Coequalisers in formal topology

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Formal topology in the sense of Martin-Löf and Sambin (Sambin 1987, 2003) may be considered as a predicative version of constructive locale theory (Johnstone 1982, Joyal and Tierney 1984). In order for the theory to permit the usual topological constructions, such as quotienting, gluing subspaces and attaching maps, it is enough that the category of formal topologies and continuous mappings has finite limits and finite colimits. See e.g. (Palmgren 2003) for a survey of earlier results, e.g. the construction of products and coproducts and equalisers. In this paper we provide the missing piece: construction of coequalisers.

In the category of locales the coequaliser of a pair of morphisms can easily be constructed as an equaliser in the dual category of frames (see e.g. Borceux 1994). The straightforward translation of this construction in terms of formal topologies is

$$\{U \in \mathcal{P}(Y) : (\forall a \in X)(a \lhd F^{-1}U \iff a \lhd G^{-1}U)\}$$

for a pair of continuous mappings $F, G : X \to Y$ between formal topologies. From a predicative point of view the problem with this construction is the use of the full power set $\mathcal{P}(Y)$. We show that it can be replaced by a restricted set of subsets which may indeed be constructed in, e.g., Martin-Löf type theory.

Together with known predicative constructions of products (Coquand *et al.* 2003) and coproducts, and equalisers (Palmgren 2003) the above result gives that the set-presented formal topologies form a small complete and small cocomplete category, just as the classical topological spaces. This indicates that the category should be adequate for constructing many of the spaces studied by methods of algebraic topology.

Already in the setting of neighbourhood spaces (and thus with points) surprisingly difficult predicativity problems appear when constructing quo-

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tient spaces or coequalisers. The problem was solved independently, simultaneously and using different methods by Hajime Ishihara and the author in October 2004; see the forthcoming paper (Ishihara and Palmgren 200?).

1 The category of set-presented formal topologies

Following Bishop and Bridges (1985), and category-theoretic practice, a subset $A = (\iota, I)$ of a given set X is an injective function $\iota : I \to X$. An element x of X is *member of the subset* A, if $x = \iota(a)$ for some $a \in A$. Note that this a is necessarily unique. We write $x \in_X A$. Two subsets A and B of X *equal* if

$$x \in_X A \Longleftrightarrow x \in_X B.$$

From this arises easily notions of inclusion and the usual set-theoretical operations.

For any family $\mathcal{U}$ of types $T(t)$ $(t : U)$ there is a notion of $\mathcal{U}$*-set,* which is a set A which is isomorphic to a set of the form $(T(t), =_e)$ where

$$x =_e y \Longleftrightarrow T(e(x, y))$$

and $e : T(t) \times T(t) \to U$. For any set X there is then a notion of restricted power set $\mathcal{R}_{\mathcal{U}}(X)$. This is a set consisting of subsets $A = (I, \iota)$ of X where I is a $\mathcal{U}$-set. Such subsets are called $\mathcal{U}$*-subsets.* Two such are identified if they are equal as subsets. Unless the family of types have certain closure properties it will not be possible to perform the usual set operations on the restricted power set. We return to the question of what these properties might be later.

A set X is a *projective set* or a *choice-set* if the axiom of choice is valid on X. The latter means that for any set Y and for any relation R between X and Y if

$$(\forall x \in X)(\exists y \in Y)R(x, y)$$

then there is a function $f : X \to Y$ so that

$$(\forall x \in X)R(x, f(x)).$$

As any type in Martin-Löf type theory can be equipped with an equality relation (given by an Id-type) so that it becomes a projective set $\underline{X}$, the above choice principle is sometimes referred to as *type-theoretic choice.* The principle is frequently used in Bishop-style constructivism. We thus assume that for every set X there is a projective set $\underline{X}$ and a surjective function

$p_X : \underline{X} \to X$. Then we get the following choice principle which is sometimes useful

$$(\forall x \in X)(\exists y \in Y)R(x,y) \Longrightarrow (\exists f : \underline{X} \to Y)(\forall x \in \underline{X})R(p_X(x), f(x)). \quad (1)$$

Definition 1.1 Let S be a set, and let $\lhd$ be a relation between elements of S and subsets of S, i.e. $\lhd \subseteq S \times \mathcal{P}(S)$. Extend $\lhd$ to a relation between subsets by letting $U \lhd V$ if and only if $a \lhd V$ for all $a \in U$. For a preorder $(X, \leq)$ and a subset $U \subseteq X$, its *downwards closure* $U_{\leq}$ consists of those $x \in X$ such that $x \leq y$ for some $y \in U$. Write $a_{\leq}$ for $\{a\}_{\leq}$. When the preorder is obvious from the context we write $U \wedge V$ for $U_{\leq} \cap V_{\leq}$. A further abbreviation is $a \wedge b$ for $\{a\} \wedge \{b\}$.

