Covering of ordinals

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ABSTRACT. The paper focuses on the structure of fundamental sequences of ordinals smaller than ε_0. A first result is the construction of a monadic second-order formula identifying a given structure, whereas such a formula cannot exist for ordinals themselves. The structures are precisely classified in the pushdown hierarchy. Ordinals are also located in the hierarchy, and a direct presentation is given.

A recurrent question in computational model theory is the problem of model checking, i.e. the way to decide whether a given formula holds in a structure or not. When studying infinite structures, first-order logic only brings local properties whereas second-order logic is most of the time undecidable, so monadic second-order logic or one of its variants is often a balanced option. In the field of countable ordinals, results of Büchi [3] and Shelah [15] both brought decidability of the monadic theory via different ways. This positive outcome is tainted with the following property : the monadic theory of a countable ordinal only depends on a small portion of it, called the ω-tail [3, Th. 4.9]. In other words, many ordinals greater than ω^ω share the same monadic theories and cannot be distinguished.

Another class of structures enjoying a decidable monadic second-order theory is the pushdown hierarchy [6], which takes its source in the Muller and Schupp characterization of transition graphs of pushdown automata [11]. In the same way, each level of the hierarchy has two characterizations : an internal by higher-order pushdown automata [4], and an external presentation by graph transformations [5]. This paper will use the latter by the means of monadic interpretation and treegraph operations.

The original motivation of this paper was the localization of ordinals smaller than ε_0 in the hierarchy. Because of the above property, ordinals themselves are not easy to manipulate with monadic interpretations. There is therefore a need of structures as expressive as ordinals (in terms of interpretations) but having additional properties, such as the existence of a monadic formula precisely identifying the structure.

A well-known object answers to this request. Each countable limit ordinal may be defined as the limit of a so-called fundamental sequence. For ordinals smaller than ε_0, it is easy to have a unique definition for this sequence using the Cantor normal form. We note $\alpha \prec \beta$ when α is in the fundamental sequence of β or $\alpha + 1 = \beta$. When restricted to ordinals smaller than λ, we call the resulting structure the covering graph of λ. In Section 2, we present precisely this structure and give some of its properties. In particular, the out-degree of its vertices is studied intensively. This eventually yields a specific formula for each covering graph.

Section 3 locates the covering graph of any ordinal α smaller than ε_0 in the level n of the hierarchy, where n is the largest size of the ω-tower smaller than α. The result also applies to ordinals themselves. This was already shown for ordinals up to ω^{ω^ω} in [1]. In Section 4, the result in strengthened by proving that covering graphs are not in the lower levels; the

question is still open for ordinals. Eventually, we produce a direct presentation for towers of ω through prefix-recognizable relations of order n, but involving a more technical proof.

Similar attempts of characterization of ordinals has been made in the field of automaticity [8, 10], but in the other way around : word- and tree-automatic ordinals are shown to be respectively less than ω^ω and ω^{ω^ω}.

1 Definitions

In this paper, ordinals are often considered from a graph theory point of view. The set of vertices of α is the set of ordinals smaller than α, and the set of arcs is the relation $<$.

1.1 Graphs

Graphs are finite or infinite sets of labeled arcs. A Σ-graph is a set $G \subseteq V \times \Sigma \times V$, where V (or V_G if unclear) is the *support*, i.e. a finite or countably infinite set of *vertices*, and Σ a finite set of *labels*. An element (p, a, q) of G is called an *arc* and noted $p \xrightarrow{a} q$. Each label $a \in \Sigma$ is associated to a relation $R_a = \{(p, q) \mid p \xrightarrow{a} q\}$ on V. A finite sequence of arcs $p \xrightarrow{a_1} \ldots \xrightarrow{a_n} q$ is a *path* and noted $p \xrightarrow{a_1 \ldots a_n} q$. This is extended to languages with $p \xrightarrow{L} q$ iff $\exists u \in L$ such that $p \xrightarrow{u} q$. Isomorphism between graphs is noted $\simeq$.

The *monadic second-order* (MSO) logic is defined as usual; see for instance [9]. We take a set of (lowercase) first-order variables and a set of (uppercase) second-order variables. For a given set of labels Σ, atomic formulas are $x \in X$, $x = y$ and $x \xrightarrow{a} y$ for all $a \in \Sigma$ and x, y, X variables. Formulas are then closed by the propositional connectives $\neg, \wedge$ and the quantifier $\exists$. Graphs are seen as relational structures over the signature consisting of the relations $\{R_a\}_{a \in \Sigma}$. The set of closed monadic formulas satisfied by a graph G is noted $\mathrm{MTh}(G)$.

