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SHORTEST PATHS AVOIDING FORBIDDEN SUBPATHS

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ABSTRACT. In this paper we study a variant of the shortest path problem in graphs: given a weighted graph G and vertices s and t, and given a set X of forbidden paths in G, find a shortest s-t path P such that no path in X is a subpath of P. Path P is allowed to repeat vertices and edges. We call each path in X an *exception*, and our desired path a *shortest exception avoiding path*. We formulate a new version of the problem where the algorithm has no a priori knowledge of X, and finds out about an exception $x \in X$ only when a path containing x fails. This situation arises in computing shortest paths in optical networks. We give an algorithm that finds a shortest exception avoiding path in time polynomial in $|G|$ and $|X|$. The main idea is to run Dijkstra's algorithm incrementally after replicating vertices when an exception is discovered.

1. Introduction

One of the most fundamental combinatorial optimization problems is that of finding shortest paths in graphs. In this paper we study a variant of the shortest path problem: given a weighted graph $G(V, E)$, and vertices s and t, and given a set X of *forbidden paths* in G, find a shortest s-t path P such that no path in X is a subpath of P. We call paths in X *exceptions*, and we call the desired path a *shortest exception avoiding path*. We allow an exception avoiding path to be non-simple, i.e., to repeat vertices and edges. In fact the problem becomes hard if the solution is restricted to simple paths [20]. This problem has been called the *Shortest Path Problem with Forbidden Paths* by Villeneuve and Desaulniers [22]. Unlike them, we assume no a priori knowledge of X. More precisely, we can identify a forbidden path only after failing in our attempt to follow that path. This variant of the problem has not been studied before. It models the computation of shortest paths in optical networks, described in more detail in the "Motivation" section below. Note that when we fail to follow a path because of a newly discovered exception, we are still interested in a shortest path from s to t as opposed to a detour from the failure point. This is what is required in optical networks, because intermediate nodes do not store packets, and hence s must resend any lost packet.

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This paper presents two algorithms to compute shortest exception avoiding paths in the model where exceptions are not known a priori. The algorithms take respectively $O(kn \log n + km)$ and $O((n + L) \log(n + L) + m + dL)$ time to find shortest exception avoiding paths from s to all other vertices, where $n = |V|$, $m = |E|$, d is the largest degree of a vertex, k is the number of exceptions in X, and L is the total size of all exceptions.

Our algorithm uses a vertex replication technique similar to the one used to handle non-simple paths in other shortest path problems [6, 22]. The idea is to handle a forbidden path by replicating its vertices and judiciously deleting edges so that one copy of the forbidden path is missing its last edge and the other copy is missing its first edge. The result is to exclude the forbidden path but allow all of its subpaths. The main challenge is that vertex replication can result in an exponential number of copies of any forbidden path that overlaps the current one. Villeneuve and Desaulniers [22] address this challenge by identifying and compressing the overlaps of forbidden paths, an approach that is impossible for us since we do not have access to X. Our new idea is to couple vertex replication with the "growth" of a shortest path tree. By preserving certain structure in the shortest path tree we prove that the extra copies of forbidden paths that are produced during vertex replication are immaterial. Our algorithm is easy to implement, yet the proof of correctness and the run-time analysis are non-trivial.

1.1. Motivation

Our research on shortest exception avoiding path was motivated by a problem in optical network routing from Nortel Networks. In an optical network when a ray of light of a particular wavelength tries to follow a path P consisting of a sequence of optical fibers, it may fail to reach the endpoint of P because of various transmission impairments such as attenuation, crosstalk, dispersion and non-linearities [12, 17]. This failure may happen even though the ray is able to follow *any* subpath P' of P. This non-transitive behavior occurs because those impairments depend on numerous physical parameters of the traversed path (e.g., length of the path, type of fiber, wavelength and type of laser used, location and gain of amplifiers, number of switching points, loss per switching point, etc.), and the effect of those parameters may be drastically different in P than in P' [2]. Forbidden subpaths provide a straight-forward model of this situation.

We now turn to the issue of identifying forbidden paths. Because of the large number of physical parameters involved, and also because many of the parameters vary over the lifetime of the component [3], it is not easy to model the feasibility of a path. Researchers at Nortel suggested a model whereby an algorithm identifies a potential path, and then this path is tried out on the actual network. In case of failure, further tests can be done to pinpoint a minimal forbidden subpath. Because such tests are expensive, a routing algorithm should try out as few paths as possible. In particular it is practically impossible to identify all forbidden paths ahead of time—we have an exponential number of possible paths to examine in the network. This justifies our assumption of having no a priori knowledge of the forbidden paths, and of identifying forbidden paths only by testing feasibility of a path.

The shortest exception avoiding path problem may also have application in vehicle routing. Forbidden subpaths involving pairs of edges occur frequently ("No left turn") and can occur dynamically due to rush hour constraints, lane closures, construction, etc. Longer forbidden subpaths are less common, but can arise, for example if heavy traffic makes it impossible to turn left soon after entering a multi-lane roadway from the right. If we are routing a single vehicle it is more natural to find a detour from the point of failure when a forbidden path is discovered. This is different from our model of rerouting from s upon discovery of a forbidden path. However, in the situation when vehicles will be dispatched repeatedly, our model does apply.

