# $\mathbf{\Sigma^0_\alpha}$-Admissible Representations (Extended Abstract)

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**Abstract.** We investigate a hierarchy of representations of topological spaces by measurable functions that extends the traditional notion of admissible representations common to computable analysis. Specific instances of these representations already occur in the literature (for example, the naive Cauchy representation of the reals and the "jump" of a representation), and have been used in investigating the computational properties of discontinuous functions. Our main contribution is the integration of a recently developing descriptive set theory for non-metrizable spaces that allows many previous results to generalize to arbitrary countably based $T_0$ topological spaces. In addition, for a class of topological spaces that include the reals (with the Euclidean topology) and the power set of $\omega$ (with the Scott-topology), we give a complete characterization of the functions that are (topologically) realizable with respect to the level of the representations of the domain and codomain spaces.

## 1 Introduction

In this paper, we introduce and investigate the topological properties of a hierarchy of representations of topological spaces, which we call $\mathbf{\Sigma}^0_\alpha$-admissible representations. A partial function $\rho\colon\subseteq \omega^\omega \to X$ is called a $\mathbf{\Sigma}^0_\alpha$-admissible representation ($1 \leq \alpha < \omega_1$) of the topological space $X$ if and only if $\rho$ is $\mathbf{\Sigma}^0_\alpha$-measurable and every $\mathbf{\Sigma}^0_\alpha$-measurable partial function to $X$ is continuously reducible to $\rho$ (see Definition 4). As $\mathbf{\Sigma}^0_1$-measurable functions are exactly the continuous functions, a $\mathbf{\Sigma}^0_1$-admissible representation is the same as the traditional notion of an "admissible" representation common to computable analysis (see [12] and [9]). A well known example of a $\mathbf{\Sigma}^0_2$-admissible representation is the naive Cauchy representation of the reals [4], and examples of representations in the finite levels of the hierarchy can be obtained iteratively by taking the "jump" of a representation [14]. These representations have been used in investigating the computational properties of discontinuous functions (see [3], [14], and [5]).

Whereas previous results have focused on metrizable spaces and finite levels of the hierarchy, in this paper we will investigate these representations for arbitrary countably based $T_0$ spaces and all countable levels of the hierarchy. Perhaps one reason that previous research has been restricted to metrizable spaces

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is that the classical definition of the Borel hierarchy behaves rather poorly on non-metrizable spaces. Since the domain of a $\mathbf{\Sigma}^0_\alpha$-admissible representation is a metrizable space, we can use the classical definition of the Borel hierarchy in defining these representations, even for arbitrary topological spaces. However, to better understand their properties, a slight modification of the definition of the Borel hierarchy is needed for non-metrizable spaces. It turns out that the correct definition is the one that has only recently been used by Tang [11] in studying $\mathcal{P}(\omega)$ and more extensively studied by Selivanov (see [10] for a survey). Using this modification, it can be shown that the Borel complexity of a subset of a countably based $T_0$ space is exactly determined by the complexity of the preimage of the set under a $\mathbf{\Sigma}^0_1$-admissible representation (see Corollary 3 below). Similar properties hold for higher levels of the hierarchy, and this regularity allows us to easily characterize the types of functions that are topologically realizable with respect to these representations. In particular, we give a complete characterization for a class of topological spaces that include the reals and $\mathcal{P}(\omega)$ (see Theorem 9), and have also extended some important realizability results by Brattka [3] and Ziegler [14] to all countably based $T_0$-spaces (see Theorem 8).

A final result worth mentioning is that, given a representation $\rho\colon\subseteq \omega^\omega \to X$ of a set $X$, if there is a sequential topology $\tau$ on $X$ that makes $\rho$ a $\mathbf{\Sigma}^0_\alpha$-admissible representation, then both $\tau$ and $\alpha$ are uniquely determined (see Corollary 4). Thus, $\mathbf{\Sigma}^0_\alpha$-admissible representations provide a useful means of characterizing representations that cannot be interpretted as being admissible in the usual (continuous) sense.

We will define the Borel hierarchy for arbitrary topological spaces and review its basic properties in the next section. In Section 3 we will investigate some basic properties of $\mathbf{\Sigma}^0_\alpha$-measurable functions between topological spaces. We prove that $\mathbf{\Sigma}^0_\alpha$-admissible representations exist for all countable ordinals $\alpha$ and all countably based $T_0$ spaces in Section 4, and further investigate their properties in Section 5. Section 6 investigates which functions between topological spaces are realizable with respect to $\mathbf{\Sigma}^0_\alpha$-admissible representations, and we conclude in Section 7. Several proofs have been omitted due to a lack of space. They can be obtained by contacting the first author.

## 2 The Borel Hierarchy

In this section we define the Borel hierarchy on arbitrary topological spaces and introduce some basic properties. We will use a definition of the Borel hierarchy that differs from the classical definition (e.g., the definition in [7]) on non-metrizable spaces, but is more suitable for general topological spaces.

We let $\omega_1$ denote the least uncountable ordinal, $\omega$ the set of natural numbers (or the first infinite ordinal), and for sets $A$ and $B$ we let $A \setminus B$ denote the subset of $A$ of elements not in $B$.

**Definition 1.** *Let $X$ be a topological space. For each ordinal $\alpha$ ($1 \leq \alpha < \omega_1$) we define $\mathbf{\Sigma}^0_\alpha(X)$ inductively as follows.*

1. *$\mathbf{\Sigma}^0_1(X)$ is the set of all open subsets of $X$.*
2. *For $\alpha > 1$, $\mathbf{\Sigma}^0_\alpha(X)$ is the set of all subsets $A$ of $X$ which can be expressed in the form*
$$A = \bigcup_{i\in\omega} B_i \setminus B'_i,$$
*where for each $i$, $B_i$ and $B'_i$ are in $\mathbf{\Sigma}^0_{\beta_i}(X)$ for some $\beta_i < \alpha$.*

*We define $\mathbf{\Pi}^0_\alpha(X) = \{X \setminus A \,|\, A \in \mathbf{\Sigma}^0_\alpha(X)\}$, $\mathbf{\Delta}^0_\alpha(X) = \mathbf{\Sigma}^0_\alpha(X) \cap \mathbf{\Pi}^0_\alpha(X)$, and $\mathbf{B}(X) = \bigcup_{1\leq\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha(X)$.* □

The above definition of the Borel hierarchy is equivalent to the definition that was used by Tang [11] in studying descriptive set theory on $\mathcal{P}(\omega)$ (the power set of the natural numbers with the Scott-topology), and more systematically investigated by Selivanov (see [10] for a survey of results and an extensive list of references).

