On Oscillation-free ε-random Sequences II

Jöran Mielke and Ludwig Staiger

Institute for Computer Science
Martin-Luther-University
Halle(Saale), Germany

Abstract. It has been shown (see [10]), that there are strongly MARTIN-LÖF-ε-random ω-words that behave in terms of complexity like random ω-words. That is, in particular, the *a priori* complexity of these ε-random ω-words is bounded from below and above by linear functions with the same slope ε. In this paper we will study the set of these ω-words in terms of HAUSDORFF measure and dimension.
Additionally we find upper bounds on *a priori* complexity, monotone and simple complexity for a certain class of ω-power languages.

1 Introduction

The present paper is a continuation of [10] where it has been shown that oscillation-free ε-random sequences exist, for every computable ε, $0 < \varepsilon < 1$. To this end two methods were developed. The first one, by diluting random sequences, led to a method for a general "construction" of ε-random sequences from random sequences whereas the second one exhibited ε-random sequences as maximally complex sequences in certain computably definable sets of sequences (ω-languages).

Here we address mainly two questions. The first one is about the Hausdorff dimension and the Hausdorff measure of the set of oscillation-free ε-random sequences. As every random sequence is also ε-random the set of ε-random sequences has Hausdorff dimension 1. We prove a result analogous to Ryabko's estimate of the dimension of the set of sequences of a certain asymptotic relative complexity (cf. [6, 9]). We show that the set of oscillation-free ε-random sequences has Hausdorff dimension ε and infinite ε-dimensional Hausdorff measure.

The second problem we address is the one of obtaining oscillation-free ε-random sequences in so-called ω-power languages. Here we generalise the results for ω-powers of regular languages obtained in [10] to more general classes of ω-powers of computably enumerable languages.

2 Notation and Preliminary Results

In this section we briefly recall the concepts of HAUSDORFF measure and complexity of finite and infinite words used in this paper. For more detailed information the reader is referred to the textbooks [2] and [4]. In the following X is

Andrej Bauer, Peter Hertling, Ker-I Ko (Eds.)
6th Int'l Conf. on Computability and Complexity in Analysis, 2009, pp. 173-184
http://drops.dagstuhl.de/opus/volltexte/2009/2269

a finite alphabet with cardinality $|X| = r$. By X^* we denote the set (monoid) of words on X, including the *empty word* e, and X^ω is the set of infinite words (ω-words) over X. For $w \in X^*$ and $\eta \in X^* \cup X^\omega$ let $w \cdot \eta$ be their concatenation. We extend this concatenation in the obvious way to subsets $W \subseteq X^*$ and $B \subseteq X^* \cup X^\omega$. For a language W let $W^* := \bigcup_{n \in \mathbb{N}} W^n$ be the *submonoid* of X^* generated by W, and by $W^\omega := \{w_1 \cdots w_n \cdots \mid w_n \in W \setminus \{e\}\}$ we denote the subset of X^ω formed by concatenating words of W. We call $V/w := \{v \mid w \cdot v \in V\}$ the *left derivative* of V by w. Furthermore $|w|$ is the *length* of the word $w \in X^*$ and $\underline{l}(V) := \min\{|v| \mid v \in V\}$ denotes the length of the shortest word contained in V. For a set $B \subseteq X^* \cup X^\omega$ the set of all finite prefixes of strings in B is $\mathbf{pref}(B)$, we abbreviate $w \in \mathbf{pref}(\{\eta\})$ by $w \sqsubseteq \eta$. By $\xi[0..n]$ we denote the prefix of $\xi \in X^* \cup X^\omega$ of length n.

A real number α is right-computable (left-computable) if and only if there is a computable sequence α_i, $i \in \mathbb{N}$, of rational numbers with $\alpha_i \geq \alpha_{i+1}$ ($\alpha_i \leq \alpha_{i+1}$) for all $i \in \mathbb{N}$ and $\lim_{i\to\infty} \alpha_i = \alpha$. A number α is called computable if and only if α is left- and right-computable. A function $f : X^* \to \mathbb{R}$ is called right-computable (left-computable) if and only if there is a computable function $h : X^* \times \mathbb{N} \to \mathbb{R}$ such that $\lim_{t\to\infty} h(w,t) = f(w)$, for every $w \in X^*$, and h is decreasing (increasing) with respect to t.

A language $V \subseteq X^*$ is called a *code* provided every $w \in V^*$ has a unique factorisation $w = v_1 \dots v_n$ with $v_i \in V$ $(1 \leq i \leq n)$. If $e \notin V$ and for all $v, w \in V$ the relation $v \sqsubseteq w$ implies $v = w$ then V is called *prefix code*.