1.2. Preliminaries

We are given an directed graph $G(V, E)$ with $n = |V|$ vertices and $m = |E|$ edges where each edge $e \in E$ has a positive weight denoting its *length*. We are also given a source vertex $s \in V$, a destination vertex $t \in V$, and a set X of paths in G. The graph G together with X models a communication network in which a packet cannot follow any path in X because of the physical constraints mentioned in Sec. 1.1. We assume that the algorithm can access the set X of forbidden paths only by performing queries to an oracle. Each query is a path P, and the oracle's response is either the confirmation that P is exception avoiding, or else an exception $x \in X$ that is a subpath of P and whose last vertex is earliest in P. Ties can be broken arbitrarily. In our discussion we say "we try a path" instead of saying "we query the oracle" because the former is more intuitive. In Sec. 4 we modify our algorithm for the case of an oracle that returns *any* exception on a path (not just the one that ends earliest). This requires more calls to the oracle but gives a faster run-time.

We want to find a shortest path from s to t that does not contain any path in X as a subpath—we make the goal more precise as follows. A *path* is a sequence of vertices each joined by an edge to the next vertex in the sequence. Note that we allow a path to visit vertices and edges more than once. If a path does not visit any vertex more than once, we explicitly call it a *simple path*. A simple directed path from vertex v to vertex w in G is called a *forbidden path* or an *exception* if a packet cannot follow the path from v to w because of the physical constraints. Given a set A of forbidden paths, a path $(v_1, v_2, v_3, \dots, v_l)$ is said to *avoid* A if $(v_i, v_{i+1}, \dots v_j) \notin A$ for all i, j such that $1 \le i < j \le l$. A path P from s to t is called a *shortest A-avoiding path* if the length of P is the shortest among all A-avoiding paths from s to t. We will use the term "exception avoiding" instead of "X-avoiding" when A is equal to X, the set of all forbidden paths in G.

1.3. Related work

A shortest s-t path in a graph can be computed in $O(n \log n + m)$ time and linear space using Dijkstra's algorithm with Fibonacci heaps if all edge weights are non-negative, and in $O(mn)$ time and linear space using the Bellman-Ford algorithm otherwise [5]. When the edge weights are non-negative integers, the problem can be solved in deterministic $O(m \log\log n \log\log\log n)$ time and linear space if the graph is directed [13], and in optimal $O(m)$ time if the graph is undirected [21]. In many of these cases, there are randomized algorithms with better expected times as well as approximation schemes. See Zwick [23] for a survey of shortest path algorithms, and Cabello [4], Goldberg and Harrelson [11] and Holzer et al. [15] for some of the more recent work.

Two recent papers on shortest paths in graphs address the issue of avoiding a set of forbidden paths, assuming that all the forbidden paths are known a priori. The first paper gives a hardness result. Szeider [20] shows, using a reduction from 3-SAT, that the problem of finding a shortest *simple* exception avoiding path is NP-complete even when each forbidden path has two edges. If the forbidden paths are *not* known a priori, the hardness result still applies to the case of simple paths because the lack of prior knowledge of the forbidden paths only makes the problem harder.

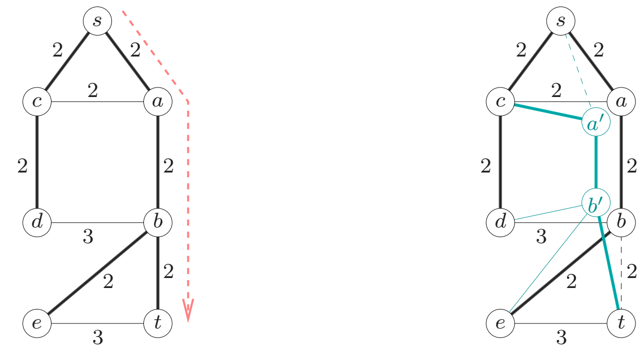
The second paper, by Villeneuve and Desaulniers [22], gives an algorithm for a shortest (possibly non-simple) exception avoiding path for the case when all the forbidden paths are known a priori. They preprocess the graph in $O((n + L) \log(n + L) + m + dL)$ time and $O(n + m + dL)$ space so that a shortest path from s to a query vertex can be found in $O(n + L)$ time. They first build a deterministic finite automaton (DFA) from the set of forbidden paths using the idea of Aho and Corasick [1], which can detect in linear time whether a given path contains any of the forbidden paths. They then "insert" the DFA into G by replicating certain vertices of G in the manner introduced by Martins [6], and then build a shortest path tree in this modified graph. Their algorithm cannot handle the case where the set of all forbidden paths is not explicitly given. Our algorithm is strictly more general, and we show in Sec. 4 that it solves their problem in roughly the same time but in less ($O(n + m + L)$) space.

