

ORDER-INVARIANT MSO IS STRONGER THAN COUNTING MSO IN THE FINITE

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ABSTRACT. We compare the expressiveness of two extensions of monadic second-order logic (MSO) over the class of finite structures. The first, counting monadic second-order logic (CMSO), extends MSO with first-order modulo-counting quantifiers, allowing the expression of queries like “the number of elements in the structure is even”. The second extension allows the use of an additional binary predicate, not contained in the signature of the queried structure, that must be interpreted as an arbitrary linear order on its universe, obtaining order-invariant MSO.

While it is straightforward that every CMSO formula can be translated into an equivalent order-invariant MSO formula, the converse had not yet been settled. Courcelle showed that for restricted classes of structures both order-invariant MSO and CMSO are equally expressive, but conjectured that, in general, order-invariant MSO is stronger than CMSO.

We affirm this conjecture by presenting a class of structures that is order-invariantly definable in MSO but not definable in CMSO.

1. Introduction

Linear orders play an important role in descriptive complexity theory since certain results relating the expressive power of logics to complexity classes, e.g., the Immerman-Vardi Theorem that LFP captures PTIME, only hold for classes of linearly ordered structures. Usually, the order only serves to systematically access all elements of the structure, and consequently to encode the configurations of a step-wise advancing computation of a Turing machine by tuples of elements of the structure. In these situations we do not actually want to make statements about the properties of the order, but merely want to have an arbitrary linear order available to express the respective coding techniques.

Furthermore, when actually working with finite structures in an algorithmic context, e.g., when evaluating queries in a relational database, we are in fact working on an implicitly ordered structure since, although relations in a database are modelled as *sets* of tuples, the relations are nevertheless stored as *ordered sequences* of tuples in memory or on a disk. As

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this linear order is always available (though, as in the case of databases, it is implementation-dependent and may even change over time as tuples are inserted or deleted), we could allow queries to make use of an additional binary predicate that is interpreted as a linear order on the universe of the structure, but require the outcome of the query not to depend on the actual ordering, but to be *order-invariant*. Precisely, given a τ -structure \mathfrak{A} , we allow queries built over an expanded vocabulary $\tau \dot{\cup} \{<\}$, and say that a query φ is *order-invariant* if $(\mathfrak{A}, <_1) \models \varphi \iff (\mathfrak{A}, <_2) \models \varphi$ for all possible relations $<_1$ and $<_2$ linearly ordering A .

Using Ehrenfeucht-Fraïssé-games for MSO, one can see that MSO on sets (i.e., structures over an empty vocabulary) is too weak to express that the universe contains an even number of elements. However, this is possible if the universe is linearly ordered: simply use the MSO sentence stating that the maximal element should be contained in the set of elements on even positions in the ordering. Obviously, such a sentence is order-invariant since rearranging the elements does not affect its truth value. Gurevich uses this observation to show that the property of Boolean algebras having an even number of atoms, although not definable in FO, is order-invariantly definable in FO (simulating the necessary MSO-quantification over sets of atoms by FO-quantification over the elements of the Boolean algebra).

If we explicitly add modulo-counting to MSO, e.g., via modulo-counting first-order quantifiers such as “there exists an even number of elements x such that ...”, we obtain *counting monadic second-order logic* (CMSO), and the question naturally arises as to whether there are properties not expressible in CMSO that can be expressed order-invariantly in MSO.

In fact, a second separation example due to Otto gives a hint in that direction. The class of structures presented in [Ott00] even separates order-invariant FO from FO extended by arbitrary unary generalised quantifiers, i.e., especially modulo-counting quantifiers, and exploits the idea of “hiding” a part of the structure such that it is only meaningfully usable for queries in presence of a linear order (or, as actually proven in the paper, in presence of an arbitrary choice function).

The expressiveness of CMSO has been studied, e.g., in [Cou90], where it is mainly compared to MSO, and in [Cou96] it is shown that, on the class of forests, order-invariant MSO is no more expressive than CMSO. As pointed out in [BS05], this can be generalised using results in [Lap98] to classes of structures of bounded tree-width. But still, this left open Courcelle’s conjecture: that order-invariant MSO is strictly stronger than CMSO for general graphs [Cou96, Conjecture 7.3].

In this paper, we present a suitable characterisation of CMSO-definability in terms of an Ehrenfeucht-Fraïssé game, and later, as the main contribution, we present a separating example showing that a special class of graphs is indeed definable by an order-invariant MSO sentence but not by a counting MSO sentence.

2. Preliminaries

Throughout the paper \mathbb{N} denotes the set of non-negative integers and $\mathbb{N}^+ := \mathbb{N} - \{0\}$. Given a non-empty finite set $M = \{m_1, \dots, m_k\} \subseteq_{\text{fin}} \mathbb{N}^+$, let $\text{lcm}(M) := \text{lcm}(m_1, \dots, m_k)$ denote the least common multiple of all elements in M ; additionally, we define $\text{lcm}(\emptyset) = 1$. For sets X and Y as well as M as before, we abbreviate that $|X| \equiv |Y| \pmod{m}$ for all $m \in M$ by using the shorthand $|X| \equiv |Y| \pmod{M}$.

We restrict our attention to finite τ -structures with a nonempty universe over a countable relational vocabulary τ , possibly with constants, and we will mainly deal with monadic second-order logic and some of its extensions. For more details concerning finite model theory, we refer to [EF95] or [Lib04].

When comparing the expressiveness of two logics \mathcal{L} and \mathcal{L}' , we say that \mathcal{L}' is at least as expressive as \mathcal{L} , denoted $\mathcal{L} \subseteq \mathcal{L}'$, if for every $\varphi \in \mathcal{L}[\tau]$ there exists a $\varphi' \in \mathcal{L}'[\tau]$ such that $\text{Mod}(\varphi) = \text{Mod}(\varphi')$, where $\text{Mod}(\varphi)$ denotes the class of all finite τ -structures satisfying φ .

2.1. Counting MSO

The notion of (modulo-)counting monadic second-order logic (CMSO) can be introduced in two different, but nonetheless equivalent, ways. The first view of CMSO is via an extension of MSO by modulo-counting first-order quantifiers.