The classical definition of the Borel hierarchy (which requires $B_i = X$ for all $i$ in the second clause of Definition 1) is not suitable for non-metrizable spaces. For example, consider the Sierpinski space $\mathcal{S} = \{\bot, \top\}$ (where $\{\top\}$ is open, but $\{\bot\}$ is not). If we used the classical definition then $\mathbf{\Sigma}^0_{2n+1}(\mathcal{S})$ is the set of open subsets of $\mathcal{S}$ and $\mathbf{\Sigma}^0_{2n+2}(\mathcal{S})$ is the closed subsets, so $\mathbf{\Sigma}^0_{2n+1}(\mathcal{S}) \not\subseteq \mathbf{\Sigma}^0_{2n+2}(\mathcal{S})$ (for $0 \leq n < \omega$). The Borel hierarchy defined in Definition 1 is equivalent to the classical definition for all metrizable spaces, and behaves as we expect it should even for non-metrizable spaces.

In the following, $X$ and $Y$ will denote arbitrary topological spaces, unless stated otherwise. The following results are easily proven, and can also be found in [10].

**Proposition 1.** *For each $\alpha$ ($1 \leq \alpha < \omega_1$),*

1. *$\mathbf{\Sigma}^0_\alpha(X)$ is closed under countable unions and finite intersections,*
2. *$\mathbf{\Pi}^0_\alpha(X)$ is closed under countable intersections and finite unions,*
3. *$\mathbf{\Delta}^0_\alpha(X)$ is closed under finite unions, finite intersections, and complementation.*

□

**Proposition 2.** *If $\beta < \alpha$ then $\mathbf{\Sigma}^0_\beta(X) \cup \mathbf{\Pi}^0_\beta(X) \subseteq \mathbf{\Delta}^0_\alpha(X)$.* □

**Proposition 3.** *For $\alpha > 2$, each $A \in \mathbf{\Sigma}^0_\alpha(X)$ can be expressed in the form $A = \bigcup_{i\in\omega} B_i$, where for each $i$, $B_i$ is in $\mathbf{\Pi}^0_{\beta_i}(X)$ for some $\beta_i < \alpha$.* □

**Proposition 4.** *If $X$ is a metrizable space, then every $A \in \mathbf{\Sigma}^0_2(X)$ is equal to a countable union of closed sets.* □

**Proposition 5.** *If $X$ is a subspace of $Y$, then $\mathbf{\Sigma}^0_\alpha(X) = \{A \cap X \,|\, A \in \mathbf{\Sigma}^0_\alpha(Y)\}$ and $\mathbf{\Pi}^0_\alpha(X) = \{A \cap X \,|\, A \in \mathbf{\Pi}^0_\alpha(Y)\}$.* □

A topological space $X$ is called a *$T_D$-space* if every singleton set $\{x\} \subseteq X$ is locally closed, i.e. $\{x\}$ is equal to the intersection of an open set and a closed set. $T_D$ is a separation axiom proposed by Aull and Thron [2] that is strictly between the $T_0$ and $T_1$ axioms.

**Proposition 6.** *For any first-countable topological space $X$,*

1. *Every singleton set $\{x\} \subseteq X$ is in $\mathbf{\Pi}^0_2(X) \iff X$ is a $T_0$-space,*
2. *Every singleton set $\{x\} \subseteq X$ is in $\mathbf{\Delta}^0_2(X) \iff X$ is a $T_D$-space,*
3. *Every singleton set $\{x\} \subseteq X$ is in $\mathbf{\Pi}^0_1(X) \iff X$ is a $T_1$-space,*
4. *Every singleton set $\{x\} \subseteq X$ is in $\mathbf{\Delta}^0_1(X) \iff X$ is a discrete space.*

□

## 3 $\mathbf{\Sigma}^0_\alpha$-measurable functions

In this section we will investigate some basic properties of $\mathbf{\Sigma}^0_\alpha$-measurable functions. Below, we will write $f\colon\subseteq X \to Y$ to indicate that $f$ is a partial function from $X$ to $Y$. The domain of definition of $f$ will be denoted $dom(f)$. We say that $f\colon\subseteq X \to Y$ is *continuous* if and only if for every open $U \subseteq Y$, there is open $V \subseteq X$ such that $f^{-1}(U) = V \cap dom(f)$. In other words, $f\colon\subseteq X \to Y$ is continuous if and only if the total function $f\colon dom(f) \to Y$ is continuous with respect to the subspace topology on $dom(f)$.

**Definition 2.** *A function $f\colon X \to Y$ is* $\mathbf{\Sigma}^0_\alpha$-measurable *if and only if for every open $U \subseteq Y$, $f^{-1}(U) \in \mathbf{\Sigma}^0_\alpha(X)$. A partial function $f\colon\subseteq X \to Y$ is said to be* $\mathbf{\Sigma}^0_\alpha$-measurable *if and only if for every open $U \subseteq Y$, there is $A \in \mathbf{\Sigma}^0_\alpha(X)$ such that $f^{-1}(U) = A \cap dom(f)$.* □

Equivalently, a partial function $f\colon\subseteq X \to Y$ is $\mathbf{\Sigma}^0_\alpha$-measurable if and only if for every open $U \subseteq Y$, $f^{-1}(U) \in \mathbf{\Sigma}^0_\alpha(dom(f))$, where $dom(f)$ is given the relative topology.