It is useful to consider the set X^ω as a metric space (Cantor space) (X^ω, ρ) of all ω-words over the alphabet X where the metric is ρ is defined as follows

$$\rho(\xi, \eta) := \inf\{r^{-|w|} \mid w \sqsubseteq \xi \wedge w \sqsubseteq \eta\}$$

The open (and simultaneously closed) balls in (X^ω, ρ) are the sets of the form $w \cdot X^\omega$, where $w \in X^*$. The diameter of these balls is $d(w \cdot X^\omega) = r^{-|w|}$. The closure of a set $F \subseteq X^\omega$ in (X^ω, ρ) is the set $\mathcal{C}(F) := \{\xi \mid \mathbf{pref}(\xi) \subseteq \mathbf{pref}(F)\}$.

We define Hausdorff measure and Hausdorff dimension for subsets of (X^ω, ρ). For every language $F \subseteq X^\omega$ and every $0 \leq \varepsilon \leq 1$ the equation

$$\mathbb{L}_\varepsilon(F) := \lim_{n\to\infty} \inf \left\{ \sum_{v \in V} r^{-\varepsilon \cdot |v|} \mid F \subseteq V \cdot X^\omega \wedge \underline{l}(V) \geq n \right\}$$

defines the ε-dimensional Hausdorff measure of F. The measure $\mathbb{L}_1$ is the usual Lebesgue measure. The following property of the Hausdorff measure is well-known.

Corollary 1. *Let $F \subseteq X^\omega$. If $\mathbb{L}_\varepsilon(F) < \infty$ then for every $\delta > 0$ it holds $\mathbb{L}_{\varepsilon+\delta}(F) = 0$ and if $0 < \mathbb{L}_\varepsilon(F)$ then for every $\delta > 0$ it holds $\mathbb{L}_{\varepsilon-\delta}(F) = \infty$.*

The Hausdorff dimension of F is defined as follows

$$\dim F = \sup\{\varepsilon \mid \mathbb{L}_\varepsilon(F) = \infty \vee \varepsilon = 0\} = \inf\{\varepsilon \mid \mathbb{L}_\varepsilon(F) = 0\}$$

Next we introduce the complexities used in this paper. Consider a semi-measure m on X^*, that is, a function $m : X^* \to \mathbb{R}$ which satisfies $m(\varepsilon) \leq 1$ and

$m(w) \geq \sum_{x \in X} m(wx)$, for $w \in X^*$. If $m(w) = \sum_{x \in X} m(wx)$ the function m is called a measure. In [13] Levin proved the existence of a universal left-computable semi-measure $\mathbf{M}$, that is, a left-computable semi-measure which satisfies

$$\exists c_m \, \forall w \in X^* \qquad m(w) \leq c_m \cdot \mathbf{M}(w), \tag{1}$$

for all left-computable semi-measures m. Then the *a priori* complexity is defined as $\mathrm{KA}(w) = \lfloor -\log_r \mathbf{M}(w) \rfloor$ (cf. [4, 11]).

For the definition of the *monotone complexity* Km we refer the reader to [4, 12]. Here we need only the following property.

Corollary 2 ([4]). *Let μ be a computable measure on X^*. Then there is a constant c_μ such that*

$$\mathrm{Km}(w) \leq -\log \mu(w) + c_\mu$$

holds for all $w \in X^*$.

Plain (cf. [4]) or simple (cf. [11]) program size complexity defines the complexity of a finite string to be the length of a shortest program which prints the string. Let $\varphi : X^* \to X^*$ be a partial computable function. The complexity of a word $w \in X^*$ with respect to φ is defined as

$$\mathrm{K}_\varphi(w) := \inf\{|\pi| \mid \pi \in X^* \wedge \varphi(\pi) = w\}\,. \tag{2}$$

It is well-known that there is an optimal partial computable function $\mathfrak{U}$, that is, a function satisfying

$$\exists c_\varphi \forall w (w \in X^* \to \mathrm{K}_\mathfrak{U}(w) \leq \mathrm{K}_\varphi(w) + c_\varphi) \tag{3}$$

for every partial computable function φ. In the sequel we fix an optimal function $\mathfrak{U}$ and denote the complexity with respect to this function by KS.

The complexity of an infinite word ξ is a function mapping natural numbers n to the complexity of the n-length prefix of ξ.

Definition 1. *Let $\xi \in X^\omega$.*

1. *The function* $\mathrm{KS}(\xi[\cdot]) : \mathbb{N} \to \mathbb{N}$ *is called* plain *or* simple complexity *of* ξ.
2. *The function* $\mathrm{Km}(\xi[\cdot]) : \mathbb{N} \to \mathbb{N}$ *is called* monotone complexity *of* ξ.
3. *The function* $\mathrm{KA}(\xi[\cdot]) : \mathbb{N} \to \mathbb{N}$ *is called* a priori complexity *of* ξ.

We follow here, except for the monotone complexity, the notation of Uspensky and Shen in [11]. In [1] strongly Martin-Löf-ε-random ω-words were introduced as follows.