Definition 2.1. Let τ be a signature and $M \subseteq \mathbb{N}^+$ a set of moduli, then

- every formula $\varphi \in \text{MSO}[\tau]$ is also a formula in $\text{CMSO}^{(M)}[\tau]$, and
- if $\varphi(x) \in \text{CMSO}^{(M)}[\tau]$ and $m \in M$, then $\exists^{(m)}x.\varphi(x) \in \text{CMSO}^{(M)}[\tau]$.

If we do not restrict the set of modulo-counting quantifiers being used, we get the full language $\text{CMSO}[\tau] = \text{CMSO}^{(\mathbb{N}^+)}[\tau]$. The semantics of MSO formulae is as expected, and we have $\mathfrak{A} \models \exists^{(m)}x.\varphi(x)$ if and only if $|\{a \in A : \mathfrak{A} \models \varphi(a)\}| \equiv 0 \pmod{m}$. The quantifier rank $\text{qr}(\psi)$ of a $\text{CMSO}[\tau]$ formula ψ is defined as for MSO-formulae with the additional rule that $\text{qr}(\exists^{(m)}x.\varphi(x)) = 1 + \text{qr}(\varphi)$, i.e., we do not distinguish between different kinds of quantifiers.

In this paper we use an alternative but equivalent definition of CMSO, namely the extension of the MSO language by monadic second-order predicates $C^{(m)}$ which hold true of a set X if and only if $|X| \equiv 0 \pmod{m}$. As in the definition above, formulae of the fragment $\text{CMSO}^{(M)}[\tau]$ may only use predicates $C^{(m)}$ where $m \in M$. The back-and-forth translation can be carried out along the following equivalences which increase the quantifier rank by at most one in each step:

$$\begin{aligned} \exists^{(m)}x.\varphi(x) &\equiv \exists X(C^{(m)}(X) \wedge \forall x(Xx \leftrightarrow \varphi(x))) \quad \text{and} \\ C^{(m)}(X) &\equiv \exists^{(m)}x.Xx. \end{aligned}$$

Furthermore, the introduction of additional predicates $C^{(m,r)}$ (or, equivalently, additional modulo-counting quantifiers $\exists^{(m,r)}$) stating for a set X that $|X| \equiv r \pmod{m}$ does not increase the expressive power since they can be simulated as follows (with only a constant increase of quantifier rank):

$$C^{(m,r)}(X) \equiv \exists X_0("X_0 \subseteq X" \wedge "|X_0| = r" \wedge "C^{(m)}(X \setminus X_0)"),$$

where all subformulae are easily expressible in MSO.

Later, we will introduce an Ehrenfeucht-Fraïssé game capturing the expressiveness of CMSO with this extended set of second-order predicates.

2.2. Order-invariance

Let τ be a relational vocabulary and $\varphi \in \text{MSO}[\tau \dot{\cup} \{<\}]$, i.e., φ may contain an additional relation symbol $<$. Then φ is called *order-invariant on a class \mathcal{C} of τ -structures* if, and only if, $(\mathfrak{A}, <_1) \models \varphi \iff (\mathfrak{A}, <_2) \models \varphi$ for all $\mathfrak{A} \in \mathcal{C}$ and all linear orders $<_1$ and $<_2$ on A .

Although, in general, it is undecidable whether a given MSO-formula is order-invariant in the finite, we will speak of the *order-invariant fragment of MSO*, denoted by $\text{MSO}[<]_{inv}$, that contains all formulae that are order-invariant on the class of all finite structures.

It is an easy observation that every CMSO formula is equivalent over the class of all finite structures to an order-invariant MSO formula by translating counting quantifiers in the following way:

$$\begin{aligned} \exists^{(q)}x.\varphi(x) &:= \exists X \exists X_0 \dots \exists X_{q-1} \\ &\left(\begin{array}{l} \forall x (Xx \leftrightarrow \varphi(x)) \wedge \text{“}\{X_0, \dots, X_{q-1}\} \text{ is a partition of } X\text{”} \\ \wedge \exists x (X_0x \wedge \forall y (Xy \rightarrow x \leq y)) \wedge \exists x (X_{q-1}x \wedge \forall y (Xy \rightarrow x \geq y)) \\ \wedge \forall x \forall y \left(S_{\varphi, <}(x, y) \rightarrow \left(\bigwedge_{i=0}^{q-1} X_i x \leftrightarrow X_{i+1} (\text{mod } q)y \right) \right) \end{array} \right) \end{aligned}$$

where $S_{\varphi, <}$ defines the successor relation induced by an arbitrary order $<$ on the universe of the structure restricted to the set X of elements for which φ holds.

Note that the quantifier rank of the translated formula is not constant but bounded by the parameter in the counting quantifier.

3. An Ehrenfeucht-Fraïssé game for CMSO

The Ehrenfeucht-Fraïssé game capturing expressiveness of MSO parameterised by the quantifier-rank (cf. [EF95, Lib04]) can be naturally extended to a game capturing the expressiveness of CMSO parameterised by the quantifier rank and the set of moduli being used in the cardinality predicates or counting quantifiers.

Viewing CMSO as MSO with additional quantifiers $\exists^{(m)}x.\varphi(x)$ for all m in a fixed set M leads to a new type of move described, e.g., in the context of extending FO by modulo-counting quantifiers in [Nur00]. Since a modulo-counting quantifier actually combines notions of a first-order and a monadic second-order quantifier in the sense that it makes a statement about the cardinality of a certain *set* of elements, but on the other hand, it behaves like a first-order quantifier binding an *element* variable and making a statement about that particular element, the move capturing modulo-counting quantification consists of two phases. First, Spoiler and Duplicator select sets of elements S and D in the structures such that $|S| \equiv |D| \pmod{M}$, and in the second phase, Spoiler and Duplicator select elements a and b such that $a \in S$ if and only if $b \in D$. After the move, reflecting the first-order nature of the quantifier, only the two selected elements a and b are remembered and contribute to the next position in the game, whereas the information about the chosen sets is discarded.

We prefer viewing CMSO via second-order cardinality predicates, yielding an Ehrenfeucht-Fraïssé game that allows a much clearer description of winning strategies. Since we do not have additional quantifiers, we have exactly the same types of moves as in the Ehrenfeucht-Fraïssé game for MSO, and we merely modify the winning condition to take the new predicates into account.