We now mention two problems that seem related to ours, but do not in fact provide solutions to ours. The first one is maintaining shortest paths in a dynamic graph, i.e., where nodes or edges may fail [7, 9, 14], or edge weights may change (e.g., [7, 8]). Forbidden paths cannot be modeled by deleting edges or by modifying edge costs because *all* edges in a particular forbidden path may be essential—see Fig. 1 for an example. The second seemingly related problem is finding the k shortest paths in a graph. This was the subject of Martins [6] who introduced the vertex replication technique that we use in our algorithm. There is considerable work on this problem, see Eppstein [10] for a brief survey. But the k shortest path problem is again different from our situation because a forbidden subpath may be a bottleneck that is present in all of the k shortest paths even for $k \in \Omega(2^{n/2})$, see Villeneuve and Desaulniers [22].

In the context of optical networks researchers have studied many theoretical problems. See Ramaswami and Sivarajan [19] for details on optical networks, and Lee and Shayman [17] and McGregor and Shepherd [18] for a brief survey of the theoretical problems that have been investigated. In the previous work, the effect of physical constraints on paths in optical networks is either not considered at all (e.g., Khuller et al. [16]), or simply modeled by a known constant upper bound on the length of such a path (e.g., Gouveia et al [12], Lee and Shayman [17] and McGregor and Shepherd [18]). To the best of our knowledge, none of the previous work on shortest paths in optical networks considers the fact that it is practically infeasible to know a priori all the forbidden paths in the network, i.e., all the constraints in X. Our paper handles the issue of physical constraints from a different and much more practical perspective.

2. Algorithm for a shortest s-t path

In our algorithm we begin with a shortest path tree rooted at s, ignoring the exceptions. We then "try out" the path from s to t in the tree. If the path is free of exceptions, we are done. Otherwise, to take the newly discovered exception into account, we modify the graph using path replication as described in the Introduction, and we modify the shortest path tree to match. In general, we maintain a modified graph and a shortest path tree in the graph that gives a shortest path in the original graph from s to every other vertex avoiding all the currently-known exceptions. We will first illustrate the idea with an example. Consider the graph G in Fig. 1(a), where the integers denote edge weights, and the dashed arrow marks



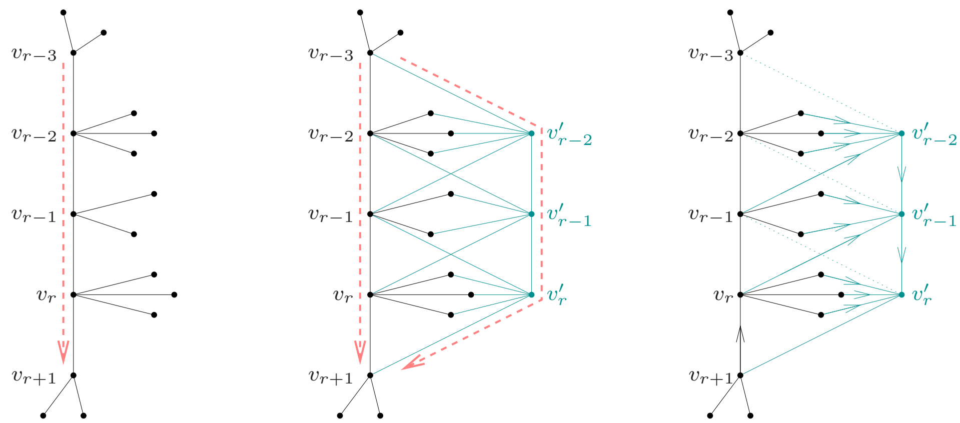
(a) (b)

Figure 1: (a) Shortest paths and (b) shortest x-avoiding paths in a graph, where $x = (s, a, b, t)$.

the forbidden path $x = (s, a, b, t)$. Note that for simplicity we have used undirected edges in the figure to denote bidirectional edges. It is not hard to see that $P = (s, c, a, b, t)$ is the shortest x-avoiding path from s to t. To find P, we first construct a shortest path tree rooted at s (marked using the heavy edges in Fig. 1(a)), and then try the path (s, a, b, t) in the tree. The path fails because it contains x, so we use a *vertex replication technique* similar to the one by Martins [6] to make duplicates of vertices a and b and delete edges (s, a') and (b, t), as shown in Fig. 1(b). We then construct a shortest path tree rooted at s (marked using the heavy edges in Fig. 1(b)) in the modified graph, and try the path (s, c, a', b', t) which "represents" the path P in G. We are done if x is the only forbidden path in G. Note that this approach can double the number of *undiscovered* forbidden paths. Suppose $y = (c, a, b)$ is another forbidden path in G. We have two copies of y in the modified graph: (c, a, b) and (c, a', b'), and we have to avoid both of them. Our solution to this doubling problem is to "grow" the shortest path tree in such a way that at most one of these two copies is encountered in future. Our algorithm is as follows:

1 construct the shortest path tree T_0 rooted at s in $G_0 = G$;
2 let $i = 1$;
3 send a packet from s to t through the path in T_0;
4 **while** *the packet fails to reach* t **do**
5   let x_i be the exception that caused the failure;
6   construct G_i from G_{i-1} by replicating the intermediate vertices of x_i and then deleting selected edges;
7   construct the shortest path tree T_i rooted at s in G_i using T_{i-1};
8   send a packet from s to t through the path in T_i;
9   let $i = i + 1$;