For any fixed $\alpha > 1$, the $\mathbf{\Sigma}^0_\alpha$-measurable functions are not closed under composition. To characterize how composition behaves, we will need ordinal addition. Addition on ordinals is defined recursively as follows:

1. $\alpha + 0 = \alpha$
2. $\alpha + (\beta + 1) = (\alpha + \beta) + 1 =$ the successor of $\alpha + \beta$.
3. $\alpha + \lambda = \lim_{\beta<\lambda}(\alpha + \beta)$ for limit ordinal $\lambda$.

Note that ordinal addition is non-commutative. For example, $1+\omega = \omega \neq \omega+1$. Also note that if $\alpha < \beta$, then there is a unique ordinal $\gamma$ such that $\alpha + \gamma = \beta$.

Composing with continuous functions does not change the level of a function. For that reason it would have been more convenient for our purposes to define the Borel Hierarchy so that open sets and continuous functions were of level 0 (the additive identity for ordinals). To simplify the statement of some of the following theorems and proofs, we will often make use of the following "hat" notation, so that we can treat the Borel Hierarchy as if we defined the open sets to be at level 0.

**Definition 3.** *For $0 \leq \alpha < \omega_1$, define $\widehat{\alpha} = \alpha + 1$ if $\alpha < \omega$ and $\widehat{\alpha} = \alpha$ if $\alpha \geq \omega$.*

□

Note that $\alpha < \beta \iff \widehat{\alpha} < \widehat{\beta}$ and $\widehat{\alpha+\beta} = \widehat{\alpha} + \beta$ hold for any countable ordinals $\alpha$ and $\beta$.

**Lemma 1.** *Let $X$ and $Y$ be countably based $T_0$ spaces. If $f\colon\subseteq X \to Y$ is $\mathbf{\Sigma}^0_{\widehat{\alpha}}$-measurable ($0 \leq \alpha < \omega_1$) and $A \in \mathbf{\Sigma}^0_{\widehat{\beta}}(Y)$ ($0 \leq \beta < \omega_1$), then $f^{-1}(A) \in \mathbf{\Sigma}^0_{\widehat{\alpha+\beta}}(dom(f))$.* □

**Theorem 1.** *Let $X$, $Y$, and $Z$ be countably based $T_0$ spaces, $f\colon\subseteq X \to Y$ a $\mathbf{\Sigma}^0_{\widehat{\alpha}}$-measurable function ($0 \leq \alpha < \omega_1$), and $g\colon\subseteq Y \to Z$ a $\mathbf{\Sigma}^0_{\widehat{\beta}}$-measurable function ($0 \leq \beta < \omega_1$). Then $g \circ f\colon\subseteq X \to Z$ is $\mathbf{\Sigma}^0_{\widehat{\alpha+\beta}}$-measurable.* □

In particular, if $f$ is $\mathbf{\Sigma}^0_2$-measurable and $g$ is $\mathbf{\Sigma}^0_\omega$-measurable, then due to the non-commutativity of ordinal addition, $g \circ f$ is $\mathbf{\Sigma}^0_\omega$-measurable but $f \circ g$ is $\mathbf{\Sigma}^0_{\omega+1}$-measurable (assuming the compositions make sense).

The following is due to Wadge (this is Theorem 22.10 in [7]). We let $\omega^\omega$ denote the Baire space.

**Proposition 7 (Wadge).** *If $B \subseteq \omega^\omega$ is in $\mathbf{B}(\omega^\omega) \setminus \mathbf{\Pi}^0_{\widehat{\alpha}}(\omega^\omega)$ ($0 \leq \alpha < \omega_1$), then for any $A \in \mathbf{\Sigma}^0_{\widehat{\alpha}}(\omega^\omega)$ there is continuous total $f\colon\omega^\omega \to \omega^\omega$ such that $A = f^{-1}(B)$.* □

We will need the following generalization of Wadge's results that characterize reductions using measurable functions.

**Theorem 2.** *For $0 \leq \alpha < \omega_1$ and $0 \leq \beta < \omega_1$, if $B \in \mathbf{B}(\omega^\omega) \setminus \mathbf{\Pi}^0_{\widehat{\beta}}(\omega^\omega)$, then for any $A \in \mathbf{\Sigma}^0_{\widehat{\alpha+\beta}}(\omega^\omega)$ there exists a $\mathbf{\Sigma}^0_{\widehat{\alpha}}$-measurable total function $f\colon\omega^\omega \to \omega^\omega$ such that $A = f^{-1}(B)$.* □

## 4 Existence of $\mathbf{\Sigma}^0_{\boldsymbol{\alpha}}$-admissible representations

The goal of this section is to show that every countably based $T_0$ space has a $\mathbf{\Sigma}^0_\alpha$-admissible representation for $1 \leq \alpha < \omega_1$ (Theorem 3 below). We also show the complexity of converting between representations of different levels (Theorem 4), and consider representations of representations of a space (Corollary 2), which is a generalization of Ziegler's "jump" of a representation [14].

**Definition 4.** *A* $\mathbf{\Sigma}^0_\alpha$-admissible representation *of a topological space $X$ is a $\mathbf{\Sigma}^0_\alpha$-measurable partial function $\rho\colon\subseteq \omega^\omega \to X$ such that for every $\mathbf{\Sigma}^0_\alpha$-measurable partial function $f\colon\subseteq \omega^\omega \to X$, there exists continuous $g\colon\subseteq \omega^\omega \to \omega^\omega$ such that $f = \rho \circ g$.* □

Note that the above definition implies that $\mathbf{\Sigma}^0_\alpha$-admissible representations are always surjective. Clearly, a $\mathbf{\Sigma}^0_1$-admissible representation is equivalent to what is usually called an "admissible representation" in the computable analysis literature (see, e.g., [12] and [9]). The above definition applies to arbitrary topological spaces, but most of our results will focus on countably based spaces.

We let $\mathcal{S}$ denote the Sierpinski space, which has only two points $\top$ and $\bot$, and where $\{\top\}$ is open but $\{\bot\}$ is not open.

