An open question on the existence of Gabor frames in general linear position

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Abstract. Uncertainty principles for functions defined on finite Abelian groups generally relate the cardinality of a function to the cardinality of its Fourier transform. We examine how the cardinality of a function is related to the cardinality of its short–time Fourier transform. We illustrate that for some cyclic groups of small order, both, the Fourier and the short–time Fourier case, show a remarkable resemblance. We pose the question whether this correspondence holds for all cyclic groups.

Keywords. Gabor systems, erasure channels, time–frequency dictionaries, short–time Fourier transform, uncertainty principle.

1 Introduction

Recent results by Tao [11] and Meshulam [9] relate the cardinality of the support of a function on a finite Abelian group to the support of its Fourier transform. This paper shall discuss corresponding results for a common joint time-frequency representation, the short-time Fourier transform. That is, the cardinality of its support be bounded below in terms of the sparsity of the input. The short-time Fourier transform measures the local frequency content of a function with respect to a given window function. Due to this dependence, only weak bounds can be expected to hold for all window functions. We have established stronger results by instead, restricting our attention to bounds that hold for almost every window function, that is, we allow an exceptional set of measure zero [7]. For example, Theorem 2 shows that for any group of prime order and for almost every window function on the group, the sum of the cardinality of the support of the analyzed function and the cardinality of the support of its short-time Fourier transform exceeds the square of the order of the group. The corresponding statement turns out to be false in the case of arbitrary Abelian groups (for instance $\mathbb{Z}_2^2$, see [7]). However, we pose the question whether a generalization to cyclic groups is possible.

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2 Preliminaries

Let G be a finite Abelian group with dual group $\widehat{G}$ consisting of the group homomorphisms $\xi : G \mapsto S^1$. The space of functions $\{f : G \longrightarrow \mathbb{C}\}$ will be denoted by $\mathbb{C}^G$, and the *support size* of a function is $\|f\|_0 := |\{x : f(x) \neq 0\}|$. In contrast to the ℓ^p-norms $\|\cdot\|_p$ for $p \geq 1$, $\|\cdot\|_0$ is not a norm. We normalize such that the *Fourier transform* is given $\widehat{f}(\xi) = \sum_{x\in G} f(x)\,\overline{\xi(x)}$ for $f \in \mathbb{C}^G, \xi \in \widehat{G}$.

The *translation operator* $T_x, x \in G$ is the unitary operator on $\mathbb{C}^G$ given by $(T_x f)(t) = f(t-x)$. The *modulation operator* $M_\xi, \xi \in \hat{G}$ is the unitary operator on $\mathbb{C}^G$ defined by $(M_\xi f)(t) = f(t) \cdot \xi(t)$ (pointwise product). A well known fact is that $\widehat{M_\xi f} = T_\xi \hat{f}$. For $\lambda = (x, \xi) \in G \times \hat{G}$, the *time-frequency shift* $\pi(\lambda)$ is the unitary operator on $\mathbb{C}^G$ given by $\pi(\lambda) f = T_x \circ M_\xi f$. The collection of functions $\{\pi(\lambda) g : \lambda \in G \times \hat{G}\}, g \in \mathbb{C}^G$ is called a *Gabor system* with window function g.

Let $g \in \mathbb{C}^G \backslash \{0\}$ be a window function. The *short-time Fourier transform* with respect to g is given by

$$V_g f(x,\xi) = \langle f, M_\xi T_x g\rangle = \sum_{y\in G} f(y)\overline{g(y-x)\xi(y)}, \quad f \in \mathbb{C}^G, (x,\xi) \in G \times \widehat{G}.$$

The linear mapping $V_g : \mathbb{C}^G \longrightarrow \mathbb{C}^{G\times\widehat{G}}$ has a matrix representation that will be denoted by $A_{G,g}$.

A finite set of vectors in $\mathbb{C}^G$ is *in general linear position* if the elements in any collection of at most $|G|$ of these vectors are linearly independent.

3 The main question

Results by Tao [11] and Frenkel [5] relate the support sizes of a function and its Fourier transform for groups of prime order as follows:

Theorem 1. *Let $G = \mathbb{Z}_p$ with p prime. Then $\|f\|_0 + \|\widehat{f}\|_0 \geq |G| + 1$ holds for all $f \in \mathbb{C}^G \backslash \{0\}$.*

Our corresponding results for the short-time Fourier transform[7,**?**]) is

Theorem 2. *Let $G = \mathbb{Z}_p$, p prime. For almost every $g \in \mathbb{C}^G$,*

$$\|f\|_0 + \|V_g f\|_0 \geq |G|^2 + 1 \tag{1}$$

for all $f \in \mathbb{C}^G \setminus \{0\}$.