In the above algorithm, the only lines that need further discussion are Lines 6 and 7; details are in Sections 2.1 and 2.2 respectively. In the rest of the paper, whenever we focus on a particular iteration $i > 0$, we use the following notation: (i) the path from s to t in T_{i-1}, i.e., the path along which we try to send the packet to t in Line 4 in the iteration, is $(s, v_1, v_2, \ldots, v_p, t)$, and (ii) the exception that prevented the packet from reaching t in the iteration is $x_i = (v_{r-l}, v_{r-l+1}, \ldots, v_r, v_{r+1})$, which consists of $l + 1$ edges.



(a) G_{i-1} (b) G'_{i-1} (c) G_i

Figure 2: Modifying G_{i-1} to G_i: (a) The part of G_{i-1} at an exception $(v_{r-3}, v_{r-2}, v_{r-1}, v_r, v_{r+1})$, with $l = 3$. (b) Replicating vertices to create G'_{i-1}. The dashed paths show two of the 8 copies of the exception. (c) Deleting edges to create G_i. The dotted lines denote deleted edges.

2.1. Modifying the graph

The modification of G_{i-1} into G_i (Line 6) in the ith iteration eliminates exception x_i while preserving all the x_i-avoiding paths in G_{i-1}. We do the modification in two steps.

In the first step, we create a graph G'_{i-1} by replicating the intermediate vertices of x_i (i.e., the vertices $v_{r-l+1}, v_{r-l+2}, \ldots, v_r$). We also add appropriate edges to the replica v' of a vertex v. Specifically, when we add v' to G_{i-1}, we also add the edges of appropriate weights between v' and the neighbors of v. It is easy to see that if a path in G_{i-1} uses $l' \le l$ intermediate vertices of x_i, then there are exactly $2^{l'}$ copies of the path in G'_{i-1}. We say that a path in G'_{i-1} is x_i-avoiding if it contains none of the 2^l copies of x_i.

In the second step, we build a spanning subgraph G_i of G'_{i-1} by deleting a few edges from G'_{i-1} in such a way that all copies of x_i in G'_{i-1} are eliminated, but all x_i-avoiding paths in G'_{i-1} remain unchanged. To build G_i from G'_{i-1}, we delete the edges (v_{j-1}, v'_j) and (v'_j, v_{j-1}) for all $j \in [r-l+1, r]$. We also delete the edge (v_r, v_{r+1}), all the outgoing edges from v'_r except (v'_r, v_{r+1}), and all the outgoing edges from v'_j except (v'_j, v'_{j+1}) for all $j \in [r-l+1, r-1]$. Figure 2 shows how the "neighborhood" of an exception changes from G_{i-1} to G_i. As before, the undirected edges in the figure are bidirectional.

Observation 2.1. Graph G_i has no copy of x_i.

In Sec. 3.1 we will prove that G_i still contains all the x_i-avoiding paths of G_{i-1}.

The vertices in G_i $[G'_{i-1}]$ that exist also in G_{i-1} (i.e., the ones that are not replica vertices) are called the *old vertices of G_i [respectively G'_{i-1}]*. Note that the vertices of G_0 exist in G_i for all $i \ge 0$. These vertices are called the *original vertices of G_i*.

2.2. Constructing the tree

In Line 7 of our algorithm we construct a tree T_i that contains a shortest x_i-avoiding path from s to every other vertex in G_{i-1}. Tree T_i is rooted at s, and its edges are directed away from s. Not every shortest path tree rooted at s in G_i will work. In order to guarantee termination of the algorithm, T_i must be similar to T_{i-1}, specifically, every x_i-avoiding path from s in T_{i-1} must be present in T_i. The necessity of this restriction is explained in Sec. 3.3.

We construct the required T_i by preserving as much of T_{i-1} as possible. We apply Dijkstra's algorithm starting from the part of T_{i-1} that can be preserved. Let V' be the set of vertices that are either replica vertices in G_i, or descendants of v_{r+1} in T_{i-1}. We first set the weight of each $v \in V'$ to infinity, and temporarily set $T_i = T_{i-1} - V'$. Then, for each $v \in V'$, we set the weight of v to the minimum, over all edges (u, v), of the sum of the weight of u and the length of (u, v). Finally, we initialize the queue used in Dijkstra with all the vertices in V' and run the main loop of Dijkstra's algorithm. Each iteration of the loop adds one vertex in V' to the temporary T_i. When the queue becomes empty, we get the final tree T_i.

3. Correctness and analysis

3.1. Justifying the graph modification

In this section we prove the following lemma, which uses the notion of a *corresponding path*. Consider any path P_i in G_i. By substituting every vertex in P_i that is not present in G_{i-1} with the corresponding old vertex in G_{i-1}, we get the corresponding path P_{i-1} in G_{i-1}. This is possible because any "new" edge in G_i is a replica of an edge in G_{i-1}. We define the corresponding path P_j in G_j for all $j < i$ by repeating this argument.