Definition 2. *A computably enumerable set $\mathcal{V} \subseteq X^* \times \mathbb{N}$ is referred to as a strong* Martin-Löf-*ε-test provided*

1. *$\forall i (V_{i+1} \cdot X^\omega \subseteq V_i \cdot X^\omega)$, where $V_i := \{v \mid (v, i) \in \mathcal{V}\}$ and*
2. *$\forall i \, \forall C (C \subseteq V_i \wedge C$ is prefix code $\to \sum_{v \in C} r^{-\varepsilon \cdot |v|} < r^{-i})$.*

An ω-word $\xi \in X^\omega$ is called strongly Martin-Löf-ε-random *if and only if $\xi \notin \bigcap_{i \in \mathbb{N}} V_i \cdot X^\omega$ for all strong* Martin-Löf-*ε-tests.*

We mention the following equivalence between strong Martin-Löf-ε-tests and *a priori* complexity.

Lemma 1 ([1]). *Let $0 < \varepsilon \le 1$ be a computable number. Then an ω-word $\xi \in X^\omega$ is strongly* Martin-Löf*-ε-random if and only if* $\mathrm{KA}(\xi[0..n]) \ge_{\mathrm{a.e.}} \varepsilon \cdot n - O(1)$.

Ryabko showed in [6] the following result on the set of ω-words having a bounded asymptotic lower complexity (see also [7]).

Theorem 1 ([6]).

$$\dim \left\{ \xi \mid \xi \in X^\omega \wedge \liminf_{n\to\infty} \frac{\mathrm{KA}(\xi[0..n])}{n} \le \varepsilon \right\} = \varepsilon$$

Depending on the ε-dimensional measure of an ω-language we obtain a lower bound on the complexity of the most complex ω-words in that ω-language.

Theorem 2 ([5]). *Let $F \subseteq X^\omega$ and $\mathbb{L}_\varepsilon(F) > 0$. Then for all $c > -\log \mathbb{L}_\varepsilon(F)$ there is a $\xi_c \in F$ such that* $\mathrm{KA}(\xi_c[0..n]) \ge_{\mathrm{a.e.}} \varepsilon \cdot n - c$.

ω-words which, analogously to random ω-words, satisfy also a linear upper bound for *a priori* complexity are referred to as oscillation-free.

Definition 3 ([10]). *An ω-word ξ is called* oscillation-free strongly Martin-Löf-ε-random *if and only if ξ is strongly* Martin-Löf*-ε-random and there is a constant c such that* $\mathrm{KA}(\xi[0..n]) \le \varepsilon \cdot n + c$ *holds.*

3 The Measure of the Set of ε-random Sequences

We start with mappings that preserve some properties of the measure of a language and the behaviour of the complexity-function of an ω-word.

Definition 4. *A computable function $\varphi : X^* \to X^*$ is called* dilution function *provided φ is prefix-monotone, one-to-one and $|\varphi(w)| = |\varphi(w')|$ for all $w, w' \in X^n$. A function $g : \mathbb{N} \to \mathbb{N}$ is called* modulus-function *for φ provided $|\varphi(w)| = g(|w|)$ for every $w \in X^*$.*

Every dilution function φ defines a mapping $\overline{\varphi} : X^\omega \to X^\omega$ in the following way: $\mathbf{pref}(\overline{\varphi}(\xi)) = \mathbf{pref}(\varphi(\mathbf{pref}(\xi)))$. The following is an example of a dilution function.

Example 1. Dilution functions can be defined inductively by inserting a fixed string in front of every letter. Let $X = \{0, 1\}$. Then $\varphi(e) := e$ and $\varphi(wx) := \varphi(w)0x$ for every $w \in X^*$ and $x \in X$ defines a dilution function with $\frac{1}{2}$-modulus.

In this paper we are interested in the following dilution functions.

Definition 5. *Let ε with $0 < \varepsilon < 1$ be a computable real. A computable function g is called* ε-modulus *if and only if there is a constant c such that $|\varepsilon \cdot g(n) - n| \le c$, for all $n \in \mathbb{N}$.*

The mapping $g(n) := \lceil \frac{n}{\varepsilon} \rceil$ is an example for an ε-modulus. If φ is a dilution function with ε-modulus then for every $w \in X^*$ holds

$$-c \leq \varepsilon \cdot |\varphi(w)| - |w| \leq c\,.$$

We obtain our first result on the relation of the measure of a language and its image.