Both results are sharp in the sense that all pairs satisfying the respective bound will correspond to the support sizes of a function and its transform. In particular, for all $1 \leq k \leq |G|$, $1 \leq l \leq |G|^2$ with $k + l \geq |G|^2 + 1$ and an appropriate window g there exists f with $\|f\|_0 = k$ and $\|V_g f\|_0 = l$. A comparison of both results shows that for $a, b \in \mathbb{Z}_p$, the pair of numbers $(a, p^2 - b)$ can be realized as $(\|f\|_0, \|V_g f\|_0)$ if and only if $(a, p - b)$ can be realized as $(\|f\|_0, \|\widehat{f}\|_0)$

Theorem 1 was generalized by Meshulam [9] to groups of arbitrary order.

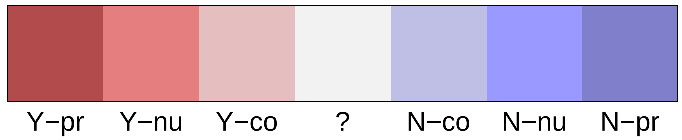

Fig. 1. Color coding used in Figures 2–4 to describe membership to subsets of $\mathbb{N}^2$. Y-pr indicates a proof is known that the corresponding value is in the set considered. Y-nu implies that there is numerical evidence that the value is in the set and Y-co indicates that we conjecture that the value is in the set. N-pr indicates a proof is known that the corresponding value is not in the set, and N-nu and N-co are defined accordingly. The color adjacent to ? implies that no judgement is made here.

Meshulam's theorem is not sharp in the sense discussed above: not every combination of numbers satisfying the bound are feasible in the sense that can be realized as a pair $(\|f\|_0, \|\widehat{f}\|_0)$. In Figure 2, we depict which combinations are feasible for the three groups of order 8. In particular, one sees that the picture differs for different groups of the same order.

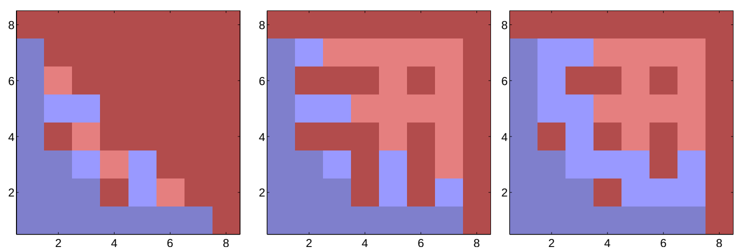

Fig. 2. The set $\{(\|f\|_0, \|\widehat{f}\|_0),\ f \in \mathbb{C}^G \backslash \{0\}\}$ for all abelian groups of order 8. The groups (from left to right) are $\mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2^3$. The color code is justified by the results in [7].

A corresponding generalization of Theorem 2 is not known.

The few numerical results that we have at hand (depicted in Figure 3) suggest that for cyclic groups, a correspondence similar to the prime order case holds true. We formulate the question:

Question 1. Is it true that for any cyclic group $\mathbb{Z}_n$ and $a, b \in \mathbb{Z}_n$, the pair of numbers $(a, n^2 - b)$ can be realized as $(\|f\|_0, \|V_g f\|_0)$ for some $f \in \mathbb{C}^{\mathbb{Z}_n}$ if and only if $(a, n - b)$ can be realized as $(\|f\|_0, \|\widehat{f}\|_0)$?

For non-cyclic groups, the answer is negative, for example $\mathbb{Z}_2^2$ as shown by Figure 4. An affirmative answer to this question would imply that for all $f \neq 0$ (and

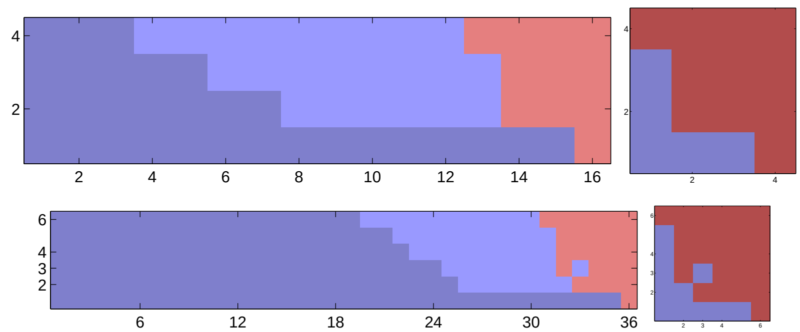

Fig. 3. The set $\{(\|f\|_0, \|V_g f\|_0),\ f \in \mathbb{C}^G \backslash \{0\}\}$ for appropriately chosen $g \in \mathbb{C}^G \backslash \{0\}$ for $G = \mathbb{Z}_4,\ \mathbb{Z}_6$. For comparison, the right column shows the set $\{(\|f\|_0, \|\widehat{f}\|_0),\ f \in \mathbb{C}^G \backslash \{0\}\}$.