Definition 1.2 A *formal topology* $\mathcal{S}$ is a pre-ordered set $S = (S, \leq)$ (of so-called *basic neighbourhoods*) together with a relation $\lhd \subseteq S \times \mathcal{P}(S)$, the *covering relation,* satisfying the four conditions

(R) $a \in U$ implies $a \lhd U$, (L) $a \lhd U$, $a \lhd V$ implies $a \lhd U \wedge V$,
(T) $a \lhd U$, $U \lhd V$ implies $a \lhd V$, (E) $a \leq b$ implies $a \lhd \{b\}$.

The topology is *set-presented* if there is a family of subsets $C(a,i)$ of S, where $i \in I(a)$ and $a \in S$ such that

$$a \lhd U \Longleftrightarrow (\exists i \in I(a))\, C(a,i) \subseteq U.$$

Equivalently, we may express this as: there is a family $C(w)$ ($w \in I$) of subsets of S and a function $p : I \to S$ so that

$$a \lhd U \Longleftrightarrow (\exists w \in I)\, p(w) = a \;\&\; C(w) \subseteq U. \quad (2)$$

A continuous mapping between formal topologies is a certain relation between their basic neighbourhoods. To define the concept we introduce some notation. For a relation $R \subseteq S \times T$ the *inverse image of* $V \subseteq T$ *under the relation* R is, as usual,

$$R^{-1}[V] =_{\text{def}} \{a \in S : (\exists b \in V)\, a\, R\, b\}$$

Notice that, in general, $R^{-1}[U] \subseteq R^{-1}[V]$ whenever $U \subseteq V$, and

$$R^{-1}[\cup_{i \in I} U_i] = \cup_{i \in I} R^{-1}[U_i].$$

The relation R is naturally extended to subsets as follows. For $U \subseteq S$, let $U\, R\, b$ mean $(\forall u \in U)\, uRb$, and for $V \subseteq T$, we let $a\, R\, V$ mean $a \lhd R^{-1}[V]$.

Definition 1.3 Let $\mathcal{S} = (S, \leq, \lhd)$ and $\mathcal{T} = (T, \leq', \lhd')$ be formal topologies. A relation $R \subseteq S \times T$ is a *continuous mapping,* or *continuous morphism,* from $\mathcal{S}$ to $\mathcal{T}$ (and we write $R : \mathcal{S} \to \mathcal{T}$) if

(A1) $a\,R\,b$, $b \lhd' V$ implies $a\,R\,V$,

(A2) $a \lhd U$, $U\,R\,b$, implies $a\,R\,b$,

(A3) $a\,R\,T$, for all $a \in S$,

(A4) $a\,R\,V$, $a\,R\,W$ implies $a\,R\,(V_{\leq'} \cap W_{\leq'})$.

Remark 1.4 Note that by $b \lhd' \{b\}$, (A1) and (A2)

$$\{a\}\,R\,b \Longleftrightarrow a\,R\,b \Longleftrightarrow a \lhd R^{-1}\{b\} \Longleftrightarrow a\,R\,\{b\}.$$

Moreover (A4) may be replaced by the condition

(A4') $a\,R\,b, a\,R\,c \Longrightarrow a\,R\,(b_{\leq'} \cap c_{\leq'})$.

The next properties are useful for checking closure under composition. Denote by $\tilde{U} = \{a : a \lhd U\}$ — the saturation of U in the topology.

