Abstract. We study the action on modulation spaces of Fourier multipliers with symbols $e^{i\mu(\xi)}$, for real-valued functions $\mu$ having unbounded second derivatives. We show that if $\mu$ satisfies the usual symbol estimates of order $\alpha \geq 2$, or if $\mu$ is a positively homogeneous function of degree $\alpha$, the corresponding Fourier multiplier is bounded as an operator between the weighted modulation spaces $M^p_q$ and $M^p_q$, for every $1 \leq p, q \leq \infty$ and $\delta \geq d(\alpha - 2)|\frac{1}{2} - \frac{1}{2}|$. Here $\delta$ represents the loss of derivatives. The above threshold is shown to be sharp for all homogeneous functions $\mu$ whose Hessian matrix is non-degenerate at some point.

1. Introduction and statement of the results

The results presented here are part of a joint work with Fabio Nicola and Silvia Rivetti [9]. A Fourier multiplier is formally an operator of the type

$$
\sigma(D)f(x) = \int_{\mathbb{R}^d} e^{2\pi i x \xi} \sigma(\xi) \hat{f}(\xi) d\xi,
$$

where $\hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \xi} f(x) dx$ is the Fourier transform. The function $\sigma$ is called symbol of the multiplier. Whereas the action of these operators on $L^2(\mathbb{R}^d)$ is clear (by Parseval’s formula), their study in $L^p$, $p \neq 2$, for several classes of symbols is a fundamental topic in Harmonic Analysis, with important applications to partial differential equations.

In particular, unimodular Fourier multipliers are defined by symbols of the type $\sigma(\xi) = e^{i\mu(\xi)}$, for real-valued functions $\mu$. They arise when solving the Cauchy problem for dispersive equations. For example, for the solution $u(t, x)$ of the Cauchy problem

$$
\begin{cases}
    i\partial_t u + |\Delta|^{\frac{\alpha}{2}} u = 0 \\
    u(0, x) = u_0(x),
\end{cases}
$$

$(t, x) \in \mathbb{R} \times \mathbb{R}^d$, we have the formula $u(t, x) = (e^{it|2\pi D|^{\alpha}} u_0)(x)$. The cases $\alpha = 1, 2, 3$ are of particular interest because they correspond to the (half-)wave equation, the Schrödinger equation and (essentially) the Airy equation, respectively.

2000 Mathematics Subject Classification. 35S30, 47G30, 42C15.

Key words and phrases. Fourier multipliers, modulation spaces, short-time Fourier transform.
Unimodular Fourier multiplier generally do not preserve any Lebesgue space $L^p$, except for $p = 2$. It is then natural to study boundedness properties on other function spaces arising in Fourier analysis. This was recently done in [1] for the modulation spaces $M^{p,q}$, $1 \leq p, q \leq \infty$. These spaces were introduced by H. Feichtinger in 1980 (see [7]) and since then have found many applications in Time-frequency analysis, see e.g. Gröchenig’s book [8] where the precise definition can be found. Here it suffices to observe that, for heuristic purposes, distributions in $M^{p,q}$ may be regarded as functions which locally have the same regularity as a function in $F^q_L$ (the space of distributions whose Fourier transform is in $L^q$), but at infinity decay like a function in $L^p$.

Now, it was shown in [1], among other things, that symbols of the type $\sigma(\xi) = e^{i|\xi|^\alpha}$, with $0 \leq \alpha \leq 2$, give rise to bounded operators on all $M^{p,q}$, $1 \leq p, q \leq \infty$. This can be rephrased by saying that the obstruction to the boundedness on $L^p$ is just local in nature. Indeed if we keep the $L^p$ decay but we measure the local regularity by any Fourier-Lebesgue space $F^q_L$ (which is of course preserved by unimodular Fourier multipliers) instead of $L^p$, boundedness is recaptured. Moreover, the conclusion extends to symbols $\sigma(\xi) = e^{i\mu(\xi)}$ where $\mu$ is a positively homogeneous function of degree $\alpha \in [0, 2]$, smooth away from the origin, or even a smooth functions on $\mathbb{R}^d$ whose derivatives of order $\geq 2$ are bounded.

