Uniqueness, Continuity, and Existence of Implicit Functions in Constructive Analysis

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Abstract. We extract a quantitative variant of uniqueness from the usual hypotheses of the implicit functions theorem. This leads not only to an a priori proof of continuity, but also to an alternative, fully constructive existence proof.

1 Introduction

To show the differentiability of an implicit function one often relies on its continuity. The latter is mostly seen as a by-product of the not uncommon construction of the implicit function as the limit of a uniformly convergent sequence of continuous functions. We now show that the continuity of the implicit function is prior to its existence, and thus independent of any particular construction. More specifically, we deduce the continuity from a quantitative strengthening of the uniqueness, which in turn follows from the hypotheses one needs to impose on the equation the implicit function is expected to satisfy. The same quantitative strengthening of uniqueness enables us to ultimately give an alternative existence proof for implicit functions that is fully constructive in Bishop's sense.

We use ideas from [6], which loc.cit. have only been spelled out in the case of implicit functions with values in $\mathbb{R}$. The existence proof given in [6] therefore can rely on reasoning by monotonicity, whereas in the general case—treated in this paper—of implicit functions with values in $\mathbb{R}^m$ we need to employ an extreme value argument. Similar considerations in related contexts can be found in Sections 3.3 and 3.4 of [10] during the course of the proof of the theorem on implicit functions via the inverse mapping theorem and Banach's fixed point theorem, respectively. We refer to [11, 20] for the implicit function theorem and the open mapping theorem in computable analysis à la Weihrauch [19].

The predecessor [18] of the present paper essentially contains the same material as far as uniqueness and continuity of the implicit function are concerned. When it comes to proving existence, however, it follows the intrinsically classical argument that a continuous function on a compact set attains its minimum. This

Andrej Bauer, Peter Hertling, Ker-I Ko (Eds.)
6th Int'l Conf. on Computability and Complexity in Analysis, 2009, pp. 131-140
http://drops.dagstuhl.de/opus/volltexte/2009/2265

argument fails being practicable constructively, unless one adds the hypothesis that there quantitatively is at most one point at which the minimum can be attained. In fact, there is a heuristic principle valid [17] even in Bishop–style constructive mathematics without countable choice: if a continuous function on a complete metric space has approximate roots and in a quantitative manner at most one root, then it actually has a root. We may refer to [17] for more on this, including the principle's history with references.

As a matter of fact, however, in the case of implicit functions the required additional hypothesis is contained in the quantitative variant of uniqueness which we find at our disposal anyway. Therefore, we only need to prove that for every parameter the given equation admits approximate solutions. Altogether we achieve the existence of an exact solution at every parameter and then, by the principle of unique choice, the existence of an implicit function: as the one and only function which assigns to every parameter the solution uniquely determined by this parameter.

The present paper as a whole is conceived in the realm of Bishop's constructive mathematics [4, 5, 7, 8]. Compared with the—so-called classical—customary way of doing mathematics, the principal characteristic of the framework created by Bishop is the exclusive use of intuitionistic logic, which allows one to view Bishop's setting as a generalisation of classical mathematics [13]. Moreover, we follow [14] in doing constructive mathematics à la Bishop without countable choice, also inasmuch as we understand real numbers as located Dedekind cuts. In particular, the so-called cotransitivity property "if $x < y$, then $x < z$ or $z < y$" amounts to say that the Dedekind cut z is located whenever x, y are rational numbers, and follows by approximation in the general case.

Avoiding countable choice is further indispensable, because we want our work to be expressible in constructive Zermelo–Fraenkel set theory (CZF) as begun in [1]: countable choice does not belong to CZF. Details on this and on CZF in general can be found in [2, 12]. We will, however, use the principle of unique choice, sometimes called the principle of non–choice. By the functions–as–graphs paradigm common to set theory, unique choice is trivially in CZF.