Given a binary relation R, the *in-degree* (respectively *out-degree*) of x is the cardinality of $\{y \mid yRx\}$ (resp. $\{y \mid xRy\}$). The output degree in a graph G of $x \in V$ is the cardinal of $\{y \mid \exists a, (x, a, y) \in G\}$. The output degree of a graph is the maximal output degree of its vertices if it exists.

1.2 Ordinals

For a general introduction to ordinal theory, see [14, 13]. An order is a well-order when each non-empty subset has a smallest element. Ordinals are well-ordered by the relation $\in$, and satisfy $\forall x (x \in \alpha \Rightarrow x \subset \alpha)$. Since any well-ordered set is isomorphic to a unique ordinal, we will often consider an ordinal up to isomorphism. In terms of graphs, the set of labels of an ordinal is a singleton often noted $\Sigma = \{<\}$ and the graph respects the following monadic properties :

$$
\begin{array}{ll}
\text{(strict order)} & \left\{ \begin{array}{c} \forall p, q (\neg(p \xrightarrow{<} q \wedge q \xrightarrow{<} p)) \\ \forall p, q, r((p \xrightarrow{<} q \wedge (q \xrightarrow{<} r) \Rightarrow p \xrightarrow{<} r) \end{array} \right. \\
\text{(total order)} & \forall p, q(p \xrightarrow{<} q \vee q \xrightarrow{<} p \vee p = q) \\
\text{(well order)} & \forall X \neq \varnothing \; \exists x (x \in X \wedge \forall y (y \in X \Rightarrow (x \xrightarrow{<} y \vee x = y)))
\end{array}
$$

The ordinal arithmetics define operations on ordinals such as addition, multiplication, exponentiation. The bound of ordinals investigated here is ε_0, the smallest ordinal such that $\varepsilon_0 = \omega^{\varepsilon_0}$; therefore **the declaration "$< \varepsilon_0$" is implicit through the rest of the paper**. To simplify the writing of towers of ω, the notation $\Uparrow$ is used to note the iteration of exponentiation ie. $a \Uparrow b = a^{a^{\cdots^{a}}} \Big\} b$ times. In particular, $a \Uparrow 0 = 1$ is the (right) exponentiation identity.

Classic operations are not commutative in ordinal theory : for instance $\omega + \omega^2 = \omega^2 < \omega^2 + \omega$. This leads to many writings for a single ordinal. Fortunately, all ordinals smaller than ε_0 may uniquely be written in the Cantor normal form (CNF)

$$\alpha = \omega^{\alpha_0} + \cdots + \omega^{\alpha_k}$$

where $\alpha_k \leq \cdots \leq \alpha_0 < \alpha$. An alternative we will call *reduced* Cantor normal form (RCNF) is $\alpha = \omega^{\alpha_0}.c_0 + \cdots + \omega^{\alpha_k}.c_k$ where $\alpha_k < \cdots < \alpha_0 < \alpha$ and $c_1, \ldots, c_k$ are non-zero integers. To express ordinals smaller than ε_0 from natural numbers and ω, the only operations needed are thus addition and exponentiation.

2 Covering graphs

In this section, we define the covering graph of an ordinal as the graph of successor and fundamental sequence relations. Then, we prove some of its important properties. One of them is the finite degree property, which is worked out to bring a specific monadic formula for each covering graph, thus allowing to differentiate them.

2.1 Fundamental sequence

The cofinality [14] of any countable ordinal is ω. To each limit ordinal α we may associate a ω-sequence whose bound is α. For $\alpha \leq \varepsilon_0$, $\alpha = \beta + \omega^\gamma$ with $\beta < \alpha$, $\gamma < \alpha$ and ω^γ is the last term in the CNF of α, we define the fundamental sequence $(\alpha[n])_{n<\omega}$ as follows :

$$\alpha[n] = \begin{cases} \beta + \omega^{\gamma'}.(n+1) & \text{if } \gamma = \gamma' + 1 \\ \beta + \omega^{\gamma[n]} & \text{otherwise.} \end{cases}$$

We define $\alpha' \prec \alpha$ whenever there is k such that $\alpha' = \alpha[k]$, or if $\alpha' + 1 = \alpha$.

For instance, the fundamental sequence of ω is the sequence of integers starting from 1. The sequence of ω^ω is therefore $(\omega, \omega^2, \omega^3, \ldots)$. The fundamental sequence merged with the successor relation yields for instance

$$0 \prec 1 \prec \omega \prec \omega + 1 \prec \omega.2 \prec \omega^2 \prec \omega^\omega.$$

Taking the transitive closure of this relation gives back the original order, so there no information loss.