Lemma 2. *Let $F \subseteq X^\omega$, $0 < \varepsilon < 1$ computable and $\varphi : X^* \to X^*$ a dilution function with ε-modulus $g : \mathbb{N} \to \mathbb{N}$. There are constants $c_1, c_2 > 0$, such that*

$$c_1 \cdot \mathbb{L}(F) \leq \mathbb{L}_\varepsilon(\overline{\varphi}(F)) \leq c_2 \cdot \mathbb{L}(F)\,.$$

Proof. The first inequality is shown as follows. Let $W \subseteq X^*$ cover $\overline{\varphi}(F)$, that is, $\overline{\varphi}(F) \subseteq W \cdot X^\omega$ and let $\underline{l}(W) \geq n$. For every $w \in W$ we define v_w as the unique word with $\varphi(v_w) \sqsubseteq w \sqsubset \varphi(v_w x)$, for some $x \in X$. Since φ has an ε-modulus, we have the following:

$$|v_w| - c \leq \varepsilon \cdot |w| \leq |v_w x| + c = |v_w| + 1 + c$$

Then the set $V = \{v_w \mid w \in W\}$ covers F. Now we obtain a bound of the ε-dimensional measure of $\overline{\varphi}(F)$ by the 1-dimensional measure of F:

$$\begin{aligned}\sum_{w\in W} r^{-\varepsilon\cdot|w|} &\geq \sum_{w\in W} r^{-|v_w|-1-c} \geq r^{-1-c}\sum_{v\in V} r^{-|v|}\\ &\geq r^{-1-c}\cdot \inf\left\{\sum_{v\in V} r^{-|v|} \mid F \subseteq V\cdot X^\omega \wedge \underline{l}(V) \geq \varepsilon\cdot n - c - 1\right\}\end{aligned}$$

Taking the limit $n \to \infty$ we get our intended inequality $\mathbb{L}_\varepsilon(\overline{\varphi}(F)) \geq c_1 \cdot \mathbb{L}(F)$.

To prove the second inequality we consider a set V with minimum length $\underline{l}(V) \geq n$ that covers F. Now the set $W = \{\varphi(v) \mid v \in V\}$ covers $\overline{\varphi}(F)$ and we can estimate

$$\begin{aligned}\sum_{v\in V} r^{-|v|} &\geq r^{-c}\cdot \sum_{w\in W} r^{-\varepsilon\cdot|w|}\\ &\geq r^{-c}\cdot \inf\left\{\sum_{w\in W} r^{-\varepsilon\cdot|w|} \mid \overline{\varphi}(F) \subseteq W\cdot X^\omega \wedge \underline{l}(W) \geq g(n) - c\right\}\end{aligned}$$

Again, the limit $n \to \infty$ yields the announced inequality.

Since the constants c_1 and c_2 in Lemma 2 are positive, the following equivalence of the 1-dimensional measure of F and the ε-dimensional measure of $\overline{\varphi}(F)$ holds true.

Corollary 3. *Let $F \subseteq X^\omega$, $0 < \varepsilon < 1$ computable and $\varphi : X^* \to X^*$ a dilution function with ε-modulus $g : \mathbb{N} \to \mathbb{N}$. The measures $\mathbb{L}(F)$ and $\mathbb{L}_\varepsilon(\overline{\varphi}(F))$ are simultaneously zero, positive or infinite, respectively.*

To derive our main theorem we still need the following result from [10]. It states that the *a priori* complexity of the $\varepsilon \cdot n$-length prefix of an ω-word and the n-length prefix of its image differ not too much.

Corollary 4 ([10]). *Let ε, $0 < \varepsilon < 1$, be a computable number. Then there is a dilution function $\varphi : X^* \to X^*$ with strictly increasing ε-modulus g such that*

$$|\mathrm{KA}(\overline{\varphi}(\xi)[0..n]) - \mathrm{KA}(\xi[0..\varepsilon \cdot n])| \leq O(1) \textit{ for all } \xi \in X^\omega \textit{ and all } n \in \mathbb{N}.$$

If the ω-word ξ is chosen to be random then $\overline{\varphi}(\xi)$ is an oscillation-free Martin-Löf-ε-random ω-word.

As every (1-)random ω-word is also strongly ML-ε-random the Hausdorff dimension of the set of all strongly ML-ε-random ω-words is 1. Theorem 1 shows that the Hausdorff dimension of the set of all oscillation-free strongly Martin-Löf-ε-random ω-words is bounded from above by ε. The next theorem calculates its Hausdorff dimension and the corresponding measure.

Theorem 3. *Let $0 < \varepsilon < 1$ computable. The set F_ε of all oscillation-free strongly* Martin-Löf*-ε-random sequences has* Hausdorff *dimension ε and infinite ε-dimensional measure.*

Proof. Theorem 1 implies $\dim F_\varepsilon \leq \varepsilon$, since $\mathrm{KA}(\xi[0..n]) \leq \varepsilon \cdot n + c$ for every $\xi \in F_\varepsilon$. On the other hand, let φ be a dilution function with ε-modulus and F_1 the set of all (1-)random sequences. Then, according to Corollary 4, $\overline{\varphi}(F_1) \subseteq F_\varepsilon$. Since F_1 has positive, finite 1-dimensional measure, $\overline{\varphi}(F_1)$ has positive, finite ε-dimensional measure. Thus $\varepsilon = \dim \overline{\varphi}(F_1) \leq \dim F_\varepsilon$.

