How Discontinuous is Computing Nash Equilibria? (Extended Abstract)

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Abstract. We investigate the degree of discontinuity of several solution concepts from non-cooperative game theory. While the consideration of Nash equilibria forms the core of our work, also pure and correlated equilibria are dealt with. Formally, we restrict the treatment to two player games, but results and proofs extend to the n-player case. As a side result, the degree of discontinuity of solving systems of linear inequalities is settled.

Keywords. Game Theory, Computable Analysis, Nash Equilibrium, Discontinuity

1 Introduction

Both for applications and theoretical considerations, the algorithmic task of computing Nash equilibria from certain representations of games is of immense importance. A natural mathematical formulation of game theory uses the real numbers for payoffs and for mixed strategies, while classical models for algorithms require a restriction to countable sets. By imposing suitable restrictions and modifications to obtain countable problems, the complexity of computing a Nash equilibrium for a normal form game was proven to be PPAD-complete ([1], [2]).

Here we will use another approach: Instead of limiting the problem, we will extend the theory of computation. While the TTE-framework ([3]) is perfectly capable of formulating the task of computing Nash equilibria from normal form games, we will see that even the most trivial cases are discontinuous, and hence not computable.

To gain a deeper understanding of the problem, its degree of discontinuity will be studied. Mirroring an approach in the study of game theory using classical computational complexity, we will also examine other solution concepts such as correlated equilibria. While correlated equilibria seem to be computationally easier than Nash equilibria[1], we will prove that both concepts share a degree

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[1] In [4] several decision problems regarding Nash equilibria and correlated equilibria were compared, most of them turned out to be NP-hard for Nash equilibria and to be in P for correlated equilibria.

of discontinuity. Limitation to pure strategies yields a strictly less discontinuous problem, the classical problem can be solved by a cubic algorithm[2].

Due to space restrictions, most of the proofs are omitted here. A more comprehensive version including proofs is [10].

2   Preliminaries

2.1   Game Theory

An $n \times m$ bi-matrix game is simply given by two $n \times m$ real valued matrices A and B. Two players simultaneously pick an index, row player chooses an $i \in \{1, 2, \ldots, n\}$ and column player chooses an $j \in \{1, 2, \ldots, m\}$. Row player gets A_{ij} as a reward, column player gets B_{ij}. We consider several solution concepts defined as equilibria, where no player has an incentive to change her strategy unilaterally.

Definition 1. *A pure equilibrium for a $n \times m$ bi-matrix game (A, B) is a pair $(i, j) \in \{1, \ldots, n\} \times \{1, \ldots, m\}$ satisfying $A_{ij} \geq A_{kj}$ for all $k \in \{1, \ldots, n\}$ and $B_{ij} \geq B_{il}$ for all $l \in \{1, \ldots, m\}$.*

As pure equilibria do not exist for all games, a more general notion is introduced. If both players can randomize independently over their actions, one is led to the definition of an m-mixed strategy as an m-dimensional real valued vector s with non-negative coefficients and $\sum_{j=1}^{m} s_j = 1$. The set of m-mixed strategies will be denoted by S^m.

Definition 2. *A Nash equilibrium for an $n \times m$ bi-matrix game (A, B) is a pair $(\hat{x}, \hat{y}) \in S^n \times S^m$ satisfying $\hat{x}^T A \hat{y} \geq x^T A \hat{y}$ for all $x \in S^n$ and $\hat{x}^T B \hat{y} \geq \hat{x}^T B y$ for all $y \in S^m$.*

If $(\hat{x}, \hat{y})$ is a Nash equilibrium, again neither of the players can improve her payoff by changing her mixed strategy unilaterally. A famous result by JOHN NASH ([7]) established that Nash equilibria in bi-matrix games always exist. By identifying a pure strategy with the mixed strategy that puts weight 1 on it, pure equilibria can be considered a special case of Nash equilibria. An even more general solution concept can be obtained by allowing the individual player's randomization processes to be correlated ([8]).