Lemma 1 *The transitive closure of $\prec$ is $<$.*

Moreover, the relation is *crossing-free* as described below, which is a helpful technical tool.

Lemma 2 *If $\alpha_1 < \lambda_1 < \alpha_2$, $\alpha_1 \prec \alpha_2$ and $\lambda_1 \prec \lambda_2$, then $\lambda_2 \leq \alpha_2$.*

This is the forbidden case :

$$\alpha_1 \qquad \lambda_1 \qquad \alpha_2 \qquad \lambda_2$$
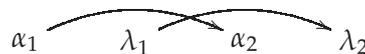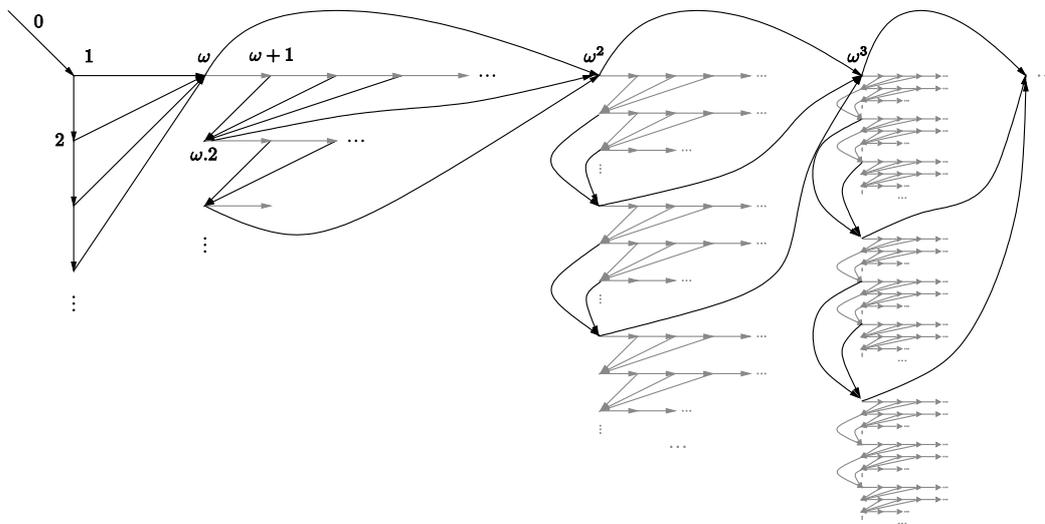
Figure 1: covering graph of ω^ω.

2.2 Covering graphs

Let $\mathcal{G}_\alpha = \{\lambda_1 \prec \lambda_2 \mid \lambda_1, \lambda_2 < \alpha\}$ be the graph of successor and fundamental sequence relation, or *covering graph* of the ordinal α. For instance, a representation of $\mathcal{G}_{\omega^\omega}$ is given in Figure 1.

We first remark the finite out-degree of the covering graphs.

Lemma 3 *For any $\omega \Uparrow (n-1) < \alpha \leq \omega \Uparrow n$ and $n > 0$, the out-degree of $\mathcal{G}_\alpha$ is n.*

In the following, we refine this property to get a characterisation of an ordinal by the degree of its vertices. We define the *degree word $u(\alpha)$* of a covering graph as follows. Consider the *greatest sequence σ* of $\mathcal{G}_\alpha$ starting from 0, i.e. $\sigma_0 = 0$ and for $k \geq 0$, σ_{k+1} is the greatest such that $\sigma_k \prec \sigma_{k+1}$. The previous lemma ensures that $\{\lambda \mid \sigma_k \prec \lambda\}$ is finite, so σ_{k+1} exists. Such a sequence may be finite.

The degree word $u(\alpha)$ is a finite or infinite word over $[0, n]$ when $\alpha \leq \omega \Uparrow n$, and its kth letter is the out-degree of σ_k in $\mathcal{G}_\alpha$.

For instance, consider $u(\omega^\omega)$. Its greatest sequence is $(0, 1, \omega, \omega^2, \omega^3, \ldots)$, where all have degree 2 in $\mathcal{G}_{\omega^\omega}$ except the first; so $u(\omega^\omega) = 12^\omega$. Now consider $u(\omega^3 + \omega^2)$: the sequence is now

$$0, 1, \omega, \omega^2, \omega^3, \omega^3 + 1, \omega^3 + \omega, \omega^3 + \omega + 1, \ldots$$

which loops into $(\ldots, \omega^3 + \omega.k, \omega^3 + \omega.k + 1, \ldots)$ so $u(\omega^3 + \omega^2) = 12221(21)^\omega$.