**Proposition 8.** *Let $A \in \mathbf{\Sigma}^0_\alpha(\omega^\omega) \setminus \mathbf{\Pi}^0_\alpha(\omega^\omega)$ and define $\rho\colon \omega^\omega \to \mathcal{S}$ so that $\rho(y) = \top$ if $y \in A$ and $\rho(y) = \bot$ if $y \notin A$. Then $\rho$ is a $\mathbf{\Sigma}^0_\alpha$-admissible representation for $\mathcal{S}$.*

*Proof.* It is clear that $\rho$ is $\mathbf{\Sigma}^0_\alpha$-measurable. Let $f\colon\subseteq \omega^\omega \to \mathcal{S}$ be a $\mathbf{\Sigma}^0_\alpha$-measurable partial function. Then $f^{-1}(\{\top\}) \in \mathbf{\Sigma}^0_\alpha(dom(f))$, so there is $B \in \mathbf{\Sigma}^0_\alpha(\omega^\omega)$ such that $f^{-1}(\{\top\}) = B \cap dom(f)$. From Proposition 7 there is continuous $g\colon \omega^\omega \to \omega^\omega$ such that $g^{-1}(A) = B$. Then for all $y \in dom(f)$, $f(y) = \top \iff g(y) \in A \iff \rho(g(y)) = \top$. Hence, by restricting the domain of $g$ if necessary, $f = \rho \circ g$. □

**Corollary 1.** *For $0 \leq \alpha < \omega_1$ and $0 \leq \beta < \omega_1$, if $\rho_{\alpha+\beta}\colon\subseteq \omega^\omega \to \mathcal{S}$ is a $\mathbf{\Sigma}^0_{\widehat{\alpha+\beta}}$-admissible representation of $\mathcal{S}$ and $\rho_\beta\colon\subseteq \omega^\omega \to \mathcal{S}$ is a $\mathbf{\Sigma}^0_{\hat{\beta}}$-admissible representation of $\mathcal{S}$, then there exists a $\mathbf{\Sigma}^0_{\hat{\alpha}}$-measurable function $f\colon\subseteq \omega^\omega \to \omega^\omega$ such that $\rho_{\alpha+\beta} = \rho_\beta \circ f$.*

*Proof.* Immediate from Theorem 2 and Proposition 8. □

**Proposition 9.** *If $X$ is a subspace of $Y$ and $\rho\colon\subseteq \omega^\omega \to Y$ is a $\mathbf{\Sigma}^0_\alpha$-admissible representation of $Y$, then $\rho_X\colon\subseteq \omega^\omega \to X$ defined as the restriction of $\rho$ to $dom(\rho_X) = \rho^{-1}(X)$, is a $\mathbf{\Sigma}^0_\alpha$-admissible representation of $X$.* □

**Proposition 10.** *If $\{X_i\}_{i\in\omega}$ and $\{Y_i\}_{i\in\omega}$ are all countably based $T_0$-spaces, and for each $i$ $f_i\colon\subseteq X_i \to Y_i$ is $\mathbf{\Sigma}^0_\alpha$-measurable ($1 \leq \alpha < \omega_1$), then $f^\omega\colon\subseteq \prod X_i \to \prod Y_i$ is $\mathbf{\Sigma}^0_\alpha$-measurable, where $\prod X_i$ and $\prod Y_i$ are given the product topologies and $f^\omega$ is defined so that $f^\omega(\xi)(i) = f_i(\xi(i))$.* □

For the following proposition, let $\phi\colon \omega^\omega \to (\omega^\omega)^\omega$ be a homeomorphism.

**Proposition 11.** *Let $X_i$ be a countably based $T_0$ space and $\rho_i\colon\subseteq \omega^\omega \to X_i$ a $\mathbf{\Sigma}^0_\alpha$-admissible representation for $X_i$ ($i \in \omega$). Then $\rho^\omega \circ \phi$ is a $\mathbf{\Sigma}^0_\alpha$-admissible representation for $\prod X_i$.*

*Proof.* The proof that $\rho^\omega \circ \phi$ is $\mathbf{\Sigma}^0_\alpha$-measurable follows from Proposition 10.

Let $f\colon\subseteq \omega^\omega \to \prod X_i$ be a $\mathbf{\Sigma}^0_\alpha$-measurable partial function. By the $\mathbf{\Sigma}^0_\alpha$-admissibility of $\rho_i\colon\subseteq \omega^\omega \to X_i$, for $i \in \omega$ there is continuous $g_i\colon\subseteq \omega^\omega \to \omega^\omega$ such that $\pi_i \circ f = \rho_i \circ g_i$, where $\pi_i\colon \prod X_i \to X$ is the $i$-th projection. Since $\pi_i$ is a total function, we must have that $dom(f) = dom(\pi_i \circ f) \subseteq dom(g_i)$ for all $i \in \omega$. Define $g\colon\subseteq \omega^\omega \to (\omega^\omega)^\omega$ so that $g(\xi)(i) = g_i(\xi)$. Then $dom(f) \subseteq dom(g)$ and

$$\rho^\omega(g(\xi))(i) = \rho_i(g(\xi)(i)) = \rho_i(g_i(\xi)) = \pi_i(f(\xi)) = f(\xi)(i),$$

so $f = \rho^\omega \circ g$. Define $h\colon\subseteq \omega^\omega \to \omega^\omega$ as $h = \phi^{-1} \circ g$. Clearly, $h$ is continuous and $f = \rho^\omega \circ g = \rho^\omega \circ \phi \circ h$. □

Since every countably based $T_0$ space is homeomorphic to a subspace of $\mathcal{S}^\omega$, we obtain the following.

**Theorem 3.** *For every countably based $T_0$ space $X$ and every $\alpha$ ($1 \leq \alpha < \omega_1$), there exists a $\mathbf{\Sigma}^0_\alpha$-admissible representation of $X$.* □

The following can be proved for $X = \mathcal{S}^\omega$ by using representations obtained from Proposition 11 and applying Corollary 1 in parallel. Subspaces of $\mathcal{S}^\omega$ are handled by restricting the functions as necessary.