To show that the ε-dimensional measure of F_ε is infinite, we find an infinite family of pairwise disjoint subsets of F_ε for which the ε-dimensional measure of every set of the family is bounded from below by the same positive constant. Let $a, b \in X$, $a \neq b$ and $k : \mathbb{N} \to \mathbb{N}$. For every $w \in X^*$ and $x \in X$ the function φ_i is defined as follows: $\varphi_i(e) = e$ and

$$\varphi_i(wx) = \begin{cases} \varphi_i(w)a^{k(|w|)}x & \text{, if } |w| \neq i \\ \varphi_i(w)b^{k(|w|)}x & \text{, if } |w| = i \end{cases}$$

Here the function k is to be defined in a way that all φ_i become computable functions with ε-modulus. Since $\varepsilon < 1$, the set $K := \{i \mid k(i) > 0\}$ is infinite. Moreover for all $i, j \in K$, $i \neq j$, the sets $\overline{\varphi}_i(X^\omega)$ and $\overline{\varphi}_j(X^\omega)$ are disjoint. Lemma 2 shows that there is a constant $c > 0$ such that $\mathbb{L}_\varepsilon(\overline{\varphi}_i(F_1)) > c$ for every $i \in \mathbb{N}$. Now we obtain

$$\mathbb{L}_\varepsilon(F_\varepsilon) \geq \mathbb{L}_\varepsilon(\bigcup_{i \in K} \overline{\varphi}_i(F_1)) = \sum_{i \in K} \mathbb{L}_\varepsilon(\overline{\varphi}_i(F_1)) = \infty\,.$$

4 Complexity Bounds for ω-power Languages

In [8] for certain ω-power languages a necessary and sufficient condition to be of non-null α-dimensional Hausdorff measure was derived. In this respect, for a language $V \subseteq X^*$, the *α-residue of V derived by w*, the value $\sum_{v \in V/w} r^{-\alpha|v|}$, plays a special rôle.

Theorem 4 ([8]). *Let $V \subseteq X^*$ be a prefix code and $\sum_{v\in V} r^{-\alpha|v|} = 1$. Then $\alpha = \dim V^\omega$, and, moreover $\mathbb{L}_\alpha(V^\omega) > 0$ if and only if the α-residues of V are bounded from above.*

Thus in view of Theorem 2 such V^ω contain sequences ξ having a linear lower complexity bound $\alpha \cdot n - c$. It is interesting now to observe that bounding the α-residues of V from below yields a linear upper bound of slope α on the complexity of ω-words in the closure $\mathcal{C}(V^\omega)$.

Lemma 3. *Let $V \subseteq X^*$ be a computably enumerable prefix code. Let α be right-computable such that $\sum_{v\in V} r^{-\alpha\cdot|v|} = a \le 1$ and the α-residues of V derived by $w \in \mathbf{pref}(V)$ are bounded from below. Then there is a constant c such that for every $\xi \in \mathcal{C}(V^\omega)$*

$$\mathrm{KA}(\xi[0..n]) \le \alpha \cdot n + c\,.$$

Proof. In the same way as in the proof of Lemma 3.9 of [10] we construct a left-computable semi-measure μ such that $\mu(w) \ge c \cdot r^{-\alpha\cdot|w|}$ and use Eq. (1). We have only to ensure that the construction works also in the case $a < 1$. The construction is as follows.

$$\mu(w) = \begin{cases} 0 & \text{, if } w \notin \mathbf{pref}(V^*) \\ \sum_{wv\in V} r^{-\alpha|wv|} & \text{, if } w \in \mathbf{pref}(V) \\ r^{-\alpha\cdot|w|} & \text{, if } w \in V^* \\ \mu(u)\cdot\mu(v) & \text{, if } w = u\cdot v \text{ with } u \in V\cdot V^* \wedge v \in \mathbf{pref}(V) \setminus V \end{cases} \tag{4}$$

Since V is a prefix code, the decomposition in the last line of the construction is unique. The equation $\mu(w) = \sum_{x\in X} \mu(wx)$ for every $w \in \mathbf{pref}(V) \setminus V$ follows directly from the second case of the construction. For $w \in V$ we have the inequality

$$\sum_{x\in X} \mu(wx) = \mu(w) \cdot \sum_{x\in X} \sum_{xv\in V} r^{-\alpha|xv|} = \mu(w) \cdot \sum_{v\in V} r^{-\alpha|v|} = \mu(w)\cdot a \le \mu(w) \tag{5}$$

The inductive construction in the last line yields the inequality in the remaining cases. To show that μ is left-computable we successively approximate the value $\mu(w)$ from below. Let V_i be the set of the first i elements in the enumeration of V and α_i the ith approximation of α from the right. We start with $\mu^{(0)}(w) := 0$ and $\mu^{(j)}(e) = 1$ for $j > 0$. Suppose that the jth approximation $\mu^{(j)}$ for all words shorter than w is already computed. If there is a $v \in V_j$ with $w = v \cdot w'$, $w' \ne e$, then $\mu^{(j)}(w) = \mu^{(j)}(v) \cdot \mu^{(j)}(w')$. Otherwise $\mu^{(j)}(w) = \sum_{v\in V_j \wedge w\sqsubseteq v} r^{-\alpha_j\cdot|v|}$.