Towards this end, we first introduce a suitable concept of partial isomorphisms between structures.

Definition 3.1. With any structure \mathfrak{A} and any set $M \subseteq_{\text{fin}} \mathbb{N}^+$ we associate the (first-order) power set structure $\mathfrak{A}^M := (\mathcal{P}(A), (C^{(m,r)})_{\substack{m \in M \\ 0 \leq r < m}})$, where the predicates $C^{(m,r)}$ are interpreted in the obvious way. (Note that first-order predicates in the power set structure \mathfrak{A}^M naturally correspond to second-order predicates in \mathfrak{A} .)

Let \mathfrak{A} and \mathfrak{B} be τ -structures, and let $M \subseteq_{\text{fin}} \mathbb{N}^+$ be a fixed set of moduli. Then the mapping $(A_1, \dots, A_s, a_1, \dots, a_t) \mapsto (B_1, \dots, B_s, b_1, \dots, b_t)$ is called a *twofold partial isomorphism between \mathfrak{A} and \mathfrak{B} with respect to M* if

- (i) $(a_1, \dots, a_t) \mapsto (b_1, \dots, b_t)$ is a partial isomorphism between $(\mathfrak{A}, A_1, \dots, A_s)$ and $(\mathfrak{B}, B_1, \dots, B_s)$ and
- (ii) $(A_1, \dots, A_s) \mapsto (B_1, \dots, B_s)$ is a partial isomorphism between \mathfrak{A}^M and \mathfrak{B}^M .

We propose the following Ehrenfeucht-Fraïssé game to capture the expressiveness of CMSO where the use of moduli is restricted to a (finite) set M and formulae of quantifier rank at most r .

Definition 3.2 (Ehrenfeucht-Fraïssé game for CMSO). Let $M \subseteq_{\text{fin}} \mathbb{N}^+$ and $r \in \mathbb{N}$. The r -round (mod M) Ehrenfeucht-Fraïssé game $\mathcal{G}_r^M(\mathfrak{A}, \mathfrak{B})$ is played by Spoiler and Duplicator on τ -structures \mathfrak{A} and \mathfrak{B} . In each turn, Spoiler can choose between the following types of moves:

- *point move*: Spoiler selects an element in one of the structures, and Duplicator answers by selecting an element in the other structure.
- *set move*: Spoiler selects a set of elements X in one of the structures, and Duplicator responds by choosing a set of elements Y in the other structure.

After $r = s + t$ rounds, when the players have chosen sets A_1, \dots, A_s and B_1, \dots, B_s as well as elements a_1, \dots, a_t and b_1, \dots, b_t in an arbitrary order, Duplicator wins the game if, and only if, $(A_1, \dots, A_s, a_1, \dots, a_t) \mapsto (B_1, \dots, B_s, b_1, \dots, b_t)$ is a twofold partial isomorphism between \mathfrak{A} and \mathfrak{B} with respect to M .

First note that, although Duplicator is required to answer a set move X by a set Y such that $|X| \equiv |Y| \pmod{M}$ in order to win, we do not have to make this explicit in the rules of the moves since these cardinality constraints are already imposed by the winning condition (X and Y would not define a twofold partial isomorphism if they did not satisfy the same cardinality predicates). Furthermore, for $M = \emptyset$ or $M = \{1\}$, the resulting game $\mathcal{G}_r^M(\mathfrak{A}, \mathfrak{B})$ corresponds exactly to the usual Ehrenfeucht-Fraïssé game for MSO.

Theorem 3.3. *Let \mathfrak{A} and \mathfrak{B} be τ -structures, $r \in \mathbb{N}$, and $M \subseteq_{\text{fin}} \mathbb{N}$. Then the following are equivalent:*

- (i) $\mathfrak{A} \equiv_r^M \mathfrak{B}$, i.e., $\mathfrak{A} \models \varphi$ if and only if $\mathfrak{B} \models \varphi$ for all $\varphi \in \text{CMSO}^{(M)}[\tau]$ with $\text{qr}(\varphi) \leq r$.
- (ii) Duplicator has a winning strategy in the r -round (mod M) Ehrenfeucht-Fraïssé game $\mathcal{G}_r^M(\mathfrak{A}, \mathfrak{B})$. ■

To prove non-definability results, we can make use of the following standard argument.

Proposition 3.4. *A class \mathcal{C} of τ -structures is not definable in CMSO if, for every $r \in \mathbb{N}$ and every $M \subseteq_{\text{fin}} \mathbb{N}^+$, there are τ -structures $\mathfrak{A}_{M,r}$ and $\mathfrak{B}_{M,r}$ such that $\mathfrak{A}_{M,r} \in \mathcal{C}$, $\mathfrak{B}_{M,r} \notin \mathcal{C}$, and $\mathfrak{A}_{M,r} \equiv_r^M \mathfrak{B}_{M,r}$.*

The following lemma, stating that the CMSO-theory of disjoint unions can be deduced from the CMSO-theories of the components, can either be proved, as carried out in [Cou90, Lemma 4.5], by giving an effective translation of sentences talking about the disjoint union of two structures into a Boolean combination of sentences each talking about the individual structures, or by using a game-oriented view showing that winning strategies for Duplicator in the games on two pairs of structures can be combined into a winning strategy on the pair of disjoint unions of the structures.

Lemma 3.5. *Let $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{B}_1$, and \mathfrak{B}_2 be τ -structures such that $\mathfrak{A}_1 \equiv_r^M \mathfrak{B}_1$ and $\mathfrak{A}_2 \equiv_r^M \mathfrak{B}_2$. Then $\mathfrak{A}_1 \dot{\cup} \mathfrak{A}_2 \equiv_r^M \mathfrak{B}_1 \dot{\cup} \mathfrak{B}_2$.*

Proof. Consider the game on $\mathfrak{A} := \mathfrak{A}_1 \dot{\cup} \mathfrak{A}_2$ and $\mathfrak{B} := \mathfrak{B}_1 \dot{\cup} \mathfrak{B}_2$. A Spoiler's point move in \mathfrak{A} (resp., in \mathfrak{B}) is answered by Duplicator according to her winning strategy in either $\mathcal{G}_r^M(\mathfrak{A}_1, \mathfrak{B}_1)$ or $\mathcal{G}_r^M(\mathfrak{A}_2, \mathfrak{B}_2)$. A set move $S \subseteq A$ (analogous for $S \subseteq B$) is decomposed into two subsets $S_1 := S \cap A_1$ and $S_2 := S \cap A_2$, and is answered by Duplicator by the set $D := D_1 \cup D_2$ consisting of the sets D_1 and D_2 chosen according to her winning strategies as responses to S_1 and S_2 in the respective games $\mathcal{G}_r^M(\mathfrak{A}_1, \mathfrak{B}_1)$ and $\mathcal{G}_r^M(\mathfrak{A}_2, \mathfrak{B}_2)$.