**Theorem 4 (Reductions between representations).** *Let $X$ be a countably based $T_0$-space. For $0 \le \alpha < \omega_1$ and $0 \le \beta < \omega_1$, if $\rho_{\alpha+\beta}\colon\subseteq \omega^\omega \to X$ is a $\mathbf{\Sigma}^0_{\widehat{\alpha+\beta}}$-admissible representation of $X$ and $\rho_\beta\colon\subseteq \omega^\omega \to X$ is a $\mathbf{\Sigma}^0_{\widehat{\beta}}$-admissible representation of $X$, then there exists a $\mathbf{\Sigma}^0_{\widehat{\alpha}}$-measurable function $f\colon\subseteq \omega^\omega \to \omega^\omega$ such that $\rho_{\alpha+\beta} = \rho_\beta \circ f$.* □

**Corollary 2 (Representations of representations).** *Let $X$ be a countably based $T_0$ space, $\rho_\beta\colon\subseteq \omega^\omega \to X$ a $\mathbf{\Sigma}^0_{\widehat{\beta}}$-admissible representation of $X$, and $\rho_\alpha\colon\subseteq \omega^\omega \to dom(\rho_\beta)$ a $\mathbf{\Sigma}^0_{\widehat{\alpha}}$-admissible representation of $dom(\rho_\beta)$, ($0 \le \alpha < \omega_1$, $0 \le \beta < \omega_1$). Then $\rho_\beta \circ \rho_\alpha\colon\subseteq \omega^\omega \to X$ is a $\mathbf{\Sigma}^0_{\widehat{\alpha+\beta}}$-admissible representation of $X$.*

*Proof.* First note that $\rho_\beta \circ \rho_\alpha$ is $\mathbf{\Sigma}^0_{\widehat{\alpha+\beta}}$-measurable by Theorem 1. Let $\rho\colon\subseteq \omega^\omega \to X$ be a $\mathbf{\Sigma}^0_{\widehat{\alpha+\beta}}$-admissible representation of $X$. By Theorem 4, there is a $\mathbf{\Sigma}^0_{\widehat{\alpha}}$-measurable $f\colon\subseteq \omega^\omega \to \omega^\omega$ such that $\rho = \rho_\beta \circ f$. We can assume without loss of generality that $range(f) \subseteq dom(\rho_\beta)$, and so by the $\mathbf{\Sigma}^0_{\widehat{\alpha}}$-admissibility of $\rho_\alpha$ there is a continuous $g\colon\subseteq \omega^\omega \to \omega^\omega$ such that $f = \rho_\alpha \circ g$. It follows that $g$ is a continuous reduction of $\rho$ to $\rho_\beta \circ \rho_\alpha$, thus $\rho_\beta \circ \rho_\alpha$ is $\mathbf{\Sigma}^0_{\widehat{\alpha+\beta}}$-admissible. □

Let $\iota'\colon\subseteq \omega^\omega \to \omega^\omega$ be a $\mathbf{\Sigma}^0_2$-admissible representation of $\omega^\omega$. By the above theorem, if $\rho\colon\subseteq \omega^\omega \to X$ is a $\mathbf{\Sigma}^0_\beta$-admissible representation ($1 \le \beta < \omega$) of a countably based $T_0$ space $X$, then $\rho \circ \iota'$ is a $\mathbf{\Sigma}^0_{\beta+1}$-admissible representation of $X$. This corresponds to Ziegler's "jump" of a representation [14]. However, it should be noted that if $\rho$ is $\mathbf{\Sigma}^0_\beta$-admissible for $\beta \ge \omega$, then $\rho \circ \iota'$ is still $\mathbf{\Sigma}^0_\beta$-measurable and thus *not* $\mathbf{\Sigma}^0_{\beta+1}$-admissible.

## 5 Properties of $\mathbf{\Sigma}^0_\alpha$-admissible representations

The main purpose of this section is to relate the Borel complexity of a subset of a space with the complexity of the preimage of the subset under a $\mathbf{\Sigma}^0_\alpha$-admissible representation. These results will be useful in the following section where we characterize the functions that are realizable with respect to these representations.

Many of the following results are heavily dependent on the following proposition by J. Saint Raymond (Lemma 17 in [8]). Although the original statement of the result was in terms of metrizable spaces, it is easy to verify that the arguments in the proof hold for more general spaces when we define the Borel hierarchy according to Definition 1.

**Proposition 12 (Saint-Raymond [8]).** *Let $\phi\colon X \to Y$ be an open continuous surjective total function with Polish fibers (i.e. $\phi^{-1}(y)$ is Polish for each $y \in Y$), where $X$ is a separable metric space and $Y$ is a countably based $T_0$ topological* *space. Then for every $A \subseteq Y$ and $1 \leq \alpha < \omega_1$, $A \in \mathbf{\Sigma}^0_\alpha(Y)$ if and only if $\phi^{-1}(A) \in \mathbf{\Sigma}^0_\alpha(X)$.* □

Since every countably based $T_0$ space has a $\mathbf{\Sigma}^0_1$-admissible representation that is open and has Polish fibers (see Corollary 15 and Proposition 16 in [4]), we find that the Borel hierarchy is preserved under $\mathbf{\Sigma}^0_1$-admissible representations of countably based $T_0$ spaces.

**Corollary 3.** *Let $X$ be a countably based $T_0$ space and $\rho{:}\subseteq \omega^\omega \to X$ a $\mathbf{\Sigma}^0_1$-admissible representation of $X$. Then for $1 \leq \alpha < \omega_1$, $A \in \mathbf{\Sigma}^0_\alpha(X)$ if and only if $\rho^{-1}(A) \in \mathbf{\Sigma}^0_\alpha(dom(\rho))$.* □

Our next goal is to generalize Corollary 3 to some $\mathbf{\Sigma}^0_\alpha$-admissible representations. Let $\omega^*$ have as a base set $\omega \cup \{\infty\}$ and the topology so that $U$ is open if and only if either $\infty \notin U$ or else $U$ is cofinite (i.e., for some $m < \omega$, $n \in U$ for all $n \geq m$). Note that $\omega^*$ is the one-point compactification of $\omega$ with the discrete topology, hence the notation (which should not be confused with the set of finite strings of natural numbers).