Let $c_{\inf} := \inf\left\{\sum_{v\in V/w} r^{-\alpha\cdot|v|} \mid w \in \mathbf{pref}(V)\right\}$. Since μ is a left-computable semi-measure, the following inequality holds true.

$$\mathbf{M}(w)\cdot c_\mu \;\ge\; \mu(w) \;=\; r^{-\alpha|w|} \cdot \sum_{v\in V/w} r^{-\alpha|v|} \;\ge\; r^{-\alpha|w|}\cdot c_{\inf}$$

Taking the negative logarithm on both sides of the inequality we obtain $\mathrm{KA}(w) \le \alpha\cdot|w| + \log\frac{c_\mu}{c_{\inf}}$ for every $w \in \mathbf{pref}(V^*)$.

The following example shows, that in Lemma 3 we cannot omit the condition that the α-residues are bounded from below. To this end we use a computable prefix code constructed in Example (6.4) of [7].

Example 2. Let X={0,1} and consider $W := \bigcup_{i\in\mathbb{N}} 0^{i+1}\cdot 1\cdot X^{i+1}\cdot 0^{4\cdot i+3}$. The language W is a prefix code. Its ω-power, W^ω, satisfies $\alpha = \dim W^\omega = \dim \mathcal{C}(W^\omega) = \frac{1}{3}$ and $\mathbb{L}_\alpha(W^\omega) = \mathbb{L}_\alpha(\mathcal{C}(W^\omega))$. For every $w \in \bigcup_{i\in\mathbb{N}} 0^{i+1}\cdot 1\cdot X^{i+1}$ we have $W/w = \{0^{4\cdot i+3}\}$. Thus $\sum_{v\in W/w} r^{-\alpha\cdot|v|} = r^{-\alpha\cdot(4\cdot i+3)}$ and, consequently, $\inf\{\sum_{v\in W/w} r^{-\alpha\cdot|v|} \mid w \in \mathbf{pref}(W)\} = 0$.
Now, in Eq. (6.13) and Proposition 6.15 of [7] it is shown that $\sup_{\xi\in W^\omega} \limsup_{n\to\infty} \frac{\mathrm{KA}(\xi[0..n])}{n} \geq \frac{1}{2} > \frac{1}{3} = \dim W^\omega$.

In connection with Theorem 4 our Lemma 3 yields a sufficient condition for ω-powers to contain oscillation-free α-random ω-words.

Corollary 5. *Let $V \subseteq X^*$ be a computably enumerable prefix code and α right-computable such that $\sum_{v\in V} r^{-\alpha\cdot|v|} = 1$ and the α-residues of V derived by $w \in \mathbf{pref}(V)$ are bounded from above and below. Then there is an oscillation-free ML-α-random ω-word in V^ω.*

The results of Section 3.2 of [10] show that Corollary 5 is valid for prefix codes which are regular languages. The subsequent example verifies that there are also non-regular prefix codes which satisfy the hypotheses of Corollary 5.

Example 3. Let $X = \{0,1\}$ and consider the Łukasiewicz language L defined by the identity $L = 0 \cup 1\cdot L^2$. This language is a prefix code and Kuich [3] showed that $\sum_{w\in L} 2^{-|w|} = 1$. Thus the language V defined by $V = 00 \cup 11\cdot V^2$ is also a prefix code and satisfies $\sum_{v\in V} 2^{-\frac{1}{2}\cdot|w|} = 1$. By induction one shows that for $v \in \mathbf{pref}(V)$ we have $V/v = w'\cdot V^k$ for suitable $k \in \mathbb{N}$ and $|w'| \leq 1$. Therefore the α-residues of V derived by $v \in \mathbf{pref}(V)$ are bounded from above and below.

For the monotone complexity Km a result similar to Lemma 3 can be obtained for a smaller class of ω-languages. We start with an auxiliary result.

Proposition 1. *1. If V is computably enumerable and $\sum_{v\in V} r^{-\alpha|v|} = 1$ then α is left-computable.*
2. If V is computably enumerable, α is right-computable and $\sum_{v\in V} r^{-\alpha|v|} = 1$ then V is computable.