2 Preliminaries

We first recall that in Bishop's setting every differentiable function comes with a continuous derivative [5, Chapter 2, Section 5]. In other words, for Bishop every differentiable function is by definition continuously differentiable. We nonetheless keep speaking of continously differentiable functions, also to facilitate any reading by a classically trained person. Note in this context that in Bishop's framework continuity means uniform continuity on every compact (that is, totally bounded and complete) subset of the domain; see [15] for a discussion of this.

Secondly, although in the work of Bishop and of his followers there barely is any talk of (partial or total) differentiability for functions of several real variables, we do not develop this concept in the present paper either. According to our opinion it is in order to take this for granted: the task of checking the classical route as far as necessary can indeed be performed in a relatively straightforward way, and is sometimes simplified by Bishop's assumption of the automatic continuity of the derivative.

For the lack of appropriate references in the constructive literature we next transfer two facts from real analysis. With Theorem 5.4 and Theorem 6.8 of [5, Chapter 2] at hand the standard proofs indeed go through constructively. (For instance, the proofs of Satz 5 and of its Corollar given in [9, I, §6] require only one addendum to the proof of the Hilfssatz: for all $K, L \in \mathbb{R}$ with $L \geqslant 0$ the implication "if $K^2 \leqslant KL$, then $K \leqslant L$" is also constructively valid. To verify this, assume that $K^2 \leqslant KL$; it suffices to prove that $K < L + \varepsilon$ for every $\varepsilon > 0$. For each $\varepsilon > 0$ either $0 < K$ or $K < \varepsilon$. In the former case, multiplying $K^2 \leqslant KL$ by $1/K > 0$ yields $K \leqslant L$; in the latter case we have $K < L + \varepsilon$ because $L \geqslant 0$.)

Lemma 1. *Let $g : W \to \mathbb{R}^n$ be a continuously differentiable mapping on an open set $W \subseteq \mathbb{R}^m$, and $c, d \in W$. If the line segment between c and d lies entirely in W, then*

$$g(d) - g(c) = \left(\int_0^1 Dg(c + t(d-c))dt \right) \cdot (d-c)\,.$$

Corollary 2. *Under the hypotheses of Lemma 1 we have*

$$\|g(d) - g(c)\| \leqslant \sup_{t \in [0,1]} \|Dg(c + t(d-c))dt\| \cdot \|d - c\| \;.$$

While Lemma 5.5 of [5, Chapter 2] is an approximative alternative of Rolle's theorem, our next lemma is a strong variant of the contrapositive.

Lemma 3. *Let $h : [c,d] \to \mathbb{R}$ be continuously differentiable, and assume that there is $r > 0$ such that $h'(x) > r$ for all $x \in [c,d]$. Furthermore assume that $c < d$. Then $h(c) < h(d)$.*

Proof. Assume that $h(d) - h(c) < \frac{r(d-c)}{4}$. By the mean value theorem [5, Theorem 5.6] there exists $\xi \in [c,d]$ such that

$$|h'(\xi)(d-c) - (h(d) - h(c))| < \frac{r(d-c)}{2}\,.$$

Then

$$\begin{aligned} h'(\xi)(d-c) - (h(d) - h(c)) &> r(d-c) - (h(d) - h(c)) \\ &> r(d-c) - \frac{r(d-c)}{4} \\ &> \frac{r(d-c)}{2}\,. \end{aligned}$$

Hence we get a contradiction, and thus $h(d) - h(c) > 0$. □

The last lemma in this section is an approximative substitute for the classical result that if a differentiable function attains its minimum at a point in the interior of a compact set, then the gradient of that function vanishes at this point.

Lemma 4. *Let $W \subseteq \mathbb{R}^n$ be an open neighbourhood of $[0,1]^n$ and $h : W \to \mathbb{R}$ a continuously differentiable function. If there is a point $\xi \in [0,1]^n$ and $s > 0$ such that*

$$h(x) > h(\xi) + s \tag{1}$$

for all $x \in \partial[0,1]^n$, then for every $\varepsilon > 0$ there exists $y \in [0,1]^n$ such that $\|\nabla h(y)\| < \varepsilon$.