**Lemma 2.** *Let $\rho{:}\subseteq \omega^\omega \to \omega^*$ be $\mathbf{\Sigma}^0_\alpha$-admissible ($1 \leq \alpha < \omega_1$). Then $S \subseteq \omega^*$ is open if and only if $\rho^{-1}(S) \in \mathbf{\Sigma}^0_\alpha(dom(\rho))$.* □

**Definition 5.** *Let $X$ be an arbitrary topological space. A subset $A \subseteq X$ is* sequentially open *if and only if for every sequence $\{x_i\}_{i\in\omega}$ that converges to $x \in A$, there is some $m$ such that $x_n \in A$ for all $n \geq m$. $X$ is a* sequential space *if and only if all sequentially open subsets of $X$ are open.* □

Note that all countably based spaces are sequential spaces (see Theorem 1.6.14 in [6]).

**Theorem 5.** *Let $X$ be a sequential $T_0$ space and $\rho{:}\subseteq \omega^\omega \to X$ be $\mathbf{\Sigma}^0_\alpha$-admissible ($1 \leq \alpha < \omega_1$). Then $U \subseteq X$ is open if and only if $\rho^{-1}(U) \in \mathbf{\Sigma}^0_\alpha(dom(\rho))$.*

*Proof.* If $U$ is open then $\rho^{-1}(U) \in \mathbf{\Sigma}^0_\alpha(dom(\rho))$ holds because $\rho$ is $\mathbf{\Sigma}^0_\alpha$-measurable.

Assume that $\rho^{-1}(U) \in \mathbf{\Sigma}^0_\alpha(dom(\rho))$ and let $\{x_i\}_{i\in\omega}$ be a sequence converging to $x \in U$. Define $f{:}\,\omega^* \to X$ so that $f(n) = x_n$ and $f(\infty) = x$. Then $f$ is clearly continuous. If $\delta$ is a $\mathbf{\Sigma}^0_\alpha$-admissible representation of $\omega^*$, then $f \circ \delta$ is $\mathbf{\Sigma}^0_\alpha$-measurable, so by the $\mathbf{\Sigma}^0_\alpha$-admissibility of $\rho$ there is continuous $g{:}\subseteq \omega^\omega \to \omega^\omega$ such that $f \circ \delta = \rho \circ g$. Since $g$ is continuous, $\delta^{-1}(f^{-1}(U)) = g^{-1}(\rho^{-1}(U)) \in \mathbf{\Sigma}^0_\alpha(dom(\delta))$. It follows that $f^{-1}(U)$ is open by Lemma 2. Since $\infty \in f^{-1}(U)$, there is $m < \omega$ such that $n \in f^{-1}(U)$ for all $n \geq m$. Therefore, $x_n \in U$ for all $n \geq m$. Since $\{x_i\}_{i\in\omega}$ and its limit $x \in U$ were arbitrary, $U$ is sequentially open, hence open because $X$ is a sequential space. □

The rest of this section extends Theorem 5 to the entire hierarchy for a special class of topological spaces.

**Lemma 3.** *Let $\rho{:}\subseteq \omega^\omega \to \omega^\omega$ be a $\mathbf{\Sigma}^0_{\widehat{\alpha}}$-admissible representation of $\omega^\omega$ ($0 \leq \alpha < \omega_1$). For $0 \leq \beta < \omega_1$ and $A \subseteq \omega^\omega$, $A \in \mathbf{\Sigma}^0_{\widehat{\beta}}(\omega^\omega)$ if and only if $\rho^{-1}(A) \in \mathbf{\Sigma}^0_{\widehat{\alpha+\beta}}(dom(\rho))$.* □

**Lemma 4.** *Let $X$ be a zero-dimensional Polish space and $\rho\colon\subseteq \omega^\omega \to X$ a $\mathbf{\Sigma}^0_{\widehat{\alpha}}$-admissible representation of $X$ ($0 \le \alpha < \omega_1$). For $0 \le \beta < \omega_1$, $A \in \mathbf{\Sigma}^0_{\widehat{\beta}}(X)$ if and only if $\rho^{-1}(A) \in \mathbf{\Sigma}^0_{\widehat{\alpha+\beta}}(dom(\rho))$.*

*Proof.* For the non-trivial part of the lemma, we can assume that $X$ is a closed subset of $\omega^\omega$ (see Theorem 7.8 in [7]) and $\rho\colon\subseteq \omega^\omega \to X$ is the restriction of a $\mathbf{\Sigma}^0_{\widehat{\alpha}}$-admissible representation $\rho'\colon\subseteq \omega^\omega \to \omega^\omega$ of $\omega^\omega$ as in Proposition 9 (i.e., $dom(\rho) = (\rho')^{-1}(X)$, and $\rho = \rho'|_{dom(\rho)}$). It follows from these assumptions that $dom(\rho) \in \mathbf{\Pi}^0_{\widehat{\alpha}}(\omega^\omega)$ because $X$ is a closed subset of $\omega^\omega$ and $\rho'$ is $\mathbf{\Sigma}^0_{\widehat{\alpha}}$-measurable.

The case $\beta = 0$ is the statement of Theorem 5, so assume $\beta \ge 1$ and $A \subseteq X$ is such that $\rho^{-1}(A) \in \mathbf{\Sigma}^0_{\widehat{\alpha+\beta}}(dom(\rho))$. By Proposition 5 there is $B \in \mathbf{\Sigma}^0_{\widehat{\alpha+\beta}}(\omega^\omega)$ such that $\rho^{-1}(A) = B \cap dom(\rho)$. Since $\alpha < \alpha + \beta$ and $dom(\rho) \in \mathbf{\Pi}^0_{\widehat{\alpha}}(\omega^\omega)$, $\rho^{-1}(A) \in \mathbf{\Sigma}^0_{\widehat{\alpha+\beta}}(\omega^\omega)$. Since $(\rho')^{-1}(A) = \rho^{-1}(A)$, it follows from Lemma 3 that $A \in \mathbf{\Sigma}^0_{\widehat{\beta}}(\omega^\omega)$ and hence $A \in \mathbf{\Sigma}^0_{\widehat{\beta}}(X)$. □

**Definition 6.** *We will say that a space $X$ has a* Polish representation *if and only if there is a $\mathbf{\Sigma}^0_1$-admissible representation $\rho\colon\subseteq \omega^\omega \to X$ of $X$ such that $dom(\rho)$ with the subspace topology is a (zero-dimensional) Polish space.*

In particular, the real numbers with the Euclidean topology and $\mathcal{P}(\omega)$ with the Scott-topology have Polish representations (an admissible representation of the reals with closed domain of definition is given in [13], and the representation $\delta\colon \omega^\omega \to \mathcal{P}(\omega)$ defined as $\delta(\xi) = \{n - 1 \,|\, \exists j(\xi(j) = n \ne 0)\}$ is total and can be shown to be admissible).