Proof. The proof of part 1 is obvious. To prove part 2 we present an algorithm to decide whether a word w is in V or not.

Let V_j be the set of the first j elements in the enumeration of V and α_j the jth approximation of α from the right.

```
Input w
j := 0
  repeat
    j := j + 1
    if w ∈ V_j then accept and exit
  until r^(-α_j|w|) + Σ_{v∈V_j} r^(-α_j|v|) > 1
reject
```

If $w \notin V$ then the repeat until loop terminates as soon as $\sum_{v \in V_j} r^{-\alpha_j |v|} > 1 - r^{-\alpha_j |w|} \geq 1 - r^{-\alpha |w|}$ because $\sum_{v \in V_j} r^{-\alpha_j |v|} \to 1$ for $j \to \infty$.

Now we can prove our result on monotone complexity.

Lemma 4. *Let $V \subseteq X^*$ be a computably enumerable prefix code. If α is right-computable such that $\sum_{v \in V} r^{-\alpha \cdot |v|} = 1$ and the α-residues of V derived by $w \in$ $\mathbf{pref}(V)$ are bounded from below then there is a constant c such that for every $\xi \in \mathcal{C}(V^\omega)$*

$$\mathrm{Km}(\xi[0..n]) \leq \alpha \cdot n + c\,.$$

Proof. Because of Proposition 1 we can assume that α is a computable real number and V is computable. We use Eq. (4) to construct μ as in the proof of Lemma 3. Since $a = 1$, equality holds in Eq. (5). Thus μ is a measure and for every $v \in V^*$ the number $\mu(v)$ is computable. Since V is a computable prefix code, for every $w \in X^*$ we can compute the unique decomposition $w = v \cdot w'$ with $v \in V^*$ and $w' \notin V \cdot X^*$. Now

$$\mu(w) = \mu(v) \cdot \left(1 - \sum_{v' \in V \wedge w \not\sqsubseteq vv'} r^{-\alpha |v'|}\right)$$

shows that μ is right-computable. If $w' \notin \mathbf{pref}(V)$ then the last factor is zero.

Again let $c_{\mathrm{inf}} := \inf \left\{ \sum_{v \in V/w} r^{-\alpha \cdot |v|} \mid w \in \mathbf{pref}(V) \right\}$. In view of Corollary 2 we get the bound

$$\mathrm{Km}(w) \leq -\log \mu(w) + c_\mu \leq \alpha \cdot |w| + c_\mu - \log c_{\mathrm{inf}}$$

for every $w \in \mathbf{pref}(V^*)$.

5 Plain Complexity

In this section we prove results analogous to Lemma 3 for the complexity KS. First we derive a preparatory result. A similar lemma, for length-conditional plain description complexity, is known from [7, 13].

Lemma 5. *Let $W \subseteq X^*$ be computably enumerable, ε, $0 < \varepsilon < 1$, be a computable real number and let $|W \cap X^l| \leq c \cdot r^{\varepsilon \cdot l}$ for some constant $c > 0$ and all $l \in \mathbb{N}$. Then*

$$\exists C \big(C \in \mathbb{N} \wedge \forall w (w \in W \to \mathrm{KS}(w) \leq \varepsilon \cdot |w| + C)\big)$$

Proof. Let $X = \{0, 1, \ldots, r-1\}$ consist of r letters. Since ε is computable, $g(n) := \lceil \frac{n}{\varepsilon} \rceil$ is a computable function. Define a partial computable function $\varphi : X^* \to X^*$ as follows.

$$\varphi(0^k 1 v) := \text{ the } v\text{th word of length } g(|v|) - k \text{ in the enumeration of } W\,. \quad (6)$$

Here we interpret a word $v \in X^n$ as a number between 0 and $r^n - 1$.

As W has at most $r^{\varepsilon\cdot(l_o+l)}$ words of length l, this enumeration process yields $\{\varphi(0^k 1v) : v \in X^n\} \supseteq W \cap X^l$ as soon as $n \geq \varepsilon\cdot(l_0+g(n)-k) = \varepsilon\cdot g(n) - \varepsilon\cdot(k-l_0)$.

Hence, $\mathrm{KS}(w) \leq \varepsilon \cdot |w| + O(1)$ for all $w \in W$.

In order to apply Lemma 5 to languages V satisfying the conditions of Lemma 3 we show that a positive lower bound to the α-residues of V implies the upper bound $|\mathbf{pref}(V^*) \cap X^l| \leq c \cdot r^{\alpha\cdot l}$ for some constant $c > 0$ and all $l \in \mathbb{N}$.