Lemma. If P_i is a shortest path from s to an original vertex v in G_i, P_0 is a shortest $\{x_1, x_2, \ldots, x_i\}$-avoiding path from s to v in G_0.

To prove the above lemma (repeated as Lemma 3.3 below), we will first prove that x_i-avoiding paths in G_{i-1} are preserved in G_i (Lemma 3.2), using the following characteristic of an x_i-avoiding path in the intermediate graph G'_{i-1}:

Lemma 3.1. *For any x_i-avoiding path P from s to v that uses only the old vertices in G'_{i-1}, there exists a copy of P in G_i that starts and ends at the old vertices s and v respectively, and possibly passes through the corresponding replicas of its intermediate vertices.*

Proof. Graph G_i contains all the edges between pairs of old vertices in G'_{i-1} except for the directed edge (v_r, v_{r+1}). Thus P can remain unchanged if it does not use this directed edge. Otherwise we will re-route any portion of P that uses the directed edge (v_r, v_{r+1}) to use the replica edge (v'_r, v_{r+1}) instead. Let $P = (s = w_1, w_2, \ldots, w_q = v)$, and (w_j, w_{j+1}) be an occurrence of (v_r, v_{r+1}) in P. Tracing P backwards from w_j, let $h \le j$ be the minimum index such that $(w_h, w_{h+1}, \ldots, w_{j+1})$ is a subpath of x_i. Because P is x_i-avoiding, w_h must be an intermediate vertex of x_i. This implies that $h > 1$, since $s = w_1$ is not an intermediate vertex of x_i because of the following reasons: (i) x_i is a path in the shortest path tree rooted at s in G_i, and (ii) there is no replica of s in G_i. Therefore w_{h-1} exists. We will reroute the portion of P between w_{h-1} and w_{j+1} by using the corresponding replica vertices in place of the subpath $(w_h, \ldots, w_j)$ of x_i. Note that the required edges exist in G_i (since P does not contain the whole exception x_i), and that the portions of P that we re-route are disjoint along P. Moreover, P starts and ends at the old vertices s and v respectively. ∎

Lemma 3.2. *Any x_i-avoiding path from s to v in G_{i-1} has a copy in G_i that starts and ends at the old vertices s and v respectively, and possibly goes through the corresponding replicas of its intermediate vertices.*

Proof. Let P be the x_i-avoiding path in G_{i-1}. As we do not delete any edge to construct G'_{i-1} from G_{i-1}, P remains unchanged in G'_{i-1}. Moreover, P uses no replica vertex in G'_{i-1}. So, Lemma 3.1 implies that P exists in G_i with the same old vertices at the endpoints, possibly going through the corresponding replicas of the intermediate vertices. ∎

Lemma 3.3. *If P_i is a shortest path from s to an original vertex v in G_i, P_0 is a shortest $\{x_1, x_2, \ldots, x_i\}$-avoiding path from s to v in G_0.*

Proof. For any $j \in [0, i]$, let $X_j = \{x_{j+1}, x_{j+2}, \ldots, x_i\}$. We show that for any j, if P_j is a shortest X_j-avoiding path in G_j, then P_{j-1} is a shortest X_{j-1}-avoiding path in G_{j-1}. The lemma then follows by induction on j, with basis $j = i$, because $X_i = \emptyset$ and thus P_i is a shortest X_i-avoiding path in G_i.

If P_j is a shortest X_j-avoiding path in G_j, P_j is X_{j-1}-avoiding because P_j is x_j-avoiding by Observation 2.1, and $X_j \cup \{x_j\} = X_{j-1}$. So, the corresponding path P_{j-1} is also X_{j-1}-avoiding. If we assume by contradiction that P_{j-1} is not a shortest X_{j-1}-avoiding path in G_{j-1}, then there exists another path P'_{j-1} from s to v in G_{j-1} which is X_{j-1}-avoiding and is shorter than P_{j-1}. Since $x_j \in X_{j-1}$, P'_{j-1} is x_j-avoiding, and hence by Lemma 3.2, there is a copy P'_j of path P'_{j-1} in G_j which has the same original vertices at the endpoints. As P'_{j-1} is X_j-avoiding, P'_j is also X_j-avoiding. This is impossible because P'_j is shorter than P_j. Therefore, P_{j-1} is a shortest X_{j-1}-avoiding path in G_{j-1}. ∎

3.2. Justifying the tree construction

To show that the "incremental" approach used in Sec. 2.2 to construct T_i is correct, we first show that the part of T_{i-1} that we keep unchanged in T_i is composed of shortest paths in G_i:

Lemma 3.4. *For every vertex v that is not a descendant of v_{r+1} in T_{i-1}, the path P from s to v in T_{i-1} is a shortest path in G_i.*

Proof. First we show that P exists in G_i. Every vertex in T_{i-1} exists in G_i as an old vertex. So, P exists in G_i through the old vertices if no edge of P gets deleted in G_i. The only edge between a pair of old vertices in G_{i-1} that gets deleted in G_i is (v_r, v_{r+1}). Since v is not a descendant of v_{r+1} in T_{i-1}, P does not use the edge (v_r, v_{r+1}). Therefore, no edge of P gets deleted in G_i. So, P exists in G_i through the old vertices.