**Theorem 6.** *Let $X$ be a countably based $T_0$ space with a Polish representation and $\rho\colon\subseteq \omega^\omega \to X$ a $\mathbf{\Sigma}^0_{\widehat{\alpha}}$-admissible representation of $X$ ($0 \le \alpha < \omega_1$). For $0 \le \beta < \omega_1$, $A \in \mathbf{\Sigma}^0_{\widehat{\beta}}(X)$ if and only if $\rho^{-1}(A) \in \mathbf{\Sigma}^0_{\widehat{\alpha+\beta}}(dom(\rho))$.*

*Proof.* For the non-trivial part of the proof, let $\delta\colon\subseteq \omega^\omega \to X$ be $\mathbf{\Sigma}^0_1$-admissible such that $dom(\delta)$ is Polish. Let $\delta'\colon\subseteq \omega^\omega \to dom(\delta)$ be a $\mathbf{\Sigma}^0_{\widehat{\alpha}}$-admissible representation of $dom(\delta)$. Since $\delta\circ\delta'$ is $\mathbf{\Sigma}^0_{\widehat{\alpha}}$-measurable, there is continuous $f\colon\subseteq \omega^\omega \to \omega^\omega$ such that $\delta \circ \delta' = \rho \circ f$.

Assume $A \subseteq X$ is such that $\rho^{-1}(A) \in \mathbf{\Sigma}^0_{\widehat{\alpha+\beta}}(dom(\rho))$. Then $(\delta')^{-1}(\delta^{-1}(A)) = f^{-1}(\rho^{-1}(A)) \in \mathbf{\Sigma}^0_{\widehat{\alpha+\beta}}(dom(\delta'))$ because $f$ is continuous (here we are using the fact that $dom(\delta') \subseteq dom(f)$). It follows from Lemma 4 that $\delta^{-1}(A) \in \mathbf{\Sigma}^0_{\widehat{\beta}}(dom(\delta))$, hence $A \in \mathbf{\Sigma}^0_{\widehat{\beta}}(X)$ from Corollary 3. □

## 6 Realizability Theorems

In this section we will investigate which functions are realizable with respect to $\mathbf{\Sigma}^0_\alpha$-admissible representations. We only consider topological realizability, and do not consider computational issues.

**Definition 7.** *Let $X$ and $Y$ be arbitrary topological spaces, and $f: X \to Y$ a function. We say that $f$ is $\langle \mathbf{\Sigma}^0_\alpha, \mathbf{\Sigma}^0_\beta \rangle$-realizable by a $\mathbf{\Sigma}^0_\gamma$-measurable function if there is a $\mathbf{\Sigma}^0_\alpha$-admissible representation $\rho_X$ of $X$ and a $\mathbf{\Sigma}^0_\beta$-admissible representation $\rho_Y$ of $Y$ and a $\mathbf{\Sigma}^0_\gamma$-measurable partial function $g: \subseteq \omega^\omega \to \omega^\omega$ such that $f \circ \rho_X = \rho_Y \circ g$. If a continuous such $g$ exists, then we say that $f$ is $\langle \mathbf{\Sigma}^0_\alpha, \mathbf{\Sigma}^0_\beta \rangle$-continuously realizable.* □

**Lemma 5.** *Let $X$ be an arbitrary topological space, and $\rho: \subseteq \omega^\omega \to X$ be a $\mathbf{\Sigma}^0_\alpha$-admissible representation of $X$ ($1 \leq \alpha < \omega_1$). Then $X$ is a $T_0$-space.*

*Proof.* Exactly like Schröder's proof for $\mathbf{\Sigma}^0_1$-admissible representations (Theorem 13 in [9]). □

**Lemma 6.** *For $1 \leq \beta < \alpha < \omega_1$, a function from the discrete two point space* **2** *to the Sierpinski space $\mathcal{S}$ is $\langle \mathbf{\Sigma}^0_\alpha, \mathbf{\Sigma}^0_\beta \rangle$-continuously realizable if and only if it is a constant function.* □

Note that the following theorem does not assume that $X$ and $Y$ are countably based.

**Theorem 7.** *Let $X$ and $Y$ be any topological spaces such that $X$ has a $\mathbf{\Sigma}^0_\alpha$-admissible representation and $Y$ has a $\mathbf{\Sigma}^0_\beta$-admissible representation, where $1 \leq \beta < \alpha < \omega_1$. Then a function from $X$ to $Y$ is $\langle \mathbf{\Sigma}^0_\alpha, \mathbf{\Sigma}^0_\beta \rangle$-continuously realizable if and only if it is a constant function.* □

Statement (3) in the following is a topological generalization of Brattka's extention [3] of the Kreitz-Weihrauch Representation Theorem [12] to all countably based $T_0$-spaces and all countable ordinals. Statements (1) and (2) are generalizations of some results by Ziegler [14].