Since A_1 and A_2 as well as B_1 and B_2 are disjoint, we have $|S| = |S_1| + |S_2|$ and $|D| = |D_1| + |D_2|$. Furthermore, $|S_1| \equiv |D_1| \pmod{M}$ and $|S_2| \equiv |D_2| \pmod{M}$ as the sets D_1 and D_2 are chosen according to Duplicator's winning strategies in the games on \mathfrak{A}_1 and \mathfrak{B}_1 , and \mathfrak{A}_2 and \mathfrak{B}_2 , respectively. Since $\equiv \pmod{M}$ is a congruence relation with respect to addition, we have that $|S| \equiv |D| \pmod{M}$. It is easily verified that the sets and elements chosen according to this strategy indeed define a twofold partial isomorphism between \mathfrak{A} and \mathfrak{B} . ■

As a direct corollary we obtain the following result that will be used in the inductive step in the forthcoming proofs.

Corollary 3.6. *Let $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{B}_1$, and \mathfrak{B}_2 be τ -structures, such that $\mathfrak{A}_1 \equiv_r^M \mathfrak{B}_1$ and $\mathfrak{A}_2 \equiv_r^M \mathfrak{B}_2$. Then $(\mathfrak{A}_1 \dot{\cup} \mathfrak{A}_2, A_1) \equiv_r^M (\mathfrak{B}_1 \dot{\cup} \mathfrak{B}_2, B_1)$.*

Proof. We consider the following $\tau \dot{\cup} \{P\}$ -expansions of the given structures: $\mathfrak{A}'_1 := (\mathfrak{A}_1, A_1)$, $\mathfrak{B}'_1 := (\mathfrak{B}_1, B_1)$, $\mathfrak{A}'_2 := (\mathfrak{A}_2, \emptyset)$, and $\mathfrak{B}'_2 := (\mathfrak{B}_2, \emptyset)$. It is immediate that

- (i) $\mathfrak{A}_1 \equiv_r^M \mathfrak{B}_1$ implies $(\mathfrak{A}_1, A_1) \equiv_r^M (\mathfrak{B}_1, B_1)$, and
- (ii) $\mathfrak{A}_2 \equiv_r^M \mathfrak{B}_2$ implies $(\mathfrak{A}_2, \emptyset) \equiv_r^M (\mathfrak{B}_2, \emptyset)$

since Duplicator can obviously win the respective Ehrenfeucht-Fraïssé games on the expanded structures using the same strategies as in the games proving the equivalences on the left-hand side. The claim follows by applying the previous lemma to the $\tau \dot{\cup} \{P\}$ -expansions. ■

It is well known that MSO exhibits a certain weakness regarding the ability to specify cardinality constraints on sets, i.e., structures over an empty vocabulary. A proof of this fact using Ehrenfeucht-Fraïssé games can be found in [Lib04]. By adapting this proof, we show that this is still the case for CMSO.

Lemma 3.7. *Let \mathfrak{A} and \mathfrak{B} be \emptyset -structures, $M \subseteq_{\text{fin}} \mathbb{N}^+$, and $r \in \mathbb{N}$. Then $\mathfrak{A} \equiv_r^M \mathfrak{B}$ if $|A|, |B| \geq (2^{r+1} - 4) \text{lcm}(M)$ and $|A| \equiv |B| \pmod{M}$.*

Proof. We prove by induction on the number of rounds that Duplicator wins the $(\text{mod } M)$ r -round Ehrenfeucht-Fraïssé game $\mathcal{G}_r^M(\mathfrak{A}, \mathfrak{B})$. For $r = 0$ and $r = 1$ the claim is obviously

true. Let $r > 1$, assume that the claim holds for $r - 1$, and consider the first move of the r -round game. We assume that Spoiler makes his move in \mathfrak{A} since the reasoning in the other case is completely symmetric.

If Spoiler makes a set move $S \subseteq A$, we consider the following cases:

- (1) $|S| < (2^r - 4) \cdot \text{lcm}(M)$ (or $|A - S| < (2^r - 4) \cdot \text{lcm}(M)$). Then Duplicator selects a set $D \subseteq B$ such that $|D| = |S|$ (or $|B - D| = |A - S|$), and hence $S \cong D$ and $A - S \equiv_{r-1}^M B - D$ (or $A - S \cong B - D$ and $S \equiv_{r-1}^M D$).
- (2) $|S|, |A - S| \geq (2^r - 4) \cdot \text{lcm}(M)$. Then Duplicator selects a set $D \subseteq B$ such that $|D| \equiv |S| \pmod{M}$ and $|D|, |B - D| \geq (2^r - 2) \cdot \text{lcm}(M)$. In fact, she chooses for D half of the elements and chooses $\ell < \text{lcm}(M)$ additional ones to fulfil the cardinality constraints $|D| \equiv |S| \pmod{M}$. Then, for the set $B - D$ of non-selected elements, we have

$$\begin{aligned} |B - D| &\geq \frac{1}{2}((2^{r+1} - 4) \text{lcm}(M)) - \ell \geq (2^r - 2) \text{lcm}(M) - \text{lcm}(M) \\ &\geq (2^r - 4) \text{lcm}(M) \end{aligned}$$

for all ℓ satisfying $0 \leq \ell < \text{lcm}(M)$. Since $|D| = |B - D| + 2\ell$, obviously $|D| \geq (2^r - 4) \text{lcm}(M)$ as well.

Thus, in both cases, by the induction hypothesis we get $S \equiv_{r-1}^M D$ and $A - S \equiv_{r-1}^M B - D$. Hence, by Corollary 3.6 $(A, S) \equiv_{r-1}^M (B, D)$, i.e., Duplicator has a winning strategy in the remaining $(r - 1)$ -round game from position (S, D) .