Definition 3. *A correlated equilibrium for a $n \times m$ bi-matrix game is a real valued $n \times m$ matrix C with non-negative entries and $\sum_{i=1}^{n} \sum_{j=1}^{m} C_{ij} = 1$ so that*

$$\sum_{j=1}^{m} A_{ij} C_{ij} \geq \sum_{j=1}^{m} A_{lj} C_{ij}$$

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[2] There are, however, several interesting hardness results for finding pure equilibria in games ([5], [6]), originating in other representations or requiring additional properties.

holds for all $i, l \in \{1, 2, \ldots, n\}$ and

$$\sum_{i=1}^{n} B_{ij} C_{ij} \geq \sum_{i=1}^{n} B_{ik} C_{ij}$$

holds for all $j, k \in \{1, 2, \ldots, m\}$.

Given a Nash equilibrium (x, y), a correlated equilibrium can be constructed as $C_{ij} = x_i y_j$, while each correlated equilibrium of this form can be obtained from a Nash equilibrium, allowing us to consider Nash equilibria as special cases of correlated equilibria. Thus, finding a correlated equilibrium has to be easier than finding a Nash equilibrium, as we just presented a reduction.

Another way of creating an easier problem consists in a restriction of the games used. A zero-sum game is a bi-matrix game of the form $(A, -A)$.

2.2 Representing Games

In order to consider games as inputs to Type-2-Machines, they have to be coded into infinite sequences. The choice of the countable alphabet used is irrelevant for the theory, to simplify proofs we will use either $\{0, 1\}$ or $\mathbb{N}$, depending on the context. The degrees of discontinuity we study are those of the realizations, that is of functions turning names of instances into names of solutions. Since all occurring representations will be admissible, topological properties carry over between sets of games and sets of names for games, etc.

As games in normal form are pairs of real matrices, and (possible) equilibria pairs of real vectors (or again real matrices), one can quickly derive suitable representations by using product and coproduct representations ([3], [9]), starting from any representation of the real numbers.

The standard representation ρ of the real numbers is chosen for various reasons; it is admissible and provides a convincing class of computable functions, in contrast to some of the alternatives ([3], [11]). Additionally, as demonstrated in [12], the representation ρ is equivalent to the representation naturally arising for the results of repeated physical measurements. For defining ρ, we fix a bijection $\nu : \mathbb{N} \to \mathbb{Q}$ with $\nu(0) = 0$, so that all the usual operations on $\mathbb{Q}$ are computable w.r.t. ν.

Definition 4. *Let $\rho(w) = x \in \mathbb{R}$ hold for $w \in \mathbb{N}^{\mathbb{N}}$, if $|\nu(w(i)) - x| \leq 2^{-i}$ holds for all $i \in \mathbb{N}$.*

Definition 5. *Let w be a Γ-name for the bi-matrix game (A, B), if*

1. $w = 0^n 1^m 0 w_2$, *when (A, B) is an $n \times m$ game*
2. $w_2 = \langle w^a, w^b \rangle$, *where $\langle \ \rangle$ denotes the usual pairing function*
3. $w^a = \langle w_{11}^a, \ldots w_{n1}^a, w_{12}^a, \ldots, w_{nm}^a \rangle$
4. $w^b = \langle w_{11}^b, \ldots w_{n1}^b, w_{12}^b, \ldots, w_{nm}^b \rangle$
5. $\rho(w_{ij}^a) = A_{ij}$
6. $\rho(w_{ij}^b) = B_{ij}$

Representations for pure, Nash and correlated equilibria can be derived in the same fashion. Detailed definitions are omitted here.

2.3   Comparing Discontinuity

As games can have multiple equilibria, we do not consider a function assigning an equilibrium to each game, but rather a multi-valued function. We will identify a multi-valued functions with the set of its choice functions. To compare the discontinuity of such sets, Type-2-Reducibility as studied in e.g. ([14], [15], [16], [17], [25], [18], [9]) is used, as well as the Level of a function (or a set of functions), introduced in [17].