Proof. For convenience we will use the supremum norm on $\mathbb{R}^n$ throughout this proof. Choose $N \in \mathbb{N}$ such that for $x, y \in [0,1]^n$, if $\|y - x\| < 2^{-N}$ then both

$$\|\nabla h(x) - \nabla h(y)\| < \frac{\varepsilon}{4} \tag{2}$$

and

$$|h(x) - h(y)| < \frac{s}{2}\,. \tag{3}$$

Let

$$G = \left\{ \left(\frac{i_1}{2^N}, \ldots, \frac{i_n}{2^N} \right) : (i_1, \ldots, i_n) \in \mathbb{N}^n \right\} \cap [0,1]^n.$$

For any $x \in G$ and $i \leqslant n$, let $x_i^{\pm}$ denote the point $x \pm 2^{-N} e_i$—i.e. the neighbouring point of x in G in the positive/negative direction of the i^{th} coordinate. For any $x \in G$ and $i \leqslant n$ fix $\lambda^+_{x,i} \in \{-1, 0, 1\}$ and $\lambda^-_{x,i} \in \{-1, 0, 1\}$, such that

$$\begin{aligned}
\lambda^+_{x,i} = 0 &\Rightarrow \left| \frac{\partial h}{\partial x_i}(x + 2^{-(N+1)} e_i) \right| < \frac{3\varepsilon}{4}, \\
\lambda^+_{x,i} = -1 &\Rightarrow \frac{\partial h}{\partial x_i}(x + 2^{-(N+1)} e_i) < -\frac{\varepsilon}{2}, \\
\lambda^+_{x,i} = 1 &\Rightarrow \frac{\partial h}{\partial x_i}(x + 2^{-(N+1)} e_i) > \frac{\varepsilon}{2}, \\
\lambda^-_{x,i} = 0 &\Rightarrow \left| \frac{\partial h}{\partial x_i}(x - 2^{-(N+1)} e_i) \right| < \frac{3\varepsilon}{4}, \\
\lambda^-_{x,i} = -1 &\Rightarrow \frac{\partial h}{\partial x_i}(x - 2^{-(N+1)} e_i) > \frac{\varepsilon}{2}, \\
\lambda^-_{x,i} = 1 &\Rightarrow \frac{\partial h}{\partial x_i}(x - 2^{-(N+1)} e_i) < -\frac{\varepsilon}{2}.
\end{aligned}$$

Notice that if $\lambda^+_{x,i} = -1$ then for all $y \in [x, x_i^+]$

$$\frac{\partial h}{\partial x_i}(y) < -\frac{\varepsilon}{4},$$

and therefore, by Lemma 3,

$$h(x) > h(x_i^+). \tag{4}$$

Similarly, when $\lambda_{x,i}^- = -1$, we obtain

$$\frac{\partial h}{\partial x_i}(y) > \frac{\varepsilon}{4}$$

for all $y \in [x_i^-, x]$, and then

$$h(x) > h(x_i^-). \tag{5}$$

Furthermore notice that, by continuity and (2),

$$\text{if } \lambda_{x,i}^+ \in \{0,1\} \text{ and } \lambda_{x,i}^- \in \{0,1\}, \text{ then } \left|\frac{\partial h}{\partial x_i}(x)\right| < \varepsilon. \tag{6}$$

Next, because of (3), we can find $x_0 \in G$ such that $|h(x_0) - h(\xi)| < s$. If there exists i such that $\lambda_{x_0,i}^+ = -1$ (or $\lambda_{x_0,i}^- = -1$), set $x_1 = (x_0)_i^+$ (or $x_1 = (x_0)_i^-$), for which $h(x_0) > h(x_1)$. Continuing this construction we will, because of (4), never visit the same point twice and never reach a point in $\partial[0,1]^n \cap G$. Therefore, we eventually reach a point $x_m \in (0,1)^n \cap G$ for which both $\lambda_{x_m,i}^+ \neq -1$ and $\lambda_{x_m,i}^- \neq -1$ for all $1 \leqslant i \leqslant n$. By (6) this implies that $\|\nabla h(x_m)\| < \varepsilon$. □