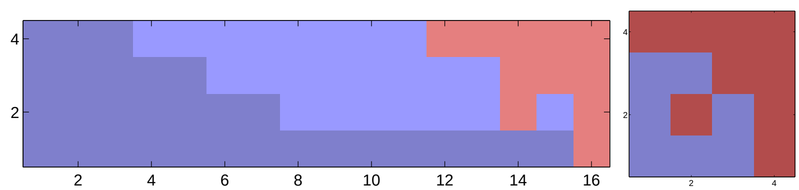

Fig. 4. The set $\{(\|f\|_0, \|V_g f\|_0),\ f \in \mathbb{C}^G \backslash \{0\}\}$ for appropriately chosen $g \in \mathbb{C}^G \backslash \{0\}$ for $G = \mathbb{Z}_2^2$.

hence $\|f\|_0 \geq 1$), $\|V_g f\|_0 \geq n^2 - n + 1$. This fact is particularly interesting, as it is equivalent to the existence of a window function $g \in \mathbb{C}^p$ such that the elements of the Gabor system $\{\pi(\lambda)g : \lambda \in \mathbb{Z}_p \times \hat{\mathbb{Z}}_p\}$ are in general linear position. Even for this special case, no more than numerical results for a few low orders are known. Understanding this case would be a big step towards answering Question 1 as well as important in its own right (see the section on applications). Hence, we pose as a second question:

Question 2. Is it true that for any cyclic group $\mathbb{Z}_n$ there exists a window function $g \in \mathbb{C}^{\mathbb{Z}_n}$ such that the elements of the Gabor system $\{\pi(\lambda)g : \lambda \in \mathbb{Z}_n \times \hat{\mathbb{Z}}_n\}$ are in general linear position?

Again, Figure 4 shows that no affirmative answer can be expected for non-cyclic groups.

4 Applications

Area of application of Theorem 2 and potential results implied by affirmative answers to the above questions include erasure channels, operator identification and recovery of signals with sparse representations [7].

In generic communication systems, information (a vector $f \in \mathbb{C}^G$) is not sent directly, but must be coded in such a way that allows recovery of f at the receiver regardless of errors and disturbances introduced by the channel. We can choose a frame $\{\varphi_k\}_{k\in K}$ for $\mathbb{C}^G$ and send the coded coefficients $\{\langle f, \varphi_k\rangle\}_{k\in K}$ (see for example [2] for the definition and properties of frames in finite-dimensional vector spaces and [6] for the definition of Gabor systems and frames in particular). If none of the transmitted coefficients are lost, a dual frame $\{\widetilde{\varphi}_k\}$ of $\{\varphi_k\}$ can be used by the receiver to recover f via the inversion formula $f = \sum_k \langle f, \varphi_k\rangle \widetilde{\varphi}_k$.

In the case of an erasure channel, some coefficients are lost during the transmission, but it is known which ones are lost. Suppose that only the coefficients $\{\langle f, \varphi_k\rangle\}_{k\in K'}, K' \subset K$ are received. The original vector f can still be recovered if and only if the subset $\{\varphi_k\}_{k\in K'}$ remains a frame for $\mathbb{C}^G$. Of course this requires $|K'| \geq |G| = \dim \mathbb{C}^G$.

Definition 1. *A frame $\mathcal{F} = \{\varphi_k\}_{k\in K}$ in $\mathbb{C}^G$ is* maximally robust to erasures *if the removal of any $l \leq |K| - |G|$ vectors from $\mathcal{F}$ leaves a frame.*

By definition, a frame is maximally robust to erasures, if and only if the frame vectors are in general linear position. Hence, Question 2 asks whether Gabor frames have this property, and Theorem 2 states that for prime order they do.

Another important application is the problem of identification of linear time-varying operators. We recall the definition of operator identification:

Definition 2. *A linear space of operators $\mathcal{H} \subseteq \{H : \mathbb{C}^A \to \mathbb{C}^B,\ H \text{ linear }\}$ is identifiable with identifier g if the linear map $\phi_g : \mathcal{H} \to \mathbb{C}^B, H \mapsto Hg$ is injective.*

A time-variant communication channel is often modeled by a linear combination of time- and frequency-shifts. Hence, identification of operators from the class $\mathcal{H}_\Lambda = \{\sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda), c_\lambda \in \mathbb{C}\}, \Lambda \in G \times \hat{G}\}$ corresponds to identifying the nature of the communication channel, which is a crucial prerequisite of successful transmission of information.