Proposition 1.5 *Let $R : \mathcal{S} \to \mathcal{T}$ be a continuous mapping. Then:*

(i) $U \lhd V$ *implies* $R^{-1}[U] \lhd R^{-1}[V]$*,*

(ii) $b\,R\,U$ *iff* $b\,R\,\tilde{U}$*,*

(iii) $R^{-1}[U]^{\sim} = R^{-1}[\tilde{U}]^{\sim}$*.* □

Let $\mathbf{FTop}_s$ be the following category of set-presented formal topologies and continuous mappings. For a formal topology $\mathcal{S} = (S, \leq, \lhd)$ we define a continuous mapping $I : \mathcal{S} \to \mathcal{S}$ (the identity) by

$$aIb \Longleftrightarrow a \lhd \{b\}.$$

For continuous mappings, $R_1 : \mathcal{S}_1 \to \mathcal{S}_2$ and $R_2 : \mathcal{S}_2 \to \mathcal{S}_3$, between formal spaces, define the composition

$$a(R_2 \circ R_1)b \Longleftrightarrow a \lhd R_1^{-1}[R_2^{-1}\{b\}].$$

This is continuous mapping $(R_2 \circ R_1) : \mathcal{S}_1 \to \mathcal{S}_3$. The category is not locally small, within any known predicative meta-theory.

2 Construction of coequalisers

Let F and G be continuous mappings $\mathcal{X} \to \mathcal{Y}$ in $\mathbf{FTop}_s$. A set $\mathcal{R}(Y)$ of subsets of Y is said to be *adequate for F and G* if (H1) – (H3) below are satisfied.

(H1) $Y \in \mathcal{R}(Y)$.

(H2) $U, V \in \mathcal{R}(Y)$ implies $U \wedge V \in \mathcal{R}(Y)$. Here $\wedge$ is taken with respect to the preorder of $\mathcal{Y}$.

(H3) For any subset U of Y with $b \in U$ such that U satisfies the equivalence

$$(\forall a \in X)(a \lhd F^{-1}U \Longleftrightarrow a \lhd G^{-1}U)$$

there is already some $V \in \mathcal{R}(Y)$ with $b \in V \subseteq U$ satisfying the equivalence.

Lemma 2.1 *Let $F, G : \mathcal{X} \to \mathcal{Y}$ be a pair of continuous morphism in $\mathbf{FTop}_s$. If $\mathcal{R}(Y)$ is adequate for the pair F and G, then the following defines a coequaliser of the pair: the formal topology $\mathcal{Q} = (Q, \leq_{\mathcal{Q}}, \lhd_{\mathcal{Q}})$ where*

$$Q = \{U \in \mathcal{R}(Y) : (\forall a \in X)(a \lhd F^{-1}U \Longleftrightarrow a \lhd G^{-1}U)\}$$

and $U \leq_{\mathcal{Q}} V$ iff $U \lhd_{\mathcal{Y}} V$, and where $U \lhd_{\mathcal{Q}} \mathcal{U}$ iff $U \lhd_{\mathcal{Y}} \cup \mathcal{U}$ for $\mathcal{U} \subseteq \mathcal{R}(Y)$. Moreover the coequalising morphism $P : \mathcal{Y} \to \mathcal{Q}$ is given by: $a\, P\, U$ iff $a \lhd_Y U$.

Proof. By (H2) it follows that Q is closed under $\wedge$. Using this it is straightforward to check that $\mathcal{Q}$ is a formal topology. It is as well set-presented since, if $C(a, i)$ $(i \in I(a))$ is the set-presentation of $\mathcal{Y}$ then we get a set-presentation (D, J) for $\mathcal{Q}$ by letting for $U \in \mathcal{R}(Y)$

$$J(U) = \{(f, g) : f \in \prod_{x \in \underline{U}} I(x),\ g \in \prod_{x \in \underline{U}} \prod_{y \in \underline{C}(p_U(x), f(x))} Q_{p(y)}\}$$

where $Q_u = \{U \in Q : u \in U\}$. Moreover,

$$D(U, (f, g)) = \{g(x, y) : x \in U, y \in \underline{C}(p(x), f(x))\}.$$

Here $p_S : \underline{S} \to S$ is the projection associated with the choice-set of S. Indeed, using the choice principle (1)