Neither the modification from G_{i-1} to G'_{i-1} nor the one from G'_{i-1} to G_i creates any "shortcut" between any pair of vertices. So, there is no way that the distance between a pair of old vertices decreases after these modifications. Since these modifications do not change P, which is a shortest path in G_{i-1}, P is a shortest path in G_i. ∎

Lemma 3.5. *The tree T_i is a shortest path tree in G_i.*

Proof. For every vertex v that is not a descendant of v_{r+1} in T_{i-1}, the path P from s to v in T_i is the same as the one in T_{i-1} and hence, a shortest path in G_i (Lemma 3.4). For all other vertices v in G_i, it follows from Dijkstra's algorithm that the path from s to v in T_i is a shortest path. ∎

Lemmas 3.3 and 3.5 together prove that our algorithm is correct provided it terminates, which we establish in the next section.

3.3. Analyzing time and space requirement

Although in every iteration we eliminate one exception by modifying the graph, we introduce copies of certain other exceptions through vertex replication. Still our algorithm does not iterate indefinitely because, as we will show in this section, the incremental construction of the shortest path tree (Sec. 2.2) guarantees that we do not discover more than one copy of any exception. We first show that any exception in G_{i-1} has at most two copies in G_i (Lemma 3.6), and then prove that one of these two copies is never discovered in the future (Lemma 3.7):

Lemma 3.6. *Let $y \neq x_i$ be any exception in G_{i-1}. If the last vertex of y is* not *an intermediate vertex of x_i, then G_i contains exactly one copy of y. Otherwise, G_i contains exactly two copies of y. In the latter case, one copy of y in G_i ends at the old vertex v and the other copy ends at the corresponding replica v'.*

Proof. Let $\pi = (w_1, w_2, \ldots, w_j)$ be a maximal sequence of vertices in y that is a subsequence of $(v_{r-l+1}, v_{r-l+2}, \ldots, v_r)$. Let w'_j be the replica of w_j in G_i. We will first show that if there is a vertex v in y right after π, then exactly one of the edges (w_j, v) and (w'_j, v) exists in G_i. Consider the subgraph of G_i induced on the set of replica vertices $\{v'_{r-l+1}, v'_{r-l+2}, \ldots, v'_r\}$: this subgraph is a directed path from v'_{r-l+1} to v'_r, and the only edge that goes out of this subgraph is (v'_r, v_{r+1}). Therefore, (i) when $(w_j, v) = (v_r, v_{r+1})$, $(w'_j, v) \in G_i$ and $(w_j, v) \notin G_i$, and (ii) otherwise, $(w_j, v) \in G_i$ and $(w'_j, v) \notin G_i$.

Now G_i has exactly two copies of π: one through the old vertices, and another through the replicas. The above claim implies that when there is a vertex v in y right after π, G_i has at most one copy of the part of y from w_1 to v. However, when π is a suffix of y, G_i has both the copies of the part of y from w_1 to w_j. The lemma then follows because any part of y that contains no intermediate vertex of x_i has exactly one copy in G_i. ∎

Lemma 3.7. *Let $y \neq x_i$ be any exception in G_{i-1} such that the last vertex of y is an intermediate vertex v of x_i. The copy of y that ends at the old vertex v in G_i is not discovered by the algorithm in any future iteration.*

Proof. The copy of the path $(s, v_1, v_2, \ldots, v_r)$ through the old vertices in G_i contains v. Let P be the part of this path from s to v. Clearly, $P \in T_{i-1}$, and P does not contain any exception because the oracle returns the exception with the earlier last vertex. So, the way we construct T_j from T_{j-1} for any iteration $j \geq i$ ensures that $P \in T_j$.

Let y_1 be the copy of y that ends at v. Now y_1 is not a subpath of P because P does not contain any exception. For any $j \geq i$, $P \in T_j$, and both P and y_1 end at the same vertex, therefore $y_1 \notin T_j$. So, a packet in iteration j will not follow y_1, and y_1 will not be discovered in that iteration. ∎

Lemma 3.8. *The* ***while*** *loop iterates at most $k = |X|$ times.*

Proof. For any iteration i, G_{i-1} contains x_i, and G_i does not contain x_i. Every exception other than x_i in G_{i-1} has either one or two copies in G_i (Lemma 3.6). By Lemma 3.7, if an exception has two copies in G_i, only one of them is relevant in the future. Thus the number of exceptions effectively decreases by one in each iteration. The lemma then follows. ∎

To determine the running time, observe that the number of vertices increases in each iteration. However, we run Dijkstra's algorithm on at most n vertices in any iteration, because the number of replica vertices added in each iteration is always less than the number of vertices in the part of the shortest path tree that is carried over from the previous tree in our incremental use of Dijkstra. Moreover, we can make sure that Dijkstra's algorithm examines at most m edges in iteration i, by deleting a few more edges from G_i after performing the graph modification described in Sec. 2.1. More precisely, for each old vertex $v \in \{v_{r-l+1}, v_{r-l+2}, \ldots, v_r\}$, since the label (i.e., the "distance" from s) put on v by Dijkstra's algorithm in the previous iterations remains unchanged later on, we can safely delete from G_i all the *incoming* edges of v without affecting future modifications. (Note that for all $j \in [r-l+1, r]$, old vertices v_j and v_{j+1} are no longer adjacent in G_i, although the edge (v_j, v_{j+1}) still exists in T_i.) It is not hard to see that the number of new edges in G_i is now equal to the number of edges deleted from G_{i-1}.