3 Uniqueness and Continuity

Situation. *Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open neighbourhoods of $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$, respectively, with $m, n \geqslant 1$. We denote the coordinates on $\mathbb{R}^n$ and $\mathbb{R}^m$ by $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$, respectively, and endow $\mathbb{R}^n \times \mathbb{R}^m$ with the norm $\|(x,y)\| = \|x\| + \|y\|$. The Jacobian of a partially differentiable function $F : U \times V \to \mathbb{R}^m$ at $(x,y) \in U \times V$ is written as*

$$DF(x,y) = \left(\frac{\partial F}{\partial x}(x,y), \frac{\partial F}{\partial y}(x,y)\right), \quad \frac{\partial F}{\partial x}(x,y) \in \mathbb{R}^{m\times n}, \quad \frac{\partial F}{\partial y}(x,y) \in \mathbb{R}^{m\times m}.$$

Finally, let $F : U \times V \to \mathbb{R}^m$ be a continuously differentiable function such that $\frac{\partial F}{\partial y}(a,b)$ is invertible; in particular $\nu > 0$ where

$$\nu = \left\|\frac{\partial F}{\partial y}(a,b)^{-1}\right\|.$$

Lemma 5. *For every $\lambda \in\,]1, +\infty[$ there are compact neighbourhoods $U_\lambda \subseteq U$ and $V_\lambda \subseteq V$ of a and b, respectively, such that for all $x \in U_\lambda$ und $y, y' \in V_\lambda$:*

$$\|y - y'\| \leqslant \lambda \cdot \nu \cdot \|F(x,y) - F(x,y')\|. \tag{7}$$

Proof. By replacing F with $\frac{\partial F}{\partial y}(a,b)^{-1} \cdot F$, we may assume that $\frac{\partial F}{\partial y}(a,b)$ is the unit matrix and therefore $\nu = 1$. Now consider

$$G : U \times V \to \mathbb{R}^m, \quad (x,y) \mapsto y - F(x,y).$$

Since G is continuously differentiable with $\frac{\partial G}{\partial y}(a,b) = 0$, there are compact neighbourhoods $U_\lambda \subseteq U$ and $V_\lambda \subseteq V$ of a and b, respectively, such that V_λ is convex and

$$\left\| \frac{\partial G}{\partial y}(x,y) \right\| \leqslant 1 - 1/\lambda \tag{8}$$

for all $(x,y) \in U_\lambda \times V_\lambda$. Then, for all $x \in U_\lambda$ and $y, y' \in V_\lambda$, we have

$$\begin{aligned} \|y - y'\| &\leqslant \|(y - G(x,y)) - (y' - G(x,y'))\| + \|G(x,y) - G(x,y')\| \\ &\leqslant \|F(x,y) - F(x,y')\| + (1 - 1/\lambda) \cdot \|y - y'\| \end{aligned}$$

by (8) and Corollary 2; whence (7) holds with $\nu = 1$. □

Throughout the following $\lambda \in]1, +\infty[$ *is arbitrary and* U_λ, V_λ *are as in Lemma 5.*

Equation (7) implies, for fixed $x \in U_\lambda$, that $y \in V_\lambda$ and $y' \in V_\lambda$ lie close together, whenever F is small at (x,y) and (x,y'). Therefore (7) can be seen as a quantitative way to express that any y with $F(x,y) = 0$ is uniquely determined by x.