A third application area is the theory of sparse representation, in particular the problem of recovering a signal which is a linear combination of a small number of frequencies from very few of its sampled values (compare [1]). In sparse representation problems, one considers dictionaries $\mathcal{D} = \{g_0, g_1, \ldots, g_{N-1}\}$ of N vectors in $\mathbb{C}^n$ and examines, for $k \leq n$, the sets

$$\Sigma_k^{\mathcal{D}} = \{f \in \mathbb{C}^n :\ f = \sum_r c_r g_r,\ \text{for all sequences } \mathbf{c} : \|\mathbf{c}\|_0 \leq k\}\,.$$

In other words $\Sigma_k^{\mathcal{D}}$ is the set of vectors (signals) in $\mathbb{C}^n$ that have k-sparse representations in the dictionary $\mathcal{D}$. A classical dictionary for $\mathbb{C}^G$ is the set of frequencies $\mathcal{D}_{\mathcal{G}} = \{\xi : \xi \in \widehat{\mathcal{G}}\}$. In this case $\Sigma_k^{\mathcal{D}} = \{\hat{f} : f \in \mathbb{C}^G : \|f\|_0 \leq k\}$. The question how many elements are enough to identify $v \in \Sigma_k^{\mathcal{D}}$ is answered by the results by Tao [11] and Meshulam [9] discussed above.

For the dictionary $\mathcal{D}_{G,g}$ which consists of the columns of $A_{G,g}$ one observes that $F \in \Sigma_k^{\mathcal{D}_{G,g}}$ if and only if $F = V_g f$ for some $f \in \mathbb{C}^G$ with $\|f\|_0 \leq k$. So here, the corresponding problem amounts to finding out how many values of $V_g f$ need to be known (or stored), in order for $V_g f$, and therefore f, to be uniquely determined by the known data. Again Theorem 2 solves the problem for groups of prime order, and so would an affirmative answer to Question 1 for an arbitrary cyclic group.

The following result [7,**?**] summarizes the answers to the questions above:

Theorem 3. *For $g \in \mathbb{C}^G \backslash \{0\}$, the following are equivalent:*

1. *For all $f \in \mathbb{C}^G \setminus \{0\}$, $\|V_g f\|_0 \geq |G|^2 - |G| + 1$.*
2. *Every minor of $A_{G,g}$ of order $|G|$ is nonzero.*
3. *The vectors from the Gabor system $\{\pi(\lambda)g : \lambda \in G \times \hat{G}\}$ are in general position.*
4. *The Gabor system, consisting of the columns of the matrix $A_{G,g}$, is an equal norm tight frame which is maximally robust to erasures.*
5. *For all $f \in \mathbb{C}^G$, $V_g f(\lambda)$ and, therefore, f is completely determined by its values on any set Λ with $|\Lambda| = |G|$.*
6. *$\mathcal{H}_\Lambda$ is identifiable by g if and only if $|\Lambda| \leq |G|$.*

Determining whether Statement 2 holds, amounts to answering Question 2. Hence, for $|G|$ prime, Theorem 2 guarantees the validity of the six statements for a generic g. In addition, we show in [7], that they hold true for some unimodular window g as well. The numerical tests depicted in Figure 3 show that the statements are true for $G = \mathbb{Z}_4, \mathbb{Z}_6$, and the numerical tests depicted in Figure 4 show that for $G = \mathbb{Z}_2^2$ it is wrong (see [7] for a counterexample).

References

1. E. J. Candes, J. Romberg, T. Tao. Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information. *IEEE Trans. Inf. Theory* **52** (2006), 489–509.
2. O. Christensen. An Introduction to Frames and Riesz Bases. Birkhäuser, 2003.
3. D. Donoho. Compressed sensing. *IEEE Trans. Inf. Theory* **52** (2006), 1289–1306.
4. D. Donoho, P. Stark. Uncertainty principles and signal recovery, *SIAM J. Appl. Math.* **49** (1989), 906–931.
5. P. E. Frenkel. Simple proof of Chebotarev's theorem on toots of unity. preprint, math.AC/0312389 (2004).
6. K. Gröchenig. Foundations of time-frequency analysis. Birkhäuser, 2001.
7. F. Krahmer, G.E. Pfander, P. Rashkov. Uncertainty in time-frequency representations on finite abelian groups and applications. *Appl. Comp. Harm. Anal.* **25** (2007), 209–225.
8. J. Lawrence, G.E. Pfander, D. Walnut. Linear independence of Gabor systems in finite dimensional vector spaces. *J. Four. Anal. Appl.* **11** (2005), 715–726.

9. R. Meshulam. An uncertainty inequality for finite abelian groups. *Eur. J. Comb.* **27** (2006), 227–254.
10. H. Rauhut. Random sampling of sparse trigonometric polynomials. *Appl. Comp. Harm. Anal.* **22** (2007), 16–42.
11. T. Tao. An uncertainty principle for cyclic groups of prime order. *Math. Res. Lett.* **12** (2005), 121–127.