If Spoiler makes a point move $s \in A$, Duplicator answers by choosing an arbitrary element $d \in B$. Similar to Case 1 above, we observe that $(\{s\}, s) \cong (\{d\}, d)$ and $A - \{s\} \equiv_{r-1}^M B - \{d\}$ by the induction hypothesis. Thus, by Lemma 3.5, $(A, s) \equiv_{r-1}^M (B, d)$ implying that Duplicator has a winning strategy for the remaining $r - 1$ rounds from position (s, d) . ■

4. The Separating Example

We will first give a brief description of our example showing that $\text{MSO}[\prec]_{inv}$ is strictly more expressive than CMSO. We consider a property of two-dimensional grids, namely that the vertical dimension divides the horizontal dimension. This property is easily definable in MSO for grids that are given as directed graphs with two edge relations, one for the horizontal edges pointing rightwards, and one for the vertical edges pointing upwards, by defining a new relation of diagonal edges combining one step rightwards and one step upwards wrapping around from the top border to the bottom border but not from the right to the left border. Note that there is a path following those diagonal edges starting from the bottom-left corner of the grid and ending in the top-right corner if, and only if, the vertical dimension divides the horizontal dimension of the grid. Thus, for our purposes, we have to weaken the structure in the sense that we hide information that remains accessible to $\text{MSO}[\prec]_{inv}$ -formulae but not to CMSO formulae.

An appropriate loss of information is achieved by replacing the two edge relations with their reflexive symmetric transitive closure, i.e., we consider grids as structures with two equivalence relations which provide a notion of *rows* and *columns* of the grid. Obviously, notions like corner and border vertices as well as the notion of an order on the rows and columns that were important for the MSO-definition of the divisibility property are lost, but clearly, all these notions can be regained in presence of an order. First, the order allows us to uniquely define an element (e.g. the \prec -least element) to be the bottom-left corner of

the grid, and second, the order induces successor relations on the set of columns and the set of rows, from which both horizontal and vertical successor vertices of any vertex can be deduced. Since the divisibility property is obviously invariant with respect to the ordering of the rows or columns, this allows for expressing it in $\text{MSO}[<]_{inv}$. In the course of this section we will develop the arguments showing that CMSO fails to express this property on the following class of grid-like structures.

Definition 4.1. A *cliquey* (k, ℓ) -grid is a $\{\sim_h, \sim_v\}$ -structure that is isomorphic to $\mathfrak{G}_{k\ell} := (\{0, \dots, k-1\} \times \{0, \dots, \ell-1\}, \sim_h, \sim_v)$, where

$$\begin{aligned}\sim_h &:= \{((x, y), (x', y')) : x = x'\} \text{ and} \\ \sim_v &:= \{((x, y), (x', y')) : y = y'\},\end{aligned}$$

i.e., \sim_h consists of exactly k equivalence classes (called *rows*), each containing ℓ elements, and \sim_v consists of exactly ℓ equivalence classes (called *columns*), each containing k elements, such that every equivalence class of \sim_h intersects every equivalence class of \sim_v in exactly one element and vice versa.

A *horizontally coloured cliquey* (k, ℓ) -grid, denoted $\mathfrak{G}_{k\ell}^{\text{col}}$, is the expansion of the $\{\sim_v\}$ -reduct of the cliquey grid $\mathfrak{G}_{k\ell}$ by unary predicates $\{P_1, \dots, P_k\}$, where the information of \sim_h is retained in the k new predicates (in the following referred to as *colours*) such that each set P_i corresponds to exactly one former equivalence class.

Note that the same class of grid-like structures has already been used by Otto in a proof showing that the number of monadic second-order quantifiers gives rise to a strict hierarchy over finite structures [Ott95].

The class is first-order definable by a sentence ψ_{grid} stating that

- \sim_v and \sim_h are equivalence relations, and
- every pair consisting of one equivalence class of \sim_h and \sim_v each has exactly one element in common

as these properties are sufficient to enforce the desired grid-like structure. Note that even the second property is first-order definable since every equivalence class is uniquely determined by each of its elements.

The following two lemmata justify the introduction of the notion of horizontally coloured cliquey grids for use in the forthcoming proofs.

Lemma 4.2. Let $\mathfrak{G}_{k\ell_1}^{\text{col}}$, $\mathfrak{G}_{k\ell_2}^{\text{col}}$, $\mathfrak{G}_{k\ell'_1}^{\text{col}}$, and $\mathfrak{G}_{k\ell'_2}^{\text{col}}$ be horizontally coloured cliquey grids such that $\mathfrak{G}_{k\ell_1}^{\text{col}} \equiv_r^M \mathfrak{G}_{k\ell'_1}^{\text{col}}$ and $\mathfrak{G}_{k\ell_2}^{\text{col}} \equiv_r^M \mathfrak{G}_{k\ell'_2}^{\text{col}}$. Then $\mathfrak{G}_{k, \ell_1 + \ell_2}^{\text{col}} \equiv_r^M \mathfrak{G}_{k, \ell'_1 + \ell'_2}^{\text{col}}$.

Proof. Note that, since there are no horizontal edges in horizontally coloured cliquey grids and the vertical dimension of all grids is k , $\mathfrak{G}_{k, \ell_1 + \ell_2}^{\text{col}}$ is the disjoint union of the two smaller horizontally coloured cliquey grids $\mathfrak{G}_{k\ell_1}^{\text{col}}$ and $\mathfrak{G}_{k\ell_2}^{\text{col}}$, and of course, the same holds for $\mathfrak{G}_{k, \ell'_1 + \ell'_2}^{\text{col}}$. Thus, the claim follows by Lemma 3.5. \blacksquare

Lemma 4.3. Let $\mathfrak{G}_{k\ell}^{\text{col}} \equiv_r^M \mathfrak{G}_{k\ell'}^{\text{col}}$. Then $\mathfrak{G}_{k\ell} \equiv_r^M \mathfrak{G}_{k\ell'}$.