We use the following definition of Type-2-Reducibility:

Definition 6. *Let A, B be multi-valued functions. Then $A \leq_2 B$ holds, iff there are continuous partial functions F, G with $w \mapsto F(w, g(G(w))) \in A$ for each $g \in B$.*

As demonstrated in [9] (for suprema) and [13] (for infima), $\leq_2$ induces a completely distributive complete lattice. We use $\lceil P_n \rceil_{n \in \mathbb{N}}$ to denote the supremum of a countable family $(P_n)_{n \in \mathbb{N}}$. This allows to consider the degree of discontinuity of finding equilibria in any game as the supremum of the degrees of discontinuity of finding equilibria in games with fixed size.

As the Level will play only a minor role in our considerations, we refer to [9] for definitions.

3   Single Player Games and Pure Equilibria

From the perspective of game theory, single player games are trivial: The acting player chooses whatever action is best for her. As a discrete computation problem, this amounts to finding a maximum in a list of integers, a task that can be solved in linear time or logarithmic space. As the problem posed over the reals is discontinuous, we will study the problems $1\textsc{Pure}_n$ and $1\textsc{Pure}$ of finding pure equilibria in single player games with n actions and without fixed game sizes. It shall be noted that single player games can be identified with $n \times 1$ bi-matrix games, justifying their inclusion.

As every $n \times 1$ bi-matrix game has a pure equilibrium, and $C_{ij} > 0$ can only hold in a correlated equilibrium C, if the entry A_{i1} is maximal in A (and thus $(i, 1)$ is a pure equilibrium), finding pure, Nash and correlated equilibria is equivalent for single player games, so the restriction to pure equilibria does not invoke any loss of generality.

The degree of discontinuity of $1\textsc{Pure}_n$ turns out to be equivalent to another family of problems, $MLPO_n$, introduced in [14] as generalizations of the *lesser limited principle of omniscience* (LPO) studied in constructive mathematics ([19]).

Definition 7. *A function $f : \{(p_1, \ldots, p_n) \in (\mathbb{N}^{\mathbb{N}})^n \mid \exists i \leq n \ p_i = 0^{\mathbb{N}}\} \to \{1, 2, \ldots, n\}$ is in $MLPO_n$, if it fulfills $p_{f(p_1, p_2, \ldots, p_n)} = 0^{\mathbb{N}}$ for all valid $(p_1, p_2, \ldots, p_n)$.*

Theorem 1. $MLPO_n \equiv_2 1\textsc{Pure}_n$

In the next step, we extend the scope of consideration to finding pure equilibria in arbitrary bi-matrix games. The relevant problems are PURE_{nm}, where the size of the game is restricted to $n \times m$, and the general case denoted by PURE. For obtaining results, reducibility to $MLPO_n$ shall be expressed by a partition property:

Lemma 1. *Let H be a multi-valued function defined on a strongly zero-dimensional metrisable space*[3] *X. Then $H \leq_2 MLPO_n$ holds, iff there are n closed sets A_i, $i \leq n$ with $X = \bigcup_{i=1}^{n} A_i$, so that for each $i \leq n$, there is an $f^i \in H$ so that $f^i_{|A_i}$ is continuous.*

Theorem 2. $\text{PURE}_{nm} \leq_2 MLPO_{n*m}$.

Proof. Given an $n \times m$ bi-matrix game (A, B), the condition for the pair (i, j) to be a pure equilibrium is $A_{ij} \geq A_{kj}$ and $B_{ij} \geq B_{il}$ for all $k \leq n$, $l \leq m$. This implies that the set $P_{nm}^{ij} = \{(A, B) \mid (i, j) \text{ is an equilibrium of } (A, B)\} \subseteq \mathbb{R}^{nm} \times \mathbb{R}^{nm}$ is closed. Due to the admissibility of Γ, the set of corresponding names for the games is also closed. As the set of $n \times m$ bi-matrix games which have a pure strategy equilibrium is the union $\bigcup_{i \leq n, j \leq m} P_{nm}^{ij}$, an application of Lemma 1 yields the claim.

Corollary 1. $1\text{PURE} \equiv_2 \text{PURE}$.

Proof. As both problems are the respective limits, considering Theorems 1 and 2 is sufficient.