Theorem 3.9. *The algorithm computes a shortest X-avoiding path in $O(kn \log n + km)$ time and $O(n + m + L)$ space.*

Proof. The correctness of the algorithm follows from Lemmas 3.3 and 3.5.

Let l_i be the number of intermediate vertices of the exception discovered at the ith iteration (thus the size of the exception is $l_i + 2$). The ith iteration adds l_i vertices. Since the algorithm iterates k times (Lemma 3.8), there are $n + \sum_{i=1}^{k} l_i < n + L$ vertices in the graph at termination. Because in each iteration the number of added edges is equal to the number of deleted edges, the space requirement is $O(n + m + L)$.

Each iteration of our algorithm takes $O(|V| \log |V| + |E|) = O(n \log n + m)$ time, and the total time requirement follows. ∎

We note that in practice, the algorithm will not discover all k of the forbidden paths. It will discover only the ones that "interfere" in getting from s to t.

4. Extensions

This section contains: (1) an algorithm to compute shortest paths from s to every other vertex in G; (2) an analysis in the case when X is given explicitly; and (3) a version of the algorithm where the oracle returns *any* exception on a query path, rather than the exception that ends earliest.

The algorithm in Sec. 2 can be extended easily to compute a shortest path from s to every other vertex in G. We simply repeat the previous algorithm for every vertex in G, but with a small change: in every iteration (except of course the first one) we use the graph and the shortest path tree constructed at the end of previous iteration. Since every exception in X is handled at most once, the **while** loop still iterates at most k times, and therefore, the time and space requirements remain the same.

Theorem 4.1. *The algorithm computes shortest X-avoiding paths from s to all other vertices in $O(kn \log n + km)$ time and $O(n + m + L)$ space.*

Our algorithm applies when X is known explicitly; taking into account the cost of sorting X so that we can efficiently query whether a path contains an exception we obtain:

Theorem 4.2. *When X is known a priori, we can preprocess the graph in $O(kn \log(kn) + km)$ time and $O(n + m + L)$ space so that we can find a shortest X-avoiding path from s to any vertex in $O(n + L)$ time.*

Recall that Villeneuve and Desaulniers [22] solved this problem in $O((n+L)\log(n+L)+m+dL)$ preprocessing time, $O(n+m+dL)$ space and $O(n+L)$ query time. Our algorithm is more space efficient than theirs. Our preprocessing is slightly slower in general, although it is slightly faster in the special case $L=\Theta(kn)$ and $m=o(dn)$ (intuitively, when the exceptions are long, and the average degree of a vertex is much smaller than the largest degree).

Finally, returning to the case where X is not known a priori, we consider a weaker oracle that returns any exception on the query path, rather than the exception that ends earliest. At the cost of querying the oracle more often, we obtain a better run-time. The idea is to query the oracle *during* the construction of a shortest path tree. The algorithm is very similar to Dijkstra's, the only difference is that it handles exceptions inside Dijkstra's loop. More precisely, right after a vertex v is dequeued and added to the current tree, we try the s-v path in the tree. If the path is exception avoiding, we update the distances of the neighbors of v and go to the next iteration, as in "traditional" Dijkstra's algorithm. Otherwise, we remove v from the current tree, perform vertex replication and edge deletion as described in Sec. 2.1, and then go to the next iteration.

Theorem 4.3. *The algorithm described above computes shortest X-avoiding paths from s to all other vertices in $O((n+L)\log(n+L)+m+dL)$ time and $O(n+m+L)$ space.*

Proof. There are at most $n+L$ vertices in the modified graph in any iteration. So, the loop in the modified Dijkstra's algorithm executes at most $n+L$ times, and the priority queue holds at most $n+L$ entries. Moreover, within Dijkstra's loop vertex replication and edge deletion take $O(dL)$ time in total. The running time then follows. The proof of correctness is similar to that of Theorem 4.1 except that the "current" shortest path tree is no longer a spanning tree in the current graph. ∎

This new algorithm is faster than the old algorithm of Theorem 4.1 in general but makes as many as $n+L$ queries to the oracle versus at most k oracle queries for the old algorithm. The old algorithm is slightly faster in the special case $L=\Theta(kn)$ and $m=o(dn)$.