More generally, similar results also hold, when $p = q$, for a class of Fourier integral operators whose phases have bounded derivatives of order $\geq 2$, see [3, 6]. However for $p \neq q$ a loss of regularity or decay may then occur; see [5] for an analysis of this phenomenon.

Now, we fix the attention on multipliers with symbols $e^{i|\xi|^\alpha}$, with $\alpha > 2$. In this case one still expects boundedness, but with a loss of regularity, namely from $M^{p,q}$ to $M^{p,q}$, for any $\delta \geq \delta(p, q)$ sufficiently large ($\delta$ represents the loss of derivatives). Here $M^{p,q}_\delta = \{ f \in S'(\mathbb{R}^d) : (1 - \Delta)^{\delta/2} f \in M^{p,q} \}$ is in fact a Sobolev-like space based on $M^{p,q}$. Since, as we already observed, Fourier-Lebesgue spaces are trivially preserved by unimodular Fourier multipliers, the obstruction to the boundedness on $M^{p,q}$ should be global in nature. As a consequence, the optimal threshold should depend on $p$ only. In fact in [1, Theorem 16(b)] it was already proved that the multiplier $e^{i|D|^\alpha}$ is bounded from $M^{p,q}_\delta$ to $M^{p,q}$ for every $\delta > d\alpha \left(\frac{1}{2} - \frac{1}{p}\right)$. The proof relied on fine classical results about boundedness of wave multipliers on $L^p$, with loss of derivatives.

The main result we report is a refinement of [1, Theorem 16(b)], with a lower threshold, and can be stated as follows (we refer to [9] for more details and proofs). Let $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$, for $\xi \in \mathbb{R}^d$.

**Theorem 1.1.** Consider a function $\mu \in C^\infty(\mathbb{R}^d)$, real-valued, satisfying

$$\quad (2) \quad |\partial^\gamma \mu(\xi)| \leq C_\gamma \langle \xi \rangle^{\alpha - 2}, \quad \forall |\gamma| \geq 2, \ \xi \in \mathbb{R}^d.$$
for some $\alpha \geq 2$. Then the multiplier
\[ e^{i\mu(D)f(x)} := \int_{\mathbb{R}^d} e^{2\pi ix\xi} e^{i\mu(\xi)} \hat{f}(\xi) d\xi \]
is bounded as an operator from $\mathcal{M}^{p,q}_b$ to $\mathcal{M}^{p,q}$ for
\[ \delta \geq d(\alpha - 2) \left| \frac{1}{p} - \frac{1}{2} \right|, \tag{3} \]
and every $1 \leq p, q \leq \infty$. The same conclusion holds true if $\mu(\xi)$ is smooth for $\xi \neq 0$ only, and positively homogeneous of degree $\alpha$.

In particular, for $\alpha = 2$, the threshold in (3) vanishes, and we recapture the above result about boundedness without loss of derivatives. Actually, the proof of Theorem 1.1 makes use of the known result for $\alpha = 2$, combined with a Littlewood-Paley decomposition of the frequency domain and the dilation properties of modulation spaces [10].

We also prove that the threshold in (3) is generally sharp. Most interesting, it is sharp for all homogeneous functions $\mu$ whose Hessian matrix is non-degenerate at some point. This highlights that the unboundedness on $\mathcal{M}^{p,q}$ is due to the presence of some curvature of the graph of $\mu$. Also, this suggests an investigation of the optimal threshold in terms of the number of principal curvatures which are identically zero. More precisely, if at every point the Hessian matrix of $\mu$ has rank at most $r$, we expect the threshold to be $r(\alpha - 2) \left| \frac{1}{p} - \frac{1}{2} \right|$. We plan to study these issues in greater details in future.

Notice that the above negative result shows that the Cauchy problem (1) is not locally wellposed in any $\mathcal{M}^{p,q}$, if $p \neq 2$ and $\alpha > 2$. For positive results in this connection we refer to [1, 2, 4, 11] and the references therein.

References


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