The same reasoning used to establish the equivalence of finding pure strategies in 1 player games and in 2 player games can directly be extended to any finite number of players. While Nash and correlated equilibria have the same degree of discontinuity as pure equilibria in single player games, we will continue to show that a higher degree of discontinuity emerges in the two player case.

4   Nash and correlated equilibria in bi-matrix games

We will now consider Nash and correlated equilibria in bi-matrix games. The problems CORR_{nm} and NASH_{nm} are the fixed size versions, CORR and NASH the general problems. An additional dimension of the problem is whether the games are zero-sum, yielding the problems ZCORR_{nm}, ZNASH_{nm} and the corresponding general problems. Straight-forward reasoning yields the reductions:

$$\text{ZCORR}_{nm} \leq_2 \text{CORR}_{nm} \leq_2 \text{NASH}_{nm} \quad \text{ZCORR}_{nm} \leq_2 \text{ZNASH}_{nm} \leq_2 \text{NASH}_{nm}$$

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[3] Examples for such spaces are $\{0, 1\}^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$ with their standard topologies. A brief characterization of strongly zero-dimensional metrisable spaces can be found in [9], for details we refer to [17] and [20].

4.1   The discontinuity of robust division

Similar to $MLPO_n$ being representative of the kind of discontinuity we face when searching for pure equilibria, we will start with considering division, which will turn out to be typical for correlated and Nash equilibria. Computing $\frac{a}{b}$ given two real numbers a, $b \neq 0$ is continuous, of course. However, testing whether $b \neq 0$ is not. A robust variant of division, which accepts division by zero and returns an arbitrary value, is not continuous anymore:

Definition 8. *Let* RDIV *be the set of functions d defined on* $\{(u, v) \mid 0 \leq \rho(u) \leq \rho(v)\}$ *satisfying* $\rho(d(u,v)) = \frac{\rho(u)}{\rho(v)}$ *for $\rho(v) > 0$.*

While $\mathrm{Lev}(\text{RDIV}) = 2$ establishes robust division as an only slightly discontinuous problem, the following result shows that robust division introduces a new kind of discontinuity not present in finding pure equilibria.

Theorem 3. RDIV $\not\leq_2$ PURE.

We will now use modifications of the game *matching pennies* as a gadget to implement divisions in a game.

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad B = -A \quad MP(a,b) = (A, B)$$

If both $a > 0$ and $b > 0$, the unique correlated equilibrium is obtained from the unique Nash equilibrium $x = y = (\frac{b}{a+b}, \frac{a}{a+b})$. If $a = 0$, $b > 0$, then (x, y) is an equilibrium, iff $y = (1, 0)$, and for $a > 0$, $b = 0$ we have $y = (0, 1)$.

Theorem 4. RDIV $\leq_2$ ZCORR$_{22}$

Proof. Given a pair of ρ-names for real numbers a, b with $0 \leq a \leq b$, a name for the game $MP(a, b - a)$ can be computed. A correlated equilibrium C of $MP(a, b - a)$ has the form:

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} xy & x(1-y) \\ (1-x)y & (1-x)(1-y) \end{pmatrix}$$

Thus, one can obtain $c_{11} + c_{21} = y = \frac{a}{b}$ for $b > 0$.

Theorem 4 in conjunction with Theorem 3 implies ZCORR$_{22}$ $\not\leq_2$ PURE, so even the simplest case of finding mixed strategies is not reducible to finding pure strategies. The problem RDIV itself cannot capture the discontinuity of finding Nash equilibria, due to $\mathrm{Lev}(\text{ZNASH}_{22}) = 4$ (s. Subsection 5.2), compelling us to derive a sequence of problems with increasing level from RDIV.

4.2   Products of Problems and Products of Games

The product of functions can be considered as computing all of them in parallel. This will allow us to specify exactly the degree of discontinuity of problems solvable by multiple robust divisions, once we defined products for multi-valued functions. The following definitions and results on the products of multi-valued functions and their discontinuity extend corresponding results from [18].