**Theorem 8.** *Let $X$ and $Y$ be countably based $T_0$ spaces, $f: X \to Y$ a total function, and $1 \leq \alpha < \omega_1$.*

1. *$f$ is $\langle \mathbf{\Sigma}^0_1, \mathbf{\Sigma}^0_\alpha \rangle$-continuously realizable if and only if $f$ is $\mathbf{\Sigma}^0_\alpha$-measurable,*
2. *$f$ is $\langle \mathbf{\Sigma}^0_\alpha, \mathbf{\Sigma}^0_\alpha \rangle$-continuously realizable if and only if $f$ is continuous,*
3. *$f$ is $\langle \mathbf{\Sigma}^0_1, \mathbf{\Sigma}^0_1 \rangle$-realizable by a $\mathbf{\Sigma}^0_\alpha$-measurable function if and only if $f$ is $\mathbf{\Sigma}^0_\alpha$-measurable.*

*Proof.* The "if" part of (1) and (2) immediately follow from the definition of admissibility. For (3), assume $f$ is $\mathbf{\Sigma}^0_\alpha$-measurable. From statement (1) it follows that $f$ is $\langle \mathbf{\Sigma}^0_1, \mathbf{\Sigma}^0_\alpha \rangle$-continuously realizable, and by Theorem 4 there is a $\mathbf{\Sigma}^0_\alpha$-measurable reduction of any $\mathbf{\Sigma}^0_\alpha$ representation of $Y$ to a $\mathbf{\Sigma}^0_1$-admissible representation of $Y$. Composing the two produces a $\mathbf{\Sigma}^0_\alpha$-measurable function that $\langle \mathbf{\Sigma}^0_1, \mathbf{\Sigma}^0_1 \rangle$-realizes $f$.

The proof of the "only if" parts are similar for all three statements, so we only prove (1). Let $\rho_X$ be a $\mathbf{\Sigma}^0_1$-admissible representation of $X$, $\rho_Y$ a $\mathbf{\Sigma}^0_\alpha$-admissible representation of $Y$, and assume $g: \subseteq \omega^\omega \to \omega^\omega$ is continuous such that $f \circ \rho_X = \rho_Y \circ g$. Let $U \subseteq Y$ be open. Then $\rho_X^{-1}(f^{-1}(U)) = g^{-1}(\rho_Y^{-1}(U)) \in \mathbf{\Sigma}^0_\alpha(dom(\rho_X))$ because $\rho_Y$ is $\mathbf{\Sigma}^0_\alpha$-measurable, $g$ is continuous, and $dom(\rho_X) \subseteq dom(g)$. By Corollary 3, it follows that $f^{-1}(U) \in \mathbf{\Sigma}^0_\alpha(X)$, hence $f$ is $\mathbf{\Sigma}^0_\alpha$-measurable (for statement (2), use Theorem 5 instead of Corollary 3). □

The following shows that, assuming that a representation of a set is admissible at some level with respect to some topology on the set, then the level of the representation and any corresponding sequential topology on the set is uniquely determined. Note, however, that it is easy to construct representations of a set that are not admissible at any level with respect to any topology on the set.

**Corollary 4.** *Let $X$ be a set with at least two elements, and let $\rho\colon\subseteq \omega^\omega \to X$ be an arbitrary function. If $\tau$ and $\tau'$ are two topologies on $X$ such that $\rho$ is $\mathbf{\Sigma}^0_\alpha$-admissible ($1 \leq \alpha < \omega_1$) with respect to $\tau$, and $\rho$ is $\mathbf{\Sigma}^0_\beta$-admissible ($1 \leq \beta < \omega_1$) with respect to $\tau'$, then $\alpha = \beta$. If in addition $\tau$ and $\tau'$ are sequential topologies then $\tau = \tau'$.* □

Finally, we give a complete characterization for the case that $X$ has a Polish representation (recall that ordinal addition is non-commutative). Note that a generalization of Theorem 6 to all countably based $T_0$-spaces would allow us to drop the "Polish representation" restriction on $X$.

**Theorem 9.** *Let $X$ and $Y$ be countably based $T_0$ spaces, and further assume $X$ has a Polish representation. For any total function $f\colon X \to Y$ and any countable ordinals $\alpha$, $\beta$ and $\gamma$, there exists a $\mathbf{\Sigma}^0_{\widehat{\gamma}}$-measurable $g\colon\subseteq \omega^\omega \to \omega^\omega$ that $\langle \mathbf{\Sigma}^0_{\widehat{\alpha}}, \mathbf{\Sigma}^0_{\widehat{\beta}}\rangle$-realizes $f$ if and only if:*

1. *$\alpha > \gamma + \beta$ and $f$ is a constant function, or*
2. *$\alpha \leq \gamma + \beta$ and $f$ is a $\mathbf{\Sigma}^0_{\widehat{\eta}}$-measurable function, where $\eta$ is (the unique ordinal) such that $\alpha + \eta = \gamma + \beta$.*

□

## 7 Conclusion

We have introduced and investigated the basic properties of a hierarchy of representations of topological spaces. $\mathbf{\Sigma}^0_\alpha$-admissible representations provide a well-behaved topological interpretation of representations that can not be interpreted as admissible in the traditional (continuous) sense (see Corollary 4). These representations are also significant for better understanding the computational properties of discontinuous functions, which has been investigated for metric spaces in [3], [14], and [5].

The first open problem is to generalize Theorem 6 to all countably based $T_0$-spaces. One difficulty in generalizing Saint-Raymond's result (Proposition 12) is that the fibers of $\mathbf{\Sigma}^0_\alpha$-admissible representations are not Polish in general.

A second open problem is to classify precisely which topological spaces have $\mathbf{\Sigma}^0_\alpha$-admissible representations. An attractive conjecture is that they are exactly the spaces with $\mathbf{\Sigma}^0_1$-admissible representations, which were completely classified by Schröder [9].

Finally, a further refinement of the hierarchy would be useful, particularly between the continuous and $\mathbf{\Sigma}^0_2$ representations. One interesting class of functions are the $\mathbf{\Delta}^0_2$-functions (i.e., preimages of open sets are $\mathbf{\Delta}^0_2$, or, equivalently, preimages of $\mathbf{\Sigma}^0_2$ sets are $\mathbf{\Sigma}^0_2$), which are closed under composition. Wadge reducibility and game semantics for these functions have been investigated by Andretta [1]. Note that a $\mathbf{\Sigma}^0_2$-admissible representation of a discrete space can be interpretted as a "$\mathbf{\Delta}^0_2$-admissible" representation, and, because they are closed under composition, a $\mathbf{\Delta}^0_2$-admissible representation can at best only determine the topology of the represented set up to $\mathbf{\Delta}^0_2$-isomorphism (i.e., a bijection that along with its inverse is a $\mathbf{\Delta}^0_2$-function).

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