Lemma 4 *For any $\alpha \leq \omega \Uparrow n$, if α is successor then $u(\alpha)$ is a finite word of $[0, n]^*$; otherwise $u(\alpha)$ is an ultimately periodic word of $[1, n]^\omega$.*

PROOF (SKETCH). If α is successor, then since the greatest sequence is unbounded, the predecessor of α is in it and the word is finite. Otherwise, we prove that $\alpha[k]$ is in the greatest sequence of α for all finite k. The sequence of degrees from 0 to $\alpha[0]$ forms the static part of the ultimately periodic word, whereas the sequences of degrees between $\alpha[k]$ and $\alpha[k+1]$ are always the same. ∎

Let $<^n_{lex}$ be the lexicographic ordering on words on $[0, n]$ based on standard order. Degree words differ for each ordinal.

Lemma 5 *If $\alpha < \alpha' \leq \omega \Uparrow n$, then $u(\alpha) <^n_{lex} u(\alpha')$.*

PROOF.   Consider $n > 0$, otherwise its degree word of α is the empty word. As before, note that the greatest sequence is unbounded, and that $\sigma_0 = \sigma'_0 = 0$. Thus if $0 < \alpha < \alpha'$ and σ' is the greatest sequence of $\mathcal{G}_{\alpha'}$, there is a smallest $n > 0$ such that $\sigma_n \neq \sigma'_n$, or σ_n doesn't exist whereas σ'_n does. In both cases, the output degree of σ_{n-1} is less in $\mathcal{G}_\alpha$ than in $\mathcal{G}_{\alpha'}$, so $u(\alpha) <^n_{lex} u(\alpha')$.   ∎

A ultimately periodic pattern can be captured by a monadic formula. This is the goal of the the following lemma.

Lemma 6 *For each finite or infinite word u over $[0, n]$ and a given ordinal α, there is a monadic formula φ^u such that $\mathcal{G}_\alpha \models \varphi^u$ iff $u = u(\alpha)$.*

PROOF.   The fact that the degree word is finite or ultimately periodic permits to use a finite number of variables. We consider the ultimately periodic case, and $u(\alpha) = uv^\omega$.

To simplify the writing, we consider the following shortcuts :

- $\tau(p, q)$ stands if q is the greatest such that $p \prec q$;
- if the output degree of p is k, then $\partial_k(p)$ is true;
- root(X, p) and end(X, p) are true when p is co-accessible (resp. accessible) from each vertex of X, with the entire path in X; root(p) looks for a root of the whole graph;
- inline(X) checks that X is a finite or infinite path;
- size$_k(X)$ stands for $|X| = k$.

All these notations stand for monadic formulas. For instance, the inline(X) property is true when there is a root in X and each vertex has output degree 1, and each except the root has input degree 1.

Now we may write the formula φ^u. For this, we need two finite sets $p_1 \ldots p_{|u|} \in U$ for the static part, $q_1 \ldots q_{|v|} \in V'$ for the beginning of the periodic part and an infinite set V with $V' \subseteq V$. We check that p_1 is the general root 0, and q_1 the root of V, which is an infinite path. Formulas τ and ∂_k force the degree of the uv part. For the periodic part, each $q \in V$ there must be the root of a finite path $X_q \subseteq V$ of size $|v| + 1$, which end has the same degree that q.   ∎

The combination of Lemmas 5 and 6 yields the following theorem.

Theorem 7 *For $\alpha \neq \alpha'$ smaller than ε_0, we have* MTh$(\mathcal{G}_\alpha) \neq$ MTh$(\mathcal{G}_{\alpha'})$.

As a consequence, there is no generic monadic interpretation (see next section for definition) from an ordinal greater than ω^ω to its covering graph. Below this limit, there is an interpretation, because it is possible to distinguish successive limit ordinals.

3 The pushdown hierarchy

In this section, the pushdown hierarchy will only be defined by monadic interpretations and the treegraph operation. For other definitions, see for instance [4]. In particular, each level can be defined as the set of transition graphs (up to some closure operation) of finite-state higher-order pushdown automata of level n (*n-hopda*), hence the name.

A major property shared by this class of graphs is the decidability of their monadic theories. Since it is also the case for countable ordinals [15, 3], it is natural to examine the intersection. Here, covering graphs and ordinals are located at each level of the hierarchy.