Definition 9. *For functions $f : X \to Y$, $g : U \to V$, define $\langle f, g \rangle : (X \times U) \to (Y \times V)$ through $\langle f, g \rangle(x, u) = (f(x), g(u))$. Define $\langle f \rangle^1 = f$ and $\langle f \rangle^{n+1} = \langle f, \langle f \rangle^n \rangle$.*

Definition 10. *For relations P, Q, define $\langle P, Q \rangle = \{ \langle f, g \rangle \mid f \in P, g \in Q \}$. Define $\langle P \rangle = P$ and $\langle P \rangle^{n+1} = \langle P, \langle P \rangle^n \rangle$.*

$\lceil P, Q \rceil \leq_2 \langle P, Q \rangle$ holds, but the converse is false in general. If $f \leq_2 g$ holds, then also $\langle f, h \rangle \leq_2 \langle g, h \rangle$. As $\langle \ \rangle$ is associative, it can be extended to any finite number of arguments in the standard way. There is a useful distributive law for $\lceil \ \rceil$ and $\langle \ \rangle$ which we will state as $\langle P, \lceil Q_i \rceil_{i \in \mathbb{N}} \rangle \equiv_2 \lceil \langle P, Q_i \rangle \rceil_{i \in \mathbb{N}}$.

For games, our notion of a product will be inspired by the model of playing two independent games at once. This will allow us to establish a link between products of relations and products of games. We will use $[\ ]$ to denote a bijection between $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$ and $\{1, 2, \ldots, nm\}$ for suitable n, m.

Definition 11. *Given an $n_1 \times m_1$ bi-matrix game (A^1, B^1) and an $n_2 \times m_2$ bi-matrix game (A^2, B^2), we define the $(n_1 n_2) \times (m_1 m_2)$ product game (A, B) through $A_{[i_1, i_2][j_1, j_2]} = A^1_{i_1 j_1} + A^2_{i_2 j_2}$ and $B_{[i_1, i_2][j_1, j_2]} = B^1_{i_1 j_1} + B^2_{i_2 j_2}$.*

The product of two games is a constant-sum game, iff both games are constant-sum[4]. If (x^1, y^1) is an equilibrium (either pure or Nash) of (A^1, B^1), and (x^2, y^2) is an equilibrium of (A^2, B^2), then (x, y) is an equilibrium (of the same type) of the product game where $x_{[i_1 i_2]} = x^1_{i_1} x^2_{i_2}$ and $y_{[m_1 m_2]} = y^1_{m_1} y^2_{m_2}$. Conversely, if (x, y) is an equilibrium of the product game, an equilibrium (x^1, y^1) for (A^1, B^1) can be obtained through $x^1_i = \sum_{k=1}^{n_2} x_{[i,k]}$ and $y^1_j = \sum_{l=1}^{m_2} y_{[j,l]}$, analogously an equilibrium (x^2, y^2) for (A^2, B^2) can be computed. Analogous statements hold for correlated equilibria.

As the product game can be computed from the constituent games, we can use the properties of the products of games to obtain the following results regarding the problem of finding equilibria:

Theorem 5. *Let* GAME $\in$ {PURE, ZCORR, ZNASH, CORR, NASH}. *Then* $\langle \text{GAME}_{nm}, \text{GAME}_{kl} \rangle \leq_2 \text{GAME}_{(nk),(ml)}$.

Theorem 6. *Let* GAME $\in$ {PURE, ZCORR, ZNASH, CORR, NASH}. *Then* $\langle \text{GAME} \rangle^n \equiv_2 \text{GAME}$ *for all $n \in \mathbb{N}$.*

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[4] As equilibria finding for constant-sum games is trivially equivalent to equilibria finding for zero-sum games, this is sufficient for our purposes.

The present paper contains two results interpretable as counterparts to Theorem 5, as they allow to reduce finding equilibria for a large game to finding equilibria in several smaller games; for mixed strategies, this will be a consequence of the main result presented in Subsection 4.3, the corresponding statement for pure strategies is given in the next theorem:

Theorem 7. $1\textsc{Pure}_{n+1} \leq_2 \langle MLPO_2 \rangle^n$.