Lemma 6. *Let $V \subseteq X^*$ be a code, $\sum_{v\in V} r^{-\alpha|v|} \leq 1$ and $\sum_{v\in V/w} r^{-\alpha|v|} \geq c' > 0$ for all $w \in \mathbf{pref}(V)$. Then $|\mathbf{pref}(V^*) \cap X^l| \leq c \cdot r^{\alpha\cdot l}$ for some constant $c > 0$ and all $l \in \mathbb{N}$.*

Proof. First observe that $w \in V^*$ if and only if $w \in V^l$ for some $l \leq |w|$. Thus $\mathbf{pref}(V^*) \cap X^l = \mathbf{pref}(V^l) \cap X^l$.

Let $a := \sum_{v\in V} r^{-\alpha|v|}$. Since V is a code, we have $a^l = \sum_{v\in V^l} r^{-\alpha|v|} = \sum_{|w|=l, w\in\mathbf{pref}(V^*)} \big(r^{-\alpha\cdot l} \cdot \sum_{v\in V^l/w} r^{-\alpha|v|}\big)$.

Now, $V^l/w \supseteq V^{l-i_w+1}/w' \supseteq (V/w' \cdot V^{l-i_w})$ where $w = v_1 \cdots v_{i_w-1} \cdot w'$ with $v_j \in V$ and $w' \in \mathbf{pref}(V)$.

Thus, $\sum_{v\in V^l/w} r^{-\alpha|v|} \geq \sum_{v\in V/w'} r^{-\alpha|v|} \cdot a^{l-i_w} \geq c' \cdot a^{l-i_w} \geq c' \cdot a^l$ and we obtain $a^l \geq r^{-\alpha\cdot l} \cdot |\mathbf{pref}(V^*) \cap X^l| \cdot c' \cdot a^l$ what proves our assertion.

Now, the fact that $\mathbf{pref}(V^*)$ is computably enumerable if only V is computably enumerable yields our result.

Lemma 7. *Let $V \subseteq X^*$ be a computably enumerable code, α be right-computable and $\sum_{v\in V} r^{-\alpha\cdot|v|} = a \leq 1$. If $\inf\Big\{\sum_{v\in V/w} r^{-\alpha\cdot|v|} \mid w \in \mathbf{pref}(V)\Big\} > 0$ then there is a constant c such that*

$$\mathrm{KS}(\xi[0..n]) \leq \alpha\cdot n + c \textit{ for every } \xi \in \mathcal{C}(V^\omega)\,.$$

References

1. Calude, C. S. , L. Staiger and S. A. Terwijn, *On partial randomness*, Annals of Pure and Applied Logic **138** (2006), 20–30.
2. Falconer, K. J. , “Fractal Geometry: Mathematical Foundations and Applications”, John Wiley & Sons, 1990.
3. Kuich W., *On the Entropy of Context-Free Languages*, Information and Control, **16** (1970), 2, 173 – 200.
4. Li, M. and P. Vitányi, “An Introduction to Kolmogorov Complexity and Its Applications”, 2nd Edition, Springer Verlag, New York, 1997.
5. Mielke, J. , *Refined Bound on Kolmogorov Complexity for ω-languages*, `in:` Proceedings of the Fifth International Conference on Computability and Complexity in Analysis (CCA 2008), (V. Brattka, R. Dillhage, T. Grubba and A. Klutsch Eds.), Electronic Notes in Theoretical Computer Science, Vol. 221 (2008) 181 – 189.
6. Ryabko, B. Ya, *Coding of combinatorial sources and Hausdorff dimension*, Soviet. Math. Dokl. **30** (1984), 1, 219–222.

7. Staiger, L. , *Kolmogorov complexity and Hausdorff dimension*, Inform. and Comput. **103** (1993), 2, 159–194.
8. Staiger, L. , *Infinite Iterated Function Systems in Cantor Space and the Hausdorff Measure of ω-Power Languages*, Int. J. of Foundations of Computer Science, **16** (2005), 4, 787–802.
9. Staiger, L., *The Kolmogorov complexity of infinite words*, Theor. Comput. Sci. Vol. 383 (2007) 187 – 199.
10. Staiger, L. , *On Oscillation-free ε-random Sequences*, `in:` Proceedings of the Fifth International Conference on Computability and Complexity in Analysis (CCA 2008), (V. Brattka, R. Dillhage, T. Grubba and A. Klutsch Eds.), Electronic Notes in Theoretical Computer Science, Vol. 221 (2008) 287 – 297.
11. Uspensky, V. A. and A. Shen, *Relations Between Varieties of Kolmogorov Complexities*, Math. Systems Theory **29** (1996), 271–292.
12. Uspensky, V. A., Semenov, A. L. and A. Shen *Can an (individual) sequence of zeroes and ones be random?*, Uspekhi Mat. Nauk **45** (1990), 1, 105–162 [Russian].
13. Zvonkin, A. K. and L. A. Levin, *The Complexity of finite objects and the Development of the Concepts of Information and Randomness by means of the Theory of Algorithms*, Russian Math. Surveys **25** (1970), 6, 83–124.