3.1   Definitions

A *monadic interpretation I* is a finite set $\{\varphi_a(x,y)\}_{a \in \Gamma}$ of monadic formulas with two free first order variables. The interpretation of a graph $G \subseteq V \times \Sigma \times V$ by I is a graph $I(G) = \{p \xrightarrow{a} q \mid p,q \in V \wedge G \vDash \varphi_a(p,q)\} \subseteq V \times \Gamma \times V$. It is helpful to have $\Gamma = \Sigma$ to allow iteration process. The set of monadic interpretations $\mathcal{I}$ is closed by composition.

A particular case of monadic interpretation is inverse rational mapping. The alphabet $\bar{\Sigma}$ is used to read the arcs backwards : $p \xrightarrow{\bar{a}} q$ iff $q \xrightarrow{a} p$. An inverse rational mapping is an interpretation such that $\varphi_a(p,q) := p \xrightarrow{L_a} q$ where L_a is a regular language over $\Sigma \cup \bar{\Sigma}$.

For instance, the transitive closure of R_a for a label a is a monadic interpretation. By Lemma 1, there is therefore an immediate monadic interpretation from $\mathcal{G}_\alpha$ to α. An important corollary of Lemma 7 is that the reverse cannot exist, or there would be a monadic formula identifying a specific ordinal smaller than ε_0, which is contradictory to the result of Büchi [3, Th. 4.9] cited in introduction.

For a more complex illustration of a monadic interpretation, we notice that the degree word allows the restriction from a greater ordinal.

Lemma 8 *If $\alpha < \alpha'$, there is a MSO interpretation I such that $\mathcal{G}_\alpha = I(\mathcal{G}_{\alpha'})$.*

PROOF.   Following the definition, we look for an interpretation $I = \{\psi_\prec\}$. We use again the fact that the degree word is unique and MSO-definable. Defining the greatest sequence of $\mathcal{G}_\alpha$ provides a MSO marking on $\mathcal{G}'_\alpha$, which bounds the set of vertices. More precisely, let $\Psi^u(p)$ be an expression similar to φ^u of the Lemma 6 but where the part $\tau(p_i, p_{i+1}) \wedge \delta_{u_i}(p_i)$ has been replaced by $\tau_{u_i}(p_i, p_{i+1})$ meaning "p_{i+1} is the u_i^{th} such that $p_i \prec p_{i+1}$"; the same goes for the q_j and for $\tau(p_{|u|}, q_1) \wedge \delta_{u_{|u|}}(p_{|u|})$. Also add the condition that p is a part of the sequence : $(\bigvee_i p = p_i) \vee p \in V$. Then $\Psi^u(p)$ is a marking of the greatest sequence associated to u. For a given α, I simply adds the condition of co-accessibility to a vertex marked by $\Psi^{u(\alpha)}$.

$$\psi_\prec(p,q) \quad := \quad p \xrightarrow{\prec} q \wedge \exists r \, (\Psi^{u(\alpha)}(r) \wedge q \xrightarrow{\prec^*} r)$$
$$\mathcal{G}_\alpha \quad = \quad \{p \xrightarrow{\prec} q \mid p \xrightarrow{\prec} q \in \mathcal{G}_{\alpha'} \wedge \exists r \, (\Psi^{u(\alpha)}(r) \wedge q \xrightarrow{\prec^*} r)\}$$

∎

The *treegraph* Treegraph(G) of a graph G is the set $\{p \xrightarrow{a} q\} \subseteq V_G^* \times (\Sigma_G \cup \{\#\}) \times V_G^*$ where $(p,q) \in V_G^*$ are sequences of vertices of G, and $a \in \Sigma_G$ either if $p = wu$, $q = wv$ and $u \xrightarrow{a} v \in G$, or if $a = \#$, $p = wu$ and $q = wuu$. One can also see the treegraph as the fixpoint of the operation which, to each vertex which is not starting point of an # arc, adds this arc leading to the location of this vertex in a copy of G. The starting graph is called the *root graph*.

One way to define the *pushdown hierarchy* (see [5] for details) is as follows.

- $\mathcal{H}_0$ is the class of graphs with finite support,

- $\mathcal{H}_n = \mathcal{I} \circ Treegraph(\mathcal{H}_{n-1})$.

For instance, $\mathcal{H}_1$ is the class of *prefix-recognizable* graphs [7] and further $\mathcal{H}_n$ classes have been proved to correspond to an extension of prefix-recognizability on higher-order stacks [4].