As we have identified $MLPO_2$ (or $1\textsc{Pure}_2$) as the basic building stone in the degree of discontinuity of finding pure strategies, the following theorem will establish the missing link in the relationship between finding pure strategies and multiple robust divisions:

Theorem 8. $MLPO_2 <_2 \textsc{rDiv}$.

To sum up the results established sofar, we have:

$$\lceil \langle 1\textsc{Pure}_2 \rangle^n \rceil_{n \in \mathbb{N}} \equiv_2 1\textsc{Pure} \equiv_2 \textsc{Pure} <_2 \lceil \langle \textsc{rDiv} \rangle^n \rceil_{n \in \mathbb{N}} \leq_2 \textsc{ZCorr}$$

4.3   Problems reducible to $\lceil \langle \mathbf{rDiv} \rangle^n \rceil_{n \in \mathbb{N}}$

The goal of this subsection is to present a way of designing reductions to $\lceil \langle \textsc{rDiv} \rangle^n \rceil_{n \in \mathbb{N}}$, and, in particular, to present a reduction from $\textsc{Nash}$. This equivalently can be considered as the task to design algorithms for a Type-2-Machine capable of making a finite number of independent queries to an oracle for $\textsc{rDiv}$. Due to Theorems 7, 8 also oracle calls to $MLPO_n$ are permitted.

We will start by providing a technical lemma similar to Lemma 1. Using the lemma, we can prove that the Fourier-Motzkin-algorithm ([21]) for solving systems of linear inequalities can be executed using continuous (even computable) operations and oracle calls to $\textsc{rDiv}$.

Lemma 2. *Let F be a multi-valued function defined on a strongly zero-dimensional metrisable space X. Then $F \leq_2 \lceil \langle \textsc{rDiv} \rangle^n \rceil_{n \in \mathbb{N}}$ holds, iff there are k closed sets A_i, $i \leq k$ with $X = \bigcup_{i=1}^{k} A_i$, so that for each $i \leq k$, there is a multi-valued function $G^i \leq_2 \lceil \langle \textsc{rDiv} \rangle^n \rceil_{n \in \mathbb{N}}$ with $\mathrm{dom}(G^i) = X$, so that for each $g^i \in G^i$ there is an $f^i \in F$ with $f^i_{|A_i} = g^i_{|A_i}$.*

Definition 12. *The problem $\textsc{BLinIneq}_{nm}$ asks for a ρ^m-name of a vector v of reals, so that $Av \leq b$ holds in addition to $0 \leq v \leq 1$, given a ρ^{nm}-name for a matrix A and a ρ^n-name for a vector b, provided that a solution exists. For simplicity, we assume that $Av \leq b$ always contains $0 \leq v \leq 1$. $\textsc{BLinIneq}$ is the problem without fixed values n, m.*

Theorem 9. $\textsc{BLinIneq} \leq_2 \lceil \langle \textsc{rDiv} \rangle^z \rceil_{z \in \mathbb{N}}$.

Proof. As BLinIneq is expressible as a supremum, it suffices to prove $\text{BLinIneq}_{nm} \leq_2 \lceil \langle \text{RDiv} \rangle^z \rceil_{z \in \mathbb{N}}$ for all $n, m \in \mathbb{N}$. For this, we use induction over m. The case $m = 0$ is trivial, so we assume $\text{BLinIneq}_{n(m-1)} \leq_2 \lceil \langle \text{RDiv} \rangle^z \rceil_{z \in \mathbb{N}}$.

For each $K \subseteq \{1, \ldots, n\}$, abbreviate $K^C := \{1, \ldots, n\} \setminus K$. The set $D_K = \{(A, b) \mid \forall k \in K \ a_{k1} \geq 0 \wedge \forall l \in K^C \ a_{l1} \leq 0\}$ is closed, and the union $\bigcup_{K \subseteq \{1, \ldots, n\}} D_K$ covers the domain of BLinIneq_{nm}. So due to Lemma 2, it is sufficient to show that BLinIneq_{nm} restricted to D_K is reducible to $\lceil \langle \text{RDiv} \rangle^z \rceil_{z \in \mathbb{N}}$ for arbitrary $K \subseteq \{1, \ldots, n\}$. In the next step we assume K to be fixed. With the same argument we can assume $|a_{k1}| \geq |a_{(k+1)1}|$ by renumbering the inequalities for each fixed sequence of increasing first coefficients.