$$\begin{aligned}
U \lhd_{\mathcal{Q}} \mathcal{U} &\iff (\forall a \in U)(\exists i \in I(a))C(a,i) \subseteq \cup\mathcal{U} \\
&\iff (\exists f \in (\Pi x \in \underline{U})I(x))(\forall a \in \underline{U})C(p(a), f(a)) \subseteq \cup\mathcal{U} \\
&\iff (\exists f \in (\Pi x \in \underline{U})I(x))(\forall a \in \underline{U})(\forall b \in \underline{C}(p(a), f(a)))(\exists V \in Q) \\
&\qquad V \in \mathcal{U} \;\&\; p(b) \in V \\
&\iff (\exists f \in (\Pi x \in \underline{U})I(x))(\exists g \in (\Pi x \in \underline{U})(\Pi y \in \underline{C}(p(x), f(x)))Q_{p(y)}) \\
&\qquad (\forall a \in \underline{U})(\forall b \in \underline{C}(p(a), f(a)))g(a,b) \in \mathcal{U}. \\
&\iff (\exists (f,g) \in J(U))D(U,(f,g)) \subseteq \mathcal{U}.
\end{aligned}$$

Next, to check that P is a continuous morphism is easy. For instance, to verify condition (A3): Trivially for any $a \in Y$ we have $a \lhd_{\mathcal{Y}} Y$. By (H1), $Y \in \mathcal{R}(Y)$ and $Y \in Q$ and it follows that $a \lhd_{\mathcal{Q}} P^{-1}[Q]$. Thus condition (A3) is verified.

The equation $P \circ F = P \circ G$ is clear by the definition of Q.

To verify the universal property of P, suppose that $H : \mathcal{Y} \to \mathcal{Z}$ is a continuous morphism such that $H \circ F = H \circ G$. Thus we have for all $a \in X$ and $b \in Z$

$$a \lhd_{\mathcal{Y}} F^{-1}[H^{-1}b] \iff a \lhd_{\mathcal{Y}} G^{-1}[H^{-1}b]. \tag{3}$$

Now define $K : \mathcal{Q} \to \mathcal{Z}$ by

$$U\, K\, c \iff_{\text{def}} (\forall a \in U)\, a\, H\, c. \tag{4}$$

We first prove that K is a morphism.

(A1): Let $U \in Q$ and suppose $U\,K\,c$ and $c \lhd_{\mathcal{Z}} W$. The since H is a morphism, $a \lhd_{\mathcal{Y}} H^{-1}W$ for any $a \in U$. The property to be shown is $U \lhd_{\mathcal{Q}} K^{-1}W$, i.e. $U \lhd_{\mathcal{Y}} \cup K^{-1}W$. Let $c \in H^{-1}W$ be arbitrary. Thus there is $b \in W$ with $c \in H^{-1}b$. Applying (3) and (H3), we obtain $V \in Q$ with $c \in V \subseteq H^{-1}b$. Thus by definition (4) we have $V\,K\,b$, so $V \in K^{-1}W$ and $c \lhd_{\mathcal{Y}} \cup K^{-1}W$. Since c was arbitrary, $H^{-1}W \lhd_{\mathcal{Y}} \cup K^{-1}W$, and thereby $a \lhd_{\mathcal{Y}} \cup K^{-1}W$ for any $a \in U$. This was what had to be shown.

(A2): immediate

(A3): follows since $Y \in Q$ by (H1).

(A4'): Suppose $U\,K\,c$ and $U\,K\,d$. Thus $U \lhd_{\mathcal{Y}} H^{-1}c$ and $U \lhd_{\mathcal{Y}} H^{-1}d$. From this follows using (A4') for H that $U \lhd_{\mathcal{Y}} H^{-1}[c \wedge d]$. Let $a \in H^{-1}[c \wedge d]$, i.e. assuming that there is $e \leq c$ and $e \leq d$ with $a\,H\,e$. By (H3) and (3) (with $b = e$) we find $W \in Q$ satisfying $W \subseteq H^{-1}e$ and $a \in W$. Thus

$W\, K\, e$ and so $W \in K^{-1}[c \wedge d]$. Since $a \in W$, we get $a \lhd_{\mathcal{Q}} K^{-1}[c \wedge d]$. Thus $H^{-1}[c \wedge d] \lhd_{\mathcal{Q}} K^{-1}[c \wedge d]$ and now $U \lhd_{\mathcal{Q}} K^{-1}[c \wedge d]$ follows.