Proof. For each fixed horizontal dimension k , there exists a one-dimensional quantifier-free interpretation of a cliquey grid in its respective horizontally coloured counterpart since we can define the horizontal equivalence relation \sim_h in terms of the colours as follows:

$$x \sim_h y \equiv \bigvee_{i=1}^k P_i x \wedge P_i y.$$

■

Actually, the argument implies that Duplicator wins a game on cliquy grids using the same strategy that is winning in the corresponding game on coloured grids since a strategy preserving the colours of selected elements especially preserves the equivalence relation \sim_h .

Before stating the main lemma, we will first prove a combinatorial result which will later help Duplicator in synthesising her winning strategy and introduce the following weakened notion of equality between numbers.

Definition 4.4. Two numbers $a, b \in \mathbb{N}$ are called *threshold t equal (mod M)*, denoted $a \stackrel{M}{=}^t b$, if

- (i) $a = b$ or
- (ii) $a, b \geq t$ and $a \equiv b \pmod{M}$.

Intuitively, $a \stackrel{M}{=}^t b$ means that the numbers are equal if they are small, or that they are at least congruent modulo all $m \in M$ if they are both at least as large as the threshold t .

Lemma 4.5. *For every $p, t \in \mathbb{N}$, and $M \subseteq_{fin} \mathbb{N}^+$, we can choose an arbitrary $T \geq p \cdot (t + \text{lcm}(M) - 1)$ such that for all sets A and B with $|A| \stackrel{M}{=}^T |B|$ and for every equivalence relation \approx_A on A of index at most p there exists an equivalence relation \approx_B on B and a bijection $g: A/\approx_A \rightarrow B/\approx_B$ satisfying $|\{a' \in A : a \approx_A a'\}| \stackrel{M}{=}^t |g(\{a' \in A : a \approx_A a'\})|$ for all $a \in A$.*

Proof. We let $\{a_1, \dots, a_{p'}\}$, where $p' \leq p$ denotes the index of \approx_A , be the set of class representatives of A/\approx_A , and we let $[a]_{\approx_A} := \{a' \in A : a' \approx_A a\}$ denote the equivalence class of a in A . Note that we will usually omit the subscript \approx_A if it is clear from the context and instead reserve the letters a and b for elements denoting equivalence classes in A and B , respectively. Furthermore, a set will be called *small* in the following if it contains less than t elements and *large* otherwise.

The equivalence relation \approx_B on B is constructed by partitioning the set into p' disjoint non-empty subsets $\{B_1, \dots, B_{p'}\}$ as follows. If $|A| = |B|$, for each class $[a_i]$, we choose a set B_i with exactly $|[a_i]|$ many elements. If $|A|, |B| \geq T$, we have to distinguish between the treatment of small and large classes. Since $|A| \geq T \geq p \cdot (t + \text{lcm}(M) - 1)$, $\text{lcm}(M) \geq 1$, and the index of \approx_A is at most p , at least one of the equivalence classes contains at least t elements, i.e., it is large, and without loss of generality, it is assumed that this is the case for $[a_1]$. For each small class $[a_i]$, we choose a set B_i with exactly $|[a_i]|$ many elements. If $[a_i]$ is large, we choose a set B_i containing $t + \ell$ many elements where ℓ is the smallest non-negative integer such that $|[a_i]| \equiv |B_i| \pmod{M}$. The number of elements selected according to these rules is at most $p \cdot (t + \text{lcm}(M) - 1) \leq T \leq |B|$. Since $[a_1]$ is large by assumption, any possibly remaining elements in B , that have not been assigned to one of the subsets $B_1, \dots, B_{p'}$ yet, can be safely added to B_1 without violating the condition that $|[a_1]| \equiv |B_1| \pmod{M}$.

This partitioning uniquely defines the equivalence relation $\approx_B := \bigcup_{i=1}^{p'} (B_i \times B_i)$ on B . By selecting an arbitrary element of each B_i we get a set of class representatives $\{b_1, \dots, b_{p'}\}$ which directly yields the bijection $g: [a_i] \mapsto [b_i]$ for all $1 \leq i \leq p'$ satisfying $|[a]| \stackrel{M}{=}^t |g([a])|$ for all $a \in A$ by construction. ■

The following lemma extends the results on CMSO-equivalence of *large enough sets* to *large enough grids* by giving a sufficient condition on the sizes of two grids for the existence of a winning strategy for Duplicator in an r -round (mod M) game on the two structures.

Due to the inductive nature of the proof that involves, in each step, a construction of equivalence classes as in the above lemma, we need as a criterion for the size, for fixed $p \in \mathbb{N}$ and $M \subseteq_{\text{fin}} \mathbb{N}^+$, a function $f_{p,M} : \mathbb{N} \rightarrow \mathbb{N}$ such that, for all $r \in \mathbb{N}^+$ and $t = f_{p,M}(r-1)$, we can choose $T = f_{p,M}(r)$ in the previous lemma. One function satisfying, for all $r \in \mathbb{N}^+$, the inequality $f_{p,M}(r) \geq p \cdot (f_{p,M}(r-1) + \text{lcm}(M) - 1)$ derived from the condition imposed on T is $f_{p,M}(r) = 2 \cdot (p^r - 1) \cdot \text{lcm}(M)$.

Lemma 4.6. *Let $M \subseteq_{\text{fin}} \mathbb{N}^+$, $r \in \mathbb{N}$ and $k > 1$ be fixed. Then for $f(r) := f_{2^k, M}(r) = (2^{kr+1} - 2) \text{lcm}(M)$, as given above, $\mathfrak{G}_{k\ell_1} \equiv_r^M \mathfrak{G}_{k\ell_2}$ if $\ell_1 \equiv_{f(r)}^M \ell_2$.*

Proof. As motivated by Lemma 4.3, we consider the r -round (mod M) Ehrenfeucht-Fraïssé game on the corresponding horizontally coloured cliquey grids $\mathfrak{G}_{k\ell_1}^{\text{col}}$ and $\mathfrak{G}_{k\ell_2}^{\text{col}}$, and we show by induction on the number of rounds that Duplicator has a winning strategy in this game.

Intuitively, the proof proceeds as follows. Spoiler's set move induces an equivalence relation on the set of columns forming the grid he plays in, and the previous lemma implies that Duplicator is able to construct an equivalence relation on the columns of the other grid which is similar in the sense that corresponding equivalence classes satisfy certain cardinality constraints. Since the grids can be regarded as disjoint unions of these equivalence classes, we can argue by induction that corresponding subparts of the two grids, being similar enough, cannot be distinguished during the remaining $r-1$ rounds of the game.