Now we rewrite the given inequalities as $a_{k1} v_1 \leq b_k - \sum_{i=2}^{m} a_{ki} v_i$ for $k \in K$ and $-b_j + \sum_{i=2}^{m} a_{ji} v_i \leq -a_{j1} v_1$ for $j \in K^C$. For each pair $k \in K$, $j \in K^C$, the corresponding inequalities can be multiplied by $-a_{j1}$ respective a_{k1}, and then contracted to:

$$a_{k1}\left(-b_j + \sum_{i=2}^{m} a_{ji} v_i\right) \leq -a_{j1}\left(b_k - \sum_{i=2}^{m} a_{ki} v_i\right)$$

Every solution to the newly created system of linear inequalities can be extended to a solution to the original system by choosing a suitable value for v_1. Due to the induction assumption, such a solution can be obtained by making oracle calls to $\lceil \langle \text{RDiv} \rangle^z \rceil_{z \in \mathbb{N}}$.

To obtain a solution for v_1, we would like to call

$$v_1 = \max(0, \min(1, \text{op}_1(\text{RDiv}(|b_1 - \sum_{i=2}^{m} a_{1i} v_i|, |a_{11}|), \text{op}_2(|\text{RDiv}(b_2 - \sum_{i=2}^{m} a_{2i} v_i|, |a_{21}|), \ldots$$

with $\text{op}_i = \min$ for $i \in K$ and $\text{op}_i = \max$ else. As the $|a_{k1}|$ are ordered as a decreasing sequence, values that arise arbitrary as result of a division by zero occur deeper inside the nested structure than significant values. While they can influence the actual value for v_1 that is chosen, it still satisfies all inequalities, if this is possible. However, the expression above contains nested calls to RDiv in form of the v_i, $2 \leq i \leq n$.

To solve the problem, one replaces v_2 with the corresponding sequence used to compute it, then v_3, and so on. By moving the max and min operators outside, and unifying all divisions, terms of the form $\text{RDiv}(P, Q)$ remain, where P is a polynomial in a_{ij}, b_j whose degree does not exceed $2n$, and Q is a polynomial in a_{ij} whose degree does not exceed n. These can be evaluated by allowed oracle calls, and the max and min operators are continuous.

As the problem BLinIneq is of considerable interest on its own, we shall note that the converse statement to Theorem 9 is also true:

Theorem 10. $\lceil \langle \text{RDiv} \rangle^z \rceil_{z \in \mathbb{N}} \leq_2 \text{BLinIneq}$.

By adapting [22, Algorithm 3.4] and applying Lemma 2 and Theorem 9 we proceed to prove the main theorem of this subsection. Again, the reasoning directly extends to more than two players.

Theorem 11. $\text{NASH} \leq_2 \lceil \langle \text{RDIV} \rangle^z \rceil_{z \in \mathbb{N}}$.

Proof. By the same reasoning as above, since NASH is the supremum $\lceil \text{NASH}_{nm} \rceil_{n,m \in \mathbb{N}}$, it suffices to show $\text{NASH}_{nm} \leq_2 \lceil \langle \text{RDIV} \rangle^z \rceil_{z \in \mathbb{N}}$ for arbitrary $n, m \in \mathbb{N}$.

By the best response condition ([22, Proposition 3.1]), a pair of mixed strategies (x, y) is a Nash equilibrium of a game if each pure strategy played with positive probability in x (in y) is a best response against y (against x). This condition can be formalized by noting that the following set is the set of games and their Nash equilibria with support in I, J:

$$\hat{G}_{I,J} = \begin{cases} (A, B, x, y) \mid j, k \in J \ l \notin J \ (x^T B)_j = (x^T B)_k \geq (x^T B)_l \ y_l = 0 \ i, p \in I \\ q \notin I \ (Ay)_i = (Ay)_p \geq (Ay)_q \ x_q = 0 \end{cases}$$

The set $\hat{G}_{I,J}$ is closed, and so is its projection $G_{I,J} = \{(A, B) \mid \exists x, y \ (A, B, x, y) \in \hat{G}_{I,J}\}$.