We finally need to prove that K is the unique morphism such that $K \circ P = H$, that is

$$a \lhd_{\mathcal{Y}} P^{-1}K^{-1}c \Longleftrightarrow a \lhd_{\mathcal{Y}} H^{-1}c. \tag{5}$$

The direction $\Rightarrow$ is clear by the definition of P and K. To prove $\Leftarrow$, assume $a \lhd_{\mathcal{Y}} H^{-1}c$. Thus $a \in H^{-1}c$. By (H3) and (3) there is $U \in Q$ with $a \in U \subseteq H^{-1}c$. Hence $U \lhd_{\mathcal{Y}} H^{-1}c$, i.e. $U\, K\, c$, so in particular we have $a \lhd_{\mathcal{Y}} \cup K^{-1}c$. Thereby $a \lhd_{\mathcal{Y}} P^{-1}K^{-1}c$.

The uniqueness of K: suppose $K_2 : \mathcal{Q} \to \mathcal{Z}$ is another morphism satisfying (5). For $U \in Q$ and $c \in Z$ we have

$$\begin{array}{rcl} U\, K_2\, c & \Leftrightarrow & U \lhd_{\mathcal{Q}} K_2^{-1}c \\ & \Leftrightarrow & (\forall a \in U) a \lhd_{\mathcal{Y}} \cup K_2^{-1}c \\ & \Leftrightarrow & (\forall a \in U) a \lhd_{\mathcal{Y}} P^{-1}K_2^{-1}c \\ & \Leftrightarrow & (\forall a \in U) a \lhd_{\mathcal{Y}} H^{-1}c \\ & \Leftrightarrow & U\, K\, c. \end{array}$$

The next to last equivalence is direct from (3). Thus $K_2 = K$. □

3 Existence of adequate, restricted power sets

Let $F, G : \mathcal{X} \to \mathcal{Y}$ be continuous morphisms between set-presentable formal topologies. Let $C(a,i)$ ($i \in I(a)$, $a \in X$) be a set-presentation of $\mathcal{X}$. Consider its equivalent form $C(w)$ ($w \in I$) with $p : I \to X$ as in (2). Now let $\mathcal{U}$ be a type-theoretic universe $\mathsf{T}(t)$ ($t : \mathsf{U}$), closed under Σ- and Π-constructions, and which is such that X, Y, I are $\mathcal{U}$-sets, the relation $\leq_{\mathcal{X}}$ is a $\mathcal{U}$-subset of $X \times X$, and the relations F and G are $\mathcal{U}$-subsets of $X \times Y$. Moreover $C(w)$ is an $\mathcal{U}$-subset of X for each $w \in I$. Moreover, we assume the universe to be closed under the W-type construction, though only a special instance is actually used.

Then form the restricted power set $\mathcal{R}_{\mathcal{U}}(Y)$ with respect to the family $\mathcal{U}$. The following is a set-theoretic collection principle.

Lemma 3.1 *Suppose $\mathcal{U}$ is as above. Let H be a $\mathcal{U}$-subset of $X \times Y$, and let A be a $\mathcal{U}$-subset of X. Then for any subset B of Y with*

$$A \subseteq H^{-1}B,$$

there is a $\mathcal{U}$-subset Z of Y with $A \subseteq H^{-1}Z$ and $Z \subseteq B$.

Proof. Suppose $A \subseteq H^{-1}B$. This is equivalent to

$$(\forall x \in \underline{A})(\exists y \in B)\; p(x)\, H\, y.$$

Thus there is some $f : \underline{A} \to B$ with $(\forall x \in \underline{A})\; p(x)\, H\, f(x)$. The image of this function $Z = \{y \in B : (\exists x \in \underline{A})\; f(x) = y\}$ is a $\mathcal{U}$-subset of Y since $\mathcal{U}$ is closed under Σ. It is clear that $A \subseteq H^{-1}Z$ and $Z \subseteq B$. $\Box$

Lemma 3.2 *Suppose $\mathcal{R}(Y) = \mathcal{R}_{\mathcal{U}}(Y)$ is as above. Take a subset U of Y such that*

$$(\forall a \in X)(a \lhd F^{-1}U \Rightarrow a \lhd G^{-1}U). \tag{6}$$

Then for any $V \in \mathcal{R}(Y)$ with $V \subseteq U$, there is $W \in \mathcal{R}(Y)$ with $W \subseteq U$ and

$$(\forall a \in X)(a \lhd F^{-1}V \Rightarrow a \lhd G^{-1}W). \tag{7}$$

Moreover, the above holds with F and G interchanged in both (6) and (7). $\Box$

Proof. Let $V \in \mathcal{R}(Y)$ with $V \subseteq U$. Then by (6) and transitivity of covers we get

$$(\forall a \in X)(a \lhd F^{-1}V \Rightarrow a \lhd G^{-1}U).$$

Using the set-presentation of the cover this may be rephrased as

$$(\forall a \in X)(a \lhd F^{-1}V \Rightarrow (\exists i \in I(a))C(a,i) \subseteq G^{-1}U).$$

