x, respectively, is given by the following formulas.

$$E[X \mid X < x] = \frac{1}{F(x)} \int_0^x t \cdot f(t) dt \qquad (17)$$

$$E[X \mid X > x] = \frac{1}{1 - F(x)} \int_x^1 t \cdot f(t) dt \qquad (18)$$