3.2 Building covering graphs

We note $p \xrightarrow{a^\bullet} q$ for the longest possible path labeled by a, and $p \xrightarrow{S} q$ a shortcut for the successor relation, i.e.

$$p \xrightarrow{a^\bullet} q \quad := \quad p \xrightarrow{a^*} q \wedge \neg \exists r \, (q \xrightarrow{a} r)$$
$$p \xrightarrow{S} q \quad := \quad p \xrightarrow{\prec} q \wedge \neg \exists r (p \xrightarrow{\prec} r \wedge r \xrightarrow{\prec^*} q).$$

Now let $I = \{\varphi_\prec\}$ and $M(p)$ respectively be the interpretation and marking

$$\varphi_\prec(p,q) \quad := \quad M(p) \wedge M(q) \wedge p \xrightarrow{\bar{\prec}^\bullet \#} q \vee p \xrightarrow{\bar{\#}^\bullet S\#} q \vee p \xrightarrow{\bar{\#}\prec\#} q$$
$$M(p) \quad := \quad \exists r : \forall q \, (r \xrightarrow{(\prec + \# + \bar{\prec})^*} q) \wedge r \xrightarrow{\prec^* \#(\bar{\prec}^* \#)^*} p$$

The marking $M(p)$ allows to start anywhere on the root graph, but as soon as a #-arc has been followed, $\prec$-arcs can only be followed backwards. We consider only goals of a #-arc.

The $\varphi_\prec(p,q)$ formula states the relation on these vertices, leaving three choices : either to follow $\prec$-arcs as long as possible (in practice, until a copy of 0) and go down one #-arc; or on the contrary, to follow # backwards as long as possible, then take the successor and one #-arc; or just to follow one # backwards, one $\prec$ and one #.

Lemma 9 $\mathcal{G}_{\omega^\alpha} = I \circ Treegraph(\mathcal{G}_\alpha)$.

For instance consider $\mathcal{G}_\omega$, which is an infinite path. A representation of its treegraph is given below (plain lines for $\prec$, dotted lines for #). The circled vertices are the ones marked by M and therefore they are the only ones kept by the interpretation φ. We are allowed to go anywhere on the root $\mathcal{G}_\omega$ structure, but as soon as we follow # we can only go backwards. This reflects the construction of a power of ω as a decreasing sequence of ordinals : we may start by any, but afterwards we only may decrease.
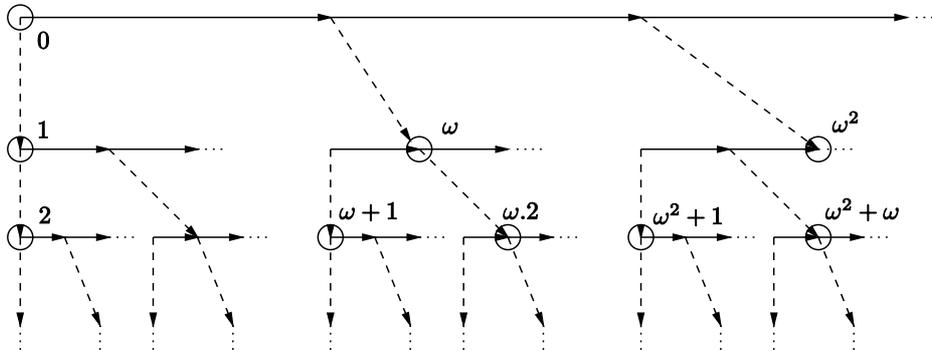
Lemma 10 *If $\alpha < \omega \Uparrow (n+1)$, then $\mathcal{G}_\alpha \in \mathcal{H}_n$.*

PROOF.   For any finite α, $\mathcal{G}_\alpha$ is in fact a finite path labeled by $\prec$ and is in $\mathcal{H}_0$. By Lemma 9 iterated n times, every $\omega^{\cdots^{\omega^k}}$ with n times ω and $1 < k < \omega$ is in $\mathcal{H}_n$. Smaller ordinals are captured by a restriction as in Lemma 8.   ∎

This proves the decidability of the monadic theory of the covering graphs. By transitive closure (Lemma 1), ordinals are also captured.

Theorem 11 *If $\alpha < \omega \Uparrow (n+1)$, then $\alpha \in \mathcal{H}_n$.*

The decidability of the monadic theory of these ordinals is well-known, but this result also shows that ordinals below ε_0 can be expressed by finite objects, namely higher-order pushdown automata. Following the steps of a well-chosen automaton (up to an operation called the ε-closure) builds exactly an ordinal. This approach is explained in Section 5.