By Lemma 3.1, and since $C(a,i) \in \mathcal{R}(X)$, the statement $C(a,i) \subseteq G^{-1}U$ is equivalent to $(\exists Z \in \mathcal{R}(Y))C(a,i) \subseteq G^{-1}(Z)\ \&\ Z \subseteq U$. Thus using the set-presentation again we get

$$(\forall a \in X)(a \lhd F^{-1}V \Rightarrow (\exists Z \in \mathcal{R}(Y))a \lhd G^{-1}Z\ \&\ Z \subseteq U).$$

Rewriting the first covering relation we get

$$(\forall a \in X)\big[(\exists i \in I(a))(C(a,i) \subseteq F^{-1}V) \Rightarrow (\exists Z \in \mathcal{R}(Y))a \lhd G^{-1}Z\ \&\ Z \subseteq U\big].$$

Let $Q(a) = (\exists i \in I(a))(C(a,i) \subseteq F^{-1}V)$. This may be rewritten again as

$$(\forall a \in \underline{X})(\forall q : Q(a))(\exists Z \in \mathcal{R}(Y))p(a) \lhd G^{-1}Z\ \&\ Z \subseteq U.$$

Using type-theoretic choice we obtain

$$H : (\Sigma a \in \underline{X})Q(a) \to \mathcal{R}(Y)$$

such that

$$(\forall a \in \underline{X})(\forall q : Q(a))p(a) \lhd G^{-1}H(a,q) \ \&\ H(a,q) \subseteq U.$$

Let

$$W = \bigcup_{(a,q):(\Sigma a \in \underline{X})Q(a)} H(a,q).$$

Now $W \in \mathcal{R}(X)$ and by transitivity

$$(\forall a \in \underline{X})(\forall q : Q(a))p(a) \lhd G^{-1}W \ \&\ W \subseteq U.$$

That is

$$(\forall a \in X)(a \lhd F^{-1}V \Rightarrow a \lhd G^{-1}W),$$

proving (7). □

Lemma 3.3 *The restricted power set $\mathcal{R}(Y) = \mathcal{R}_{\mathcal{U}}(Y)$ as constructed above is adequate for F and G.*

Proof. Condition (H1) is trivial since the subset $Y = \{x \in Y : x = x\}$ belongs to $\mathcal{R}(Y)$. Condition (H2) follows since the relation $\leq_{\mathcal{Y}}$ is a $\mathcal{U}$-subset of $Y \times Y$ and $\mathcal{U}$ is closed under Σ.

To prove (H3) a W-type, T defined by the following introduction rules, is used

(i) $0 : T$,

(ii) if $\alpha : T$, then $s_0(\alpha) : T$,

(iii) if $\alpha : T$, then $s_1(\alpha) : T$,

(iv) if $(a,i) : (\Sigma a : A)\underline{I(a)}$ and $f : \underline{C}(a,i) \to T$, then $\sup((a,i),f) : T$.

Suppose now that $U \subseteq Y$ satisfies the equivalence

$$(\forall a \in X)(a \lhd F^{-1}U \iff a \lhd G^{-1}U).$$

Let $b \in U$. We construct V_α $(\alpha : T)$ with $V_\alpha \in \mathcal{R}(Y)$ and $V_\alpha \subseteq U$ as follows.

(base) $V_0 = \{b\}$

(suc0) $(\forall a \in X)[a \lhd F^{-1}V_\alpha \Rightarrow a \lhd G^{-1}V_{s_0(\alpha)}]$

(suc1) $(\forall a \in X)[a \lhd G^{-1}V_\alpha \Rightarrow a \lhd F^{-1}V_{s_1(\alpha)}]$

(sup) For $f : \underline{C}(a,i) \to T$ let

$$V_{\sup((a,i),f)} = \bigcup_{p:\underline{C}(a,i)} V_{f(p)}.$$

By Lemma 3.2 we can find such $V_{s_0(\alpha)}$ and $V_{s_1(\alpha)}$ as in (suc0) and (suc1). Now put

$$V_\infty = \bigcup_{\alpha:T} V_\alpha.$$

Then $V_\infty \in \mathcal{R}(Y)$ since $\mathcal{U}$ is closed under W-types. (Actually, only the particular W-type T is required.) Furthermore $b \in V_\infty \subseteq U$.