The case where $\ell_1 = \ell_2$ is trivial since grids of the same dimensions are isomorphic. Thus, we assume in the following that $\ell_1, \ell_2 \geq f(r)$ and $\ell_1 \equiv \ell_2 \pmod{M}$. The claim is obviously true for $r = 0$, hence we assume that it holds for $r-1$ and proceed with the inductive step. As before, we assume without loss of generality that Spoiler makes his moves in $\mathfrak{G}_{k\ell_1}$ since the other case is symmetric.

A *coloured k -column* is a $\{\sim_v, P_1, \dots, P_k\}$ -structure isomorphic to $\mathfrak{C}_k^{\text{col}} := \mathfrak{G}_{k,1}^{\text{col}}$, such that a coloured grid can be regarded as a disjoint union of columns. Given a subset S of vertices of a grid and one of its coloured k -columns \mathfrak{C} with universe C , the *colour-type of \mathfrak{C} induced by S* is defined as the isomorphism type of the expansion $(\mathfrak{C}, S \cap C)$ denoted by $\text{tp}(\mathfrak{C}, S)$. Given a set \mathcal{F} of k -columns, each subset S of all of their vertices gives rise to an equivalence relation \approx_S on \mathcal{F} by virtue of $\mathfrak{C}_1 \approx_S \mathfrak{C}_2$ if, and only if, $\text{tp}(\mathfrak{C}_1, S) = \text{tp}(\mathfrak{C}_2, S)$. Note that the index of \approx_S is at most 2^k .

Assume, Spoiler performs a set move and chooses a subset S in $\mathfrak{G}_{k\ell_1}^{\text{col}} = \mathfrak{C}_1 \dot{\cup} \dots \dot{\cup} \mathfrak{C}_{\ell_1}$. As described above, S induces an equivalence relation \approx_S with at most 2^k equivalence classes on the set $\mathcal{F} = \{\mathfrak{C}_1, \dots, \mathfrak{C}_{\ell_1}\}$ of columns forming the grid. For $p = 2^k$, $t = f(r-1)$ and M as given, by the previous lemma, there is an equivalence relation \approx'_S on the set $\mathcal{F}' = \{\mathfrak{C}'_1, \dots, \mathfrak{C}'_{\ell_2}\}$ of columns on the Duplicator's grid $\mathfrak{G}_{k\ell_2}^{\text{col}}$ since $\ell_1, \ell_2 \geq f(r)$. Furthermore, there is a bijection g mapping equivalence classes of columns in one grid to the other.

Given that the index of both \approx_S and \approx'_S is $p' \leq p = 2^k$, we can assume $\{\mathfrak{C}_1, \dots, \mathfrak{C}_{p'}\}$ and $\{\mathfrak{C}'_1, \dots, \mathfrak{C}'_{p'}\}$ to be the sets of class representatives of \approx_S and \approx'_S , respectively. Duplicator now selects the unique set D of elements such that $\text{tp}(\mathfrak{C}, S) = \text{tp}(\mathfrak{C}', D)$ for all $1 \leq i \leq p'$, $\mathfrak{C} \in [\mathfrak{C}_i]$ and $\mathfrak{C}' \in g([\mathfrak{C}'_i])$.

For each $1 \leq i \leq p'$, we let $\langle \mathfrak{C}_i \rangle := \mathfrak{G}_{k\ell_1}^{\text{col}} \upharpoonright [\mathfrak{C}_i]$ and $\langle \mathfrak{C}'_i \rangle := \mathfrak{G}_{k\ell_2}^{\text{col}} \upharpoonright [\mathfrak{C}'_i]$ denote the substructures of the grids $\mathfrak{G}_{k\ell_1}^{\text{col}}$ and $\mathfrak{G}_{k\ell_2}^{\text{col}}$ induced by the sets of columns $[\mathfrak{C}_i]$ and $[\mathfrak{C}'_i]$, respectively. By construction, we have $||[\mathfrak{C}_i]|| =_{f(r-1)}^M ||[\mathfrak{C}'_i]||$ for all i . Thus, depending on whether $[\mathfrak{C}_i]$ (and hence $[\mathfrak{C}'_i]$) are small or large with respect to the threshold $f(r-1)$, either $\langle \mathfrak{C}_i \rangle \cong \langle \mathfrak{C}'_i \rangle$ or $\langle \mathfrak{C}_i \rangle \equiv_{r-1}^M \langle \mathfrak{C}'_i \rangle$ by the induction hypothesis. Since S and D induce the

same colour-types on the columns in $[\mathfrak{C}_i]$ and $[\mathfrak{C}'_i]$, respectively, we have

$$(\langle \mathfrak{C}_i \rangle, S \cap \text{univ}(\langle \mathfrak{C}_i \rangle)) \equiv_{r-1}^M (\langle \mathfrak{C}'_i \rangle, D \cap \text{univ}(\langle \mathfrak{C}'_i \rangle))$$

for all i , where $\text{univ}(\cdot)$ denotes the universe of the respective structure. Thus, iterating Lemma 3.5 yields that Duplicator has a winning strategy in the remaining rounds of the game $\mathcal{G}_{r-1}^M(\mathfrak{G}_{k\ell_1}^{\text{col}}, \mathfrak{G}_{k\ell_2}^{\text{col}})$ from position (S, D) .

If Spoiler makes a point move s , say in column \mathfrak{C}_1 of the grid $\mathfrak{G}_{k\ell_1}^{\text{col}}$, Duplicator picks an arbitrary element d of the same colour in her grid, say in column \mathfrak{C}'_1 . As the substructures consisting of just the columns containing the chosen elements are isomorphic, i.e., $(\mathfrak{C}_1, s) \cong (\mathfrak{C}'_1, d)$, and by the induction hypothesis we have $\mathfrak{C}_2 \dot{\cup} \dots \dot{\cup} \mathfrak{C}_{\ell_1} \equiv_{r-1}^M \mathfrak{C}'_2 \dot{\cup} \dots \dot{\cup} \mathfrak{C}'_{\ell_2}$, Duplicator can win the remaining $(r-1)$ -round game from position (s, d) by Lemma 3.5. ■

Now we have the necessary tools available to prove the main theorem.