5. Conclusion

Motivated by the practical problem of finding shortest paths in optical networks, we introduced a novel version of the shortest path problem where we must avoid forbidden paths, but we only discover the forbidden paths by trying them. We gave an easily implementable, polynomial time algorithm that uses vertex replication and incremental Dijkstra.

As we have mentioned before, in practice our algorithms will not discover all the forbidden paths in X. In fact, the running time of each of our algorithms is determined by only the forbidden paths that "interfere" in getting from s to t. An interesting open problem is to bound the number of such paths. We conjecture that in a real optical network, the number of such paths is $o(k)$, and therefore, our algorithms run much faster in practice.

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References

[1] Alfred V. Aho and Margaret J. Corasick. Efficient string matching: an aid to bibliographic search. *Commun. ACM*, 18(6):333–340, 1975.

[2] Peter Ashwood-Smith. Personal communication, 2007.

[3] Peter Ashwood-Smith, Don Fedyk, and Vik Saxena. Link viability constraints requirements for GMPLS-enabled networks. http://tools.ietf.org/html/draft-ashwood-ccamp-gmpls-constraint-reqts-00, July 2005. Internet draft, work in progress.

[4] Sergio Cabello. Many distances in planar graphs. In *Proceedings of the 17th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1213–1220, New York, NY, USA, 2006.

[5] Thomas H. Cormen, Clifford Stein, Ronald L. Rivest, and Charles E. Leiserson. *Introduction to Algorithms*. McGraw-Hill Higher Education, 2001.

[6] Ernesto de Queiros Vieira Martins. An algorithm for ranking paths that may contain cycles. *European Journal of Operational Research*, 18(1):123–130, October 1984.

[7] Camil Demetrescu, Stefano Emiliozzi, and Giuseppe F. Italiano. Experimental analysis of dynamic all pairs shortest path algorithms. In *Proceedings of the 15th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 369–378, Philadelphia, PA, USA, 2004.

[8] Camil Demetrescu, Daniele Frigioni, Alberto Marchetti-Spaccamela, and Umberto Nanni. Maintaining shortest paths in digraphs with arbitrary arc weights: an experimental study. In *Proceedings of the Fourth International Workshop on Algorithm Engineering*, pages 218–229, London, UK, 2001.

[9] Camil Demetrescu, Mikkel Thorup, Rezaul A. Chowdhury, and Vijaya Ramachandran. Oracles for distances avoiding a failed node or link. *SIAM J. Comput.*, 37(5):1299–1318, 2008.

[10] David Eppstein. Finding the k shortest paths. *SIAM J. Comput.*, 28(2):652–673, 1999.

[11] Andrew V. Goldberg and Chris Harrelson. Computing the shortest path: A^* search meets graph theory. In *Proceedings of the 16th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 156–165, Philadelphia, PA, USA, 2005.

[12] Luis Gouveia, Pedro Patrício, Amaro de Sousa, and Rui Valadas. MPLS over WDM network design with packet level QoS constraints based on ILP models. In *Proceedings of the 22nd Annual Joint Conference of the IEEE Computer and Communications Societies*, April 2003.

[13] Yijie Han. Improved fast integer sorting in linear space. In *Proceedings of the 12th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 793–796, Philadelphia, PA, USA, 2001.

[14] John Hershberger, Subhash Suri, and Amit Bhosle. On the difficulty of some shortest path problems. *ACM Trans. Algorithms*, 3(1):5, 2007.

[15] Martin Holzer, Frank Schulz, Dorothea Wagner, and Thomas Willhalm. Combining speed-up techniques for shortest-path computations. *J. Exp. Algorithmics*, 10:2.5, 2005.

[16] Samir Khuller, Kwangil Lee, and Mark A. Shayman. On degree constrained shortest paths. In *Proceedings of the 13th Annual European Symposium on Algorithms*, pages 259–270, 2005.

[17] Kwangil Lee and Mark A. Shayman. Optical network design with optical constraints in IP/WDM networks. *IEICE Transactions on Communications*, E88-B(5):1898–1905, 2005.

[18] Andrew McGregor and Bruce Shepherd. Island hopping and path colouring with applications to WDM network design. In *Proceedings of the 18th Annual ACM-SIAM Symposium on Discrete Algorithm*, pages 864–873, Philadelphia, PA, USA, January 2007.

[19] Rajiv Ramaswami and Kumar N. Sivarajan. *Optical Networks: A Practical Perspective*. Morgan Kaufmann Publishers Inc., San Francisco, CA, USA, 2002.

[20] Stefan Szeider. Finding paths in graphs avoiding forbidden transitions. *Discrete Appl. Math.*, 126(2-3):261–273, 2003.

[21] Mikkel Thorup. Undirected single-source shortest paths with positive integer weights in linear time. *J. ACM*, 46(3):362–394, 1999.

[22] Daniel Villeneuve and Guy Desaulniers. The shortest path problem with forbidden paths. *European Journal of Operational Research*, 165(1):97–107, 2005.

[23] Uri Zwick. Exact and approximate distances in graphs—a survey. In *Proceedings of the Ninth Annual European Symposium on Algorithms*, pages 33–48, London, UK, 2001.

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