Suppose now that $a \lhd F^{-1}[V_\infty]$. Thus for some $i \in I(a)$,

$$C(a,i) \subseteq F^{-1}[U_\infty] = \bigcup_{\alpha:T} F^{-1}V_\alpha.$$

Thus we have

$$(\forall x : \underline{C}(a,i))\,(\exists \alpha : T)\, p(x) \in F^{-1}V_\alpha.$$

Using type-theoretic choice we get $f : \underline{C}(a,i) \to T$ so that

$$(\forall x : \underline{C}(a,i))\, p(x) \in F^{-1}V_{f(x)}.$$

But $F^{-1}V_{f(x)} \subseteq F^{-1}V_{\sup((a,i),f)}$ and so we obtain

$$C(a,i) \subseteq F^{-1}V_{\sup((a,i),f)}.$$

Hence by the construction (suc0) we get

$$a \lhd G^{-1}V_{s_0(\sup((a,i),f))}.$$

But $V_{s_0(\sup((a,i),f))} \subseteq V_\infty$, and thereby $a \lhd G^{-1}V_\infty$. We have shown

$$(\forall a \in X)(a \lhd F^{-1}V_\infty \Rightarrow G^{-1}V_\infty).$$

The proof of the converse implication is the same, exchanging F and G and using (suc1) instead of (suc0). This establishes the lemma with V_∞ as the set satisfying the required equivalence. $\square$

A *universe forming operator* is a type construction which over any family of types builds a type universe closed under Π, Σ, W, Id and $+$-constructions (Palmgren 1998).

Theorem 3.4 *Under the assumption of universe forming operators, coequalisers exists in the category* $\mathbf{FTop}_s$.

Remark 3.5 There is a corresponding notion of universe forming operator in extensions of constructive set theory, see Rathjen *et al.* (1998). It would be of interest to investigate whether the above construction is possible to carry out in these extensions.

4 Examples

For any set S there is the *discrete formal topology* $\mathcal{D}(S) = (S, \lhd, \leq)$ given by $a \lhd U$ iff $a \in U$ and $a \leq b$ iff $a = b$. It is easily seen to be set-presented. Moreover $\mathcal{D}$ is a functor from sets to $\mathbf{FTop}_s$.

1. The two-dimensional $\mathbb{T}^2$ torus may be constructed as the coequaliser of the followings maps $\mathbb{R}^2 \times \mathbb{Z}^2 \to \mathbb{R}^2$

$$(\mathbf{x}, \mathbf{n}) \mapsto \mathbf{x},$$
$$(\mathbf{x}, \mathbf{n}) \mapsto \mathbf{x} + \mathbf{n}.$$

2. The n-dimensional real projective space $\mathbb{R}P^n$ may be constructed as a coequaliser of two maps $\mathbb{R}^{n+1} \times \mathbb{R}_{\neq 0} \to \mathbb{R}^{n+1}$

$$(\mathbf{x}, \lambda) \mapsto \mathbf{x},$$
$$(\mathbf{x}, \lambda) \mapsto \lambda\mathbf{x}.$$

3. Pushouts may be constructed using sums and coequalisers. Various glued spaces may be constructed using pushouts.

4. Suppose that $\mathcal{X}$ is formal topology whose points $P = \mathrm{Pt}(\mathcal{X})$ form a set. Then a continuous morphism

$$J : \mathcal{D}(P) \to \mathcal{X}$$

is given by $\alpha \, J \, a$ iff $a \in \alpha$. For any equivalence relation $E \subseteq P \times P$, with projections $\pi_1, \pi_2 : E \to P$, the point-wise quotient topology is the coequaliser of the continuous mappings $J \circ \mathcal{D}(\pi_1)$ and $J \circ \mathcal{D}(\pi_2)$ going from $\mathcal{D}$ to $\mathcal{X}$.

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