Theorem 4.7. $\text{CMSO} \subsetneq \text{MSO}[\prec]_{\text{inv}}$.

Proof. We show that the class $\mathcal{C} := \{\mathfrak{G}_{k\ell} : k|\ell\}$ is not definable in CMSO but order-invariantly definable in MSO by the sentence $\psi_{\text{grid}} \wedge \varphi$, where

$$\varphi = \exists \text{min} \exists c \left(\begin{array}{l} \forall x (\text{min} \leq x) \wedge \neg \exists z (E_h(c, z) \vee E_v(c, z)) \\ \wedge \forall T (\forall x \forall y (Tx \wedge \varphi_{\text{diag}}(x, y) \rightarrow Ty) \wedge T \text{min} \rightarrow Tc) \end{array} \right),$$

and

$$\begin{aligned} \varphi_{\text{diag}}(x, y) &= (\exists z (E_v(x, z) \wedge E_h(z, y))) \\ &\quad \vee (\neg \exists z E_v(x, z) \wedge \exists z (z \sim_h \text{min} \wedge z \sim_v x \wedge E_h(z, y))), \\ E_h(x, y) &= x \sim_h y \wedge \exists x_0 \exists y_0 \left(\begin{array}{l} x_0 \sim_h \text{min} \wedge y_0 \sim_h \text{min} \\ \wedge x \sim_v x_0 \wedge y \sim_v y_0 \wedge x_0 < y_0 \\ \wedge \forall z_0 (z_0 \sim_h \text{min} \rightarrow z_0 \leq x_0 \vee z_0 \geq y_0) \end{array} \right), \\ E_v(x, y) &= x \sim_v y \wedge \exists x_0 \exists y_0 \left(\begin{array}{l} x_0 \sim_v \text{min} \wedge y_0 \sim_v \text{min} \\ \wedge x \sim_h x_0 \wedge y \sim_h y_0 \wedge x_0 < y_0 \\ \wedge \forall z_0 (z_0 \sim_v \text{min} \rightarrow z_0 \leq x_0 \vee z_0 \geq y_0) \end{array} \right). \end{aligned}$$

As hinted above, the horizontal and vertical edge relations (E_h and E_v , respectively) are defined using the successor relation which is induced by an arbitrary ordering on the row (and column) containing the minimal element (min) which itself serves as the lower left corner of the grid. φ_{diag} defines diagonal steps through the grid that wrap around from the top to the bottom row. Finally, φ states that the pair consisting of the lower left corner (min) and the upper right corner (c) of the grid is contained in the transitive closure of φ_{diag} . Obviously, there is such a sawtooth-shaped path starting at min and ending exactly in the upper right corner if, and only if, $k|\ell$.

The second step consists in showing that \mathcal{C} is not definable in CMSO. Towards this goal, we show that for any choice of $r \in \mathbb{N}$ and $M \subseteq_{\text{fin}} \mathbb{N}^+$, we can find $k, \ell_1, \ell_2 \in \mathbb{N}$, such that $\mathfrak{G}_{k\ell_1} \in \mathcal{C}$, $\mathfrak{G}_{k\ell_2} \notin \mathcal{C}$, and $\mathfrak{G}_{k\ell_1} \equiv_r^M \mathfrak{G}_{k\ell_2}$ which contradicts the CMSO-definability of \mathcal{C} .

Let $r \in \mathbb{N}$ and $M \subseteq_{\text{fin}} \mathbb{N}^+$ be fixed. We choose $s \geq r+1$ such that $2^s \nmid \text{lcm}(M)$. Let $k = 2^s$, $\ell_1 = 2^{kr+1} \text{lcm}(M)$, and $\ell_2 = \ell_1 + \text{lcm}(M)$. Obviously, ℓ_1 and ℓ_2 satisfy the conditions of Lemma 4.6, and thus $\mathfrak{G}_{k\ell_1} \equiv_r^M \mathfrak{G}_{k\ell_2}$.

Furthermore, $\ell_1 = k \cdot 2^{2^s \cdot r - s + 1} \text{lcm}(M)$, hence $k \mid \ell_1$ and $\mathfrak{G}_{k\ell_1} \in \mathcal{C}$. On the other hand, $k \nmid \ell_2 = \ell_1 + \text{lcm}(M)$ by the choice of s , thus $\mathfrak{G}_{k\ell_2} \notin \mathcal{C}$. ■

5. Conclusion

We have provided a characterisation of the expressiveness of CMSO in terms of an Ehrenfeucht-Fraïssé game that naturally extends the known game capturing MSO-definability, and we have presented a class of structures that are shown, using the proposed game characterisation, to be undefinable by a CMSO-sentence yet being definable by an order-invariant MSO-sentence. This establishes that order-invariant MSO is strictly more expressive than counting MSO in the finite. Modifying the separating example by considering a variant of cliquey grids where the two separate equivalence relations are unified into a single binary relation and considering, e.g., the class of such grids where the horizontal dimension exactly matches the vertical dimension, we can also confirm Courcelle’s original conjecture.

Corollary 5.1. *CMSO-definability is strictly weaker than $\text{MSO}[\langle \rangle]_{\text{inv}}$ -definability for general graphs.*

The separating query being essentially a transitive closure query, i.e., the only place where monadic second-order quantification is used is in the definition of the transitive closure of a binary relation, we can conclude that the same class of structures yields a separation of $(\text{D})\text{TC}^1[\langle \rangle]_{\text{inv}}$ from $(\text{D})\text{TC}^1$ (the extension of FO by a (deterministic) transitive closure operator on binary relations) and even from $(\text{D})\text{TC}^1$ extended with modulo-counting predicates since $(\text{D})\text{TC}^1 \subseteq \text{MSO}$. Finding separating examples concerning higher arity $(\text{D})\text{TC}$ or even full $(\text{D})\text{TC}$ requires further investigation since, in general, $\text{MSO} \subsetneq \text{DTC}^2$.

Following an opposite line of research, it would be interesting to identify further classes of graphs, besides classes of graphs of bounded tree-width, on which $\text{MSO}[\langle \rangle]_{\text{inv}}$ is no more expressive than CMSO.

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