As every game has a Nash equilibrium, the sets $G_{I,J}$ cover the domain of NASH, so we can apply Lemma 2. To recover the Nash equilibrium (x, y) from I, J the corresponding system of linear inequalities has to be solved, which is reducible to $\lceil \langle \text{RDIV} \rangle^z \rceil_{z \in \mathbb{N}}$ as established in Theorem 9.

Corollary 2. $\text{ZCORR} \equiv_2 \text{CORR} \equiv_2 \text{ZNASH} \equiv_2 \text{NASH} \equiv_2 \lceil \langle \text{RDIV} \rangle^n \rceil_{n \in \mathbb{N}}$.

The same technique applied in the proof of Theorem 9 can also be used to show that Gaussian Elimination can be reduced to $\lceil \langle \text{RDIV} \rangle^n \rceil_{n \in \mathbb{N}}$. This shows that the reduction of Gaussian Elimination to the rank of a matrix given in [23] is strict, taking into consideration Corollary 4.

5   Additional Results

5.1   Nash and *Sep*

To shed further light on the degree of discontinuity of NASH, we will compare it to the problem *Sep* studied in [24].

Definition 13. $f \in Sep$ *holds, iff* f *is a function from*

$$\{(p, q) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \mid \forall n, m \in \mathbb{N} \ p(n) \neq q(m)\}$$

to $\mathbb{N}^{\mathbb{N}}$ *satisfying* $f(p(n)) = 0$ *and* $f(q(n)) = 1$ *for all* $n \in \mathbb{N}$.

The problem *Sep* was shown to be equivalent to finding an infinite path in an infinite binary tree and extending a linear functional from a subspace of a Banach space to the complete space following the Hahn-Banach Theorem. *Sep* can be

reduced to $\{C_1\}$, which is defined through $C_1(p)(n) = 1$, iff there is an $i \in \mathbb{N}$ with $p(i) = n$ and $C_1(p)(n) = 0$ else. The function C_1 has been introduced in [16]. In [25, Theorem 5.5], it was proven that a function is $\sum_2^0$-measurable, iff it is reducible to C_1.

In [24], $\{cf\} \not\leq_2 Sep$ was shown, which can directly to extended to prove $\{f\} \not\leq_2 Sep$ for all discontinuous functions f. In the following, we will prove that $Nash$ is strictly reducible to Sep, thereby obtaining a lower bound for Sep. For this aim, we need the level of Sep.

Theorem 12. $\mathrm{Lev}^2(Sep)$ *does not exist.*

Due to the behaviour of the level under formation of products ([18]) and suprema ([9], [17]), we know $\mathrm{Lev}^2(\textsc{Nash}) = \omega$, where ω is the smallest infinite ordinal. This is sufficient to establish $Sep \not\leq_2 \textsc{Nash}$ by [9, Theorem 5.7].

Theorem 13. $\textsc{rDiv} \leq_2 Sep$.

Theorem 14. $\langle Sep, Sep \rangle \equiv_2 Sep$.

Corollary 3. $\textsc{Nash} <_2 Sep$.

Corollary 4. $\{f\} \not\leq_2 \textsc{Nash}$ *for all discontinuous functions f.*

5.2 The Level of Nash$_{22}$

The simplest non-trivial bi-matrix games, 2×2 games, have already been investigated from a constructive point of view in [26]. Among other results, [26] contains the constructive analogue to the reduction $MLPO_2 \leq_2 \textsc{Nash}_{22}$, and the constructive analogue to determine a subset of $\mathcal{L}_0(\textsc{Nash}_{22}) \setminus \mathcal{L}_1(\textsc{Nash}_{22})$, that is the set where Nash equilibria are continuous. We will produce the TTE-counterpart by investigating the Level of $\textsc{Nash}_{22}$.

Theorem 15. $\mathrm{Lev}(\textsc{Nash}_{22}) = 4$.

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