This can be made more precise. We say that a function $H : S \to \mathbb{R}$ on a metric space S with $H \geqslant 0$ has *uniformly at most one* root [16] if

$$\forall \delta > 0 \, \exists \varepsilon > 0 \, \forall y, y' \in S \; (d(y,y') \geqslant \delta \Rightarrow H(y) \geqslant \varepsilon \vee H(y') \geqslant \varepsilon) \; .$$

If H has uniformly at most one root, then H has *at most one* root [3]: i.e.,

$$\forall y, y' \in S \; (y \neq y' \Rightarrow H(y) > 0 \vee H(y') > 0) \; .$$

If H has at most one root, then its root—if it exists at all—is uniquely determined:

$$\forall y, y' \in S \; (H(y) = 0 \wedge H(y') = 0 \Rightarrow y = y') \; .$$

Corollary 6. *For each* $x \in U_\lambda$ *the function*

$$H : V_\lambda \to \mathbb{R}, \; y \mapsto \|F(x,y)\|$$

has uniformly at most one root; in particular, for all $y, y' \in V_\lambda$,

$$F(x,y) = 0 \wedge F(x,y') = 0 \Rightarrow y = y' \, .$$

Theorem 7. *Every function* $f : U_\lambda \to V_\lambda$ *with* $F(x, f(x)) = 0$ *for all* $x \in U_\lambda$ *is continuous.*

Proof. Consider $\varepsilon > 0$ arbitrary. Since F is uniformly continuous on the compact set $U_\lambda \times V_\lambda$, there exists $\delta > 0$ such that

$$\|F(x,y) - F(x',y')\| \leqslant (\lambda \cdot \nu)^{-1} \cdot \varepsilon \, .$$

whenever $(x,y), (x',y') \in U_\lambda \times V_\lambda$ are such that $\|x - x'\| + \|y - y'\| < \delta$. In particular,

$$\|F(x, f(x'))\| \leqslant (\lambda \cdot \nu)^{-1} \cdot \varepsilon$$

for all $x, x' \in U_\lambda$ with $\|x - x'\| < \delta$ (recall that $F(x', f(x')) = 0$). Using this and (7) we get

$$\begin{aligned}\|f(x) - f(x')\| &\leqslant \lambda \cdot \nu \cdot \|F(x, f(x)) - F(x, f(x'))\| \\ &= \lambda \cdot \nu \cdot \|F(x, f(x'))\| \\ &\leqslant \varepsilon\end{aligned}$$

for all $x, x' \in U_\lambda$ with $\|x - x'\| < \delta$. Hence f is uniformly continuous. $\square$

This proof's heuristic can be explained as follows. If x and x' are close, then $F(x, f(x'))$ is close to $F(x', f(x')) = 0$, and therefore close to $F(x, f(x)) = 0$; Equation (7) now implies that $f(x)$ and $f(x')$ are close.

Following the standard argument, one can now easily show that every f as in Theorem 7 is differentiable in the interior of U_λ^0 with uniformly continuous derivative

$$Df(x) = -\frac{\partial F}{\partial y}(x, f(x))^{-1} \cdot \frac{\partial F}{\partial x}(x, f(x)) \ .$$

Note that the quantitative version (7) of uniqueness was sufficient to prove continuity, which therefore only depends on differentiability inasmuch as this is needed to prove (7).

4 Existence

Last, we present an alternative approach to the existence of the implicit function, which is—just as the proof of continuity—based on the quantitative version (7) of uniqueness, but again requires involving the partial derivative of the given equation. An additional ingredient is the following result, for whose validity in Bishop-style constructive mathematics without choice we refer to [17, Theorem 5]:

Theorem 8. *Let S be a complete metric space and $H : S \to \mathbb{R}$ uniformly continuous. If* $\inf H = 0$ *and H has uniformly at most one root, then there is $y_H \in S$ with $H(y_H) = 0$.*

Note that $\inf H = 0$ means that $H \geqslant 0$ and that H has approximate roots.

From now on we also assume that $F(a, b) = 0$. (An assumption that has not been used so far.)

Theorem 9. *There are compact neighbourhoods $U_\lambda^0 \subseteq U_\lambda$ and $V_\lambda^0 \subseteq V_\lambda$ of a and b, respectively, such that there is a function $f : U_\lambda^0 \to V_\lambda^0$ with $F(x, f(x)) = 0$ for all $x \in U_\lambda^0$.*

As a by-product of Corollary 6, there is exactly one f, which by Theorem 7 is continuous.