4   Strictness of the hierarchy for covering graphs

In this section, we strengthen Lemma 10 by proving that covering graphs cannot be in any level of the hierarchy. Let $\exp(x, n, k)$ be a tower of exponentiation of x of height n with power k on the top, where n and k are integers.

$$
\begin{aligned}
\exp(x, n, k) &= k & \text{if } n = 0 \\
&= x^{\exp(x, n-1, k)} & \text{otherwise.}
\end{aligned}
$$

In the following section, this function will be used in the cases $x = 2$ and $x = \omega$.

We examine the tree $\mathcal{T}_n$ of trace (from the root) $\{a^n b^{\exp(2, n, k)}\}$. It has the form below with $f(k) = \exp(2, n, k)$. The horizontal arcs are labeled by a and the vertical arcs by b.
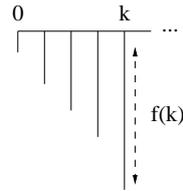
For any n, there is such a tree which is not in the level n of the hierarchy [2].

Proposition 1 *For $n \geq 1$, $\mathcal{T}_{3n} \notin \mathcal{H}_n$.*

Finding a monadic interpretation from $\mathcal{G}_\alpha$ to $\mathcal{T}_{3n}$ is therefore enough to prove $\mathcal{G}_\alpha \notin \mathcal{H}_n$. In fact, Lemma 8 already states that if $\omega \Uparrow 3n + 1 \leq \alpha$, then there is an interpretation from $\mathcal{G}_\alpha$ to $\mathcal{G}_{\omega \Uparrow 3n+1}$; so the interpretation from $\mathcal{G}_{\omega \Uparrow 3n+1}$ to $\mathcal{T}_{3n}$ is enough for a whole class of ordinals. We sketch this interpretation.

Let C_n^k be the set of ordinals smaller than $\exp(\omega, n, k)$ where each coefficient in RCNF is at most 1, except for the top-most power :

- $[0, k-1] \in C_0^k$,
- $0 \in C_n^k$,
- if $\gamma_0, \ldots, \gamma_h$ are all distinct ordinals of C_{n-1}^k, then $\omega^{\gamma_0} + \cdots + \omega^{\gamma_h} \in C_n^k$.

For instance, $C_1^3 = \{0, 1, \omega, \omega + 1, \omega^2, \omega^2 + 1, \omega^2 + \omega, \omega^2 + \omega + 1\}$;

$$\begin{aligned} C_2^2 \; = \; & \{0, 1, \omega, \omega + 1, \omega^\omega, \omega^\omega + 1, \omega^\omega + \omega, \omega^\omega + \omega + 1, \\ & \omega^{\omega+1}, \omega^{\omega+1} + 1, \omega^{\omega+1,} + \omega, \omega^{\omega+1} + \omega + 1, \\ & \omega^{\omega+1} + \omega^\omega, \omega^{\omega+1} + \omega^\omega + 1, \omega^{\omega+1} + \omega^\omega + \omega, \omega^{\omega+1} + \omega^\omega + \omega + 1\}. \end{aligned}$$

The following lemma is only a matter of cardinality of powersets.

Lemma 12 *The cardinality of the set C_n^k is $\exp(2, n, k)$.*

We abusively note $\alpha + C_n^k$ for the set $\{\alpha + \gamma \mid \gamma \in C_n^k\}$. The main difficulty of this section is to define a monadic formula for this set.

Lemma 13 *For $n > 0$, there is a monadic formula describing $\exp(\omega, n, k) + C_n^k$ in $\mathcal{G}_\alpha$, for α greater than $\exp(\omega, n, k).2$.*

These ordinals are easy to capture by previous tools. The following lemma is a natural corollary of the proof of Lemma 4, since $\exp(\omega, n, k) \prec \exp(\omega, n, k+1)$.

Lemma 14 *The greatest sequence of $\omega \Uparrow (n+1)$ is ultimately the sequence $(\exp(\omega, n, k))_{k \geq 1}$.*

We may now state the main result of this section.

Theorem 15 *If $n > 0$ and $\alpha \geq \omega \Uparrow 3n + 1$, then $\mathcal{G}_\alpha \notin \mathcal{H}_n$.*

PROOF (SKETCH).   If we concatenate the previous lemmas, it appears that

- since the greatest sequence of α is interpretable from $\mathcal{G}_\alpha$, we can extract the sequence $(\exp(\omega, 3n, k))_{k \geq 1}$ from $\mathcal{G}_{\omega \Uparrow 3n+1}$, which will be the "horizontal path" of $\mathcal{T}_{3n}$;
- for each $\exp(\omega, 3n, k)$ we can also capture the associated set $\exp(\omega, 3n, k) + C_{3n}^k$ and arrange it in path. This yields the "vertical path" hanging from $\exp(\omega, 3n, k)$ and of length $\exp(2, 3n, k)$.