Proof. Using Corollary 6 and the principle of unique choice, we only need to find compact neighbourhoods U_λ^0 and V_λ^0 of a and b, respectively, with $U_\lambda^0 \times V_\lambda^0 \subseteq U_\lambda \times V_\lambda$, such that for every $x \in U_\lambda^0$ there exists $y \in V_\lambda^0$ with $F(x,y) = 0$. We may also assume that $(a,b) = (0,0)$. Setting $(x,y') = (0,0)$ in (7), we get

$$\|y\| \leqslant \lambda \cdot \nu \cdot \|F(0,y)\| \tag{9}$$

for all $y \in U_\lambda$, since $F(0,0) = 0$. We can now find $r, s > 0$, such that

$$U_\lambda^0 = [-r,+r]^n \ , \quad V_\lambda^0 = [-s,+s]^m$$

completely lie in U_λ and V_λ respectively. By choosing r, s small enough, we may assume that

$$\left\| \frac{\partial F}{\partial y}(x,y)^{-1} \right\| \leqslant \nu + 1 \tag{10}$$

for all $(x,y) \in U_\lambda^0 \times V_\lambda^0$. Since F is uniformly continuous on the compact set $U_\lambda^0 \times V_\lambda^0$, by making r sufficiently small, we may further assume that

$$\lambda \cdot \nu \cdot \|F(x,y) - F(x',y)\| \leqslant s/3$$

for all $(x,y), (x',y) \in U_\lambda^0 \times V_\lambda^0$. If we now substitute $x' = 0$, we get

$$\lambda \cdot \nu \cdot \|F(x,y) - F(0,y)\| \leqslant s/3 \tag{11}$$

for all $(x,y) \in U_\lambda^0 \times V_\lambda^0$; if we also substitute $y = 0$, we get

$$\lambda \cdot \nu \cdot \|F(x,0)\| \leqslant s/3 \tag{12}$$

for all $x \in U_\lambda^0$. (If we were only interested in getting (12), it would suffice to point out that $F(\_,0)$ is continuous at 0 and that $F(0,0) = 0$.) Equations (9) and (11) imply that

$$2s/3 \leqslant \lambda \cdot \nu \cdot \|F(x,y)\| \tag{13}$$

for all $x \in U_\lambda^0$ and $y \in \partial V_\lambda^0$, i.e. $\|y\| = s$. Now consider $x \in U_\lambda^0$ arbitrary, but fixed. The function

$$h : V \to \mathbb{R}, \ y \mapsto \|F(x,y)\|^2$$

is differentiable with continuous derivative

$$\nabla h(y) = 2 \cdot F(x,y) \cdot \frac{\partial F}{\partial y}(x,y) \ . \tag{14}$$

By (12) and (13) we have

$$\lambda^2 \cdot \nu^2 \cdot h(0) + s^2/3 \leqslant \lambda^2 \cdot \nu^2 \cdot h(y)$$

for all $y \in \partial V_\lambda^0$; whence by virtue of Lemma 4

$$\inf_{y \in V_\lambda^0} \|\nabla h(y)\| = 0 \, .$$

In view of (10) and (14) this implies

$$\inf_{y \in V_\lambda^0} \|F(x,y)\| = 0 \, .$$

By Corollary 6 and Theorem 8, we achieve $y \in V_\lambda^0$ with $F(x,y) = 0$. $\square$

Acknowledgments. A lecture on implicit functions the second author gave at the University of Bern made him thinking first of the subject matters present in this paper. Next, D.S. Bridges pointed the authors to [6]; O. Forster, H. Kalf, and the two referees of this paper gave useful hints. The first author would like to thank the departments of mathematics of the University of Canterbury and the Universität Siegen for supporting him as an outgoing postdoctoral fellow and an incoming staff member respectively. During the preparation of this paper the second author was holding a Feodor Lynen Research Fellowship for Experienced Researchers granted by the Alexander von Humboldt Foundation from sources of the German Federal Ministry of Education and Research; he is grateful to A. Cantini and G. Sambin for their hospitality in Florence and Padua, respectively.

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