Eventually, the monadic interpretation builds exactly $\mathcal{T}_{3n}$, which is the expected result.   ∎

The covering graph $\mathcal{G}_{\varepsilon_0}$ can be defined and has unbounded degree, but has still the property of Lemma 8 : it can give any smaller ordinal via monadic interpretation, which yields the following result.

Corollary 16 *$\mathcal{G}_{\varepsilon_0}$ does not belong to the hierarchy.*

From [2] we could actually extract the lower bound $\mathcal{T}_{2n} \notin \mathcal{H}_n$. The conjecture is that $\mathcal{T}_n \notin \mathcal{H}_n$, which would allow to locate exactly each covering graph in the hierarchy.

The Theorem 15 does not apply to ordinal themselves, since there we showed that there is no interpretation from ordinals to covering graphs. Therefore, the question is still open, which leads to Conjecture 1 at the end of this paper.

5   Higher-order stack description of ordinals

The graph on the level n of the hierarchy are also graphs (up to ε-closure) of higher-order pushdown automata of level n [5], i.e. automata which use nested stacks of stacks of depth n. The construction by monadic interpretations and unfolding could be translated into a pushdown automata description. Instead of doing so, we use the equivalent notion of prefix-recognizable relations [4] from scratch. This notion offers a natural encoding of ordinals by their Cantor normal form. Nonetheless, the associated proof is still heavy.

5.1 Short presentation

This section sketches a particular case of prefix-recognizable graphs. For a complete description, see [4]. We only consider 1-stacks (usual stacks) over an alphabet of size 1, *i.e.* integers. The empty 1-stack is therefore noted 0. For all $n > 1$, a n-stack is a non-empty finite sequence of $(n-1)$-stacks, noted $[a_1, \ldots, a_m]_n$. The operations Ops_1 on a 1-stack are

$$
\begin{aligned}
\mathsf{push}_1(i) &:= i+1, \\
\mathsf{pop}_1(i+1) &:= i.
\end{aligned}
$$

For $n > 1$, the set Ops_n of operations on a n-stack include

$$
\begin{aligned}
\mathsf{copy}_n([a_1, \ldots, a_m]_n) &:= [a_1, \ldots, a_m, a_m]_n \\
\mathsf{pop}_n([a_1, \ldots, a_m]_n) &:= [a_1, \ldots, a_{m-1}]_n \\
f([a_1, \ldots, a_m]_n) &:= [a_1, \ldots, f(a_m)]_n
\end{aligned}
$$

where f is any operation on k-stacks, $k < n$.

The 2-stack containing only 0 is noted $[\ ]_2$, and the n-stack containing only $[\ ]_{n-1}$ is noted $[\ ]_n$. Let also be an identity operation id defined on all stacks.

The set Ops_n forms a monoid with the composition operation. Let $Reg(Ops_n)$ the closure of the finite subsets of this monoid under union, product and iteration, *i.e.* the set of *regular expressions* on Ops_n. To each expression $E \in Reg(Ops_n)$ we associate the set of n-stacks $S(E) = E([\ ]_n)$ and the set of relations on stacks $R(E) = \{(s, s') | s' \in E(s)\}$.

Given E and a finite set $(E_a)_{a \in \Sigma}$ in $Reg(Ops_n)$, the graph of support $S(F)$ and arcs $s \xrightarrow{a} s'$ iff $(s, s') \in R(F_a)$ is a *prefix-recognizable graph of order n*. General prefix-recognizable graphs are exactly graphs of pushdown automata of the same order.

5.2 Towers of ω

We define the expressions dom and inc which respectively fix the domain of the structure and the order relation. In the following we also will need an expression dec to perform the symmetric of inc. In one word, we want the structure $\langle S(\mathsf{dom}(\alpha)), R(\mathsf{dec}(\alpha)), R(\mathsf{inc}(\alpha)) \rangle$ to be isomorphic to the structure $\langle \alpha, >, < \rangle$.

For ω, we consider the set of all 1-stacks (i.e. integers). In this case, $\mathsf{dom}(\omega)$ is obtained by iterating push_1 on the empty stack. The other operations are also straighforward.

$$
\begin{aligned}
\mathsf{dom}(\omega) &:= \mathsf{push}_1^* \\
\mathsf{inc}(\omega) &:= \mathsf{push}_1^+ \\
\mathsf{dec}(\omega) &:= \mathsf{pop}_1^+
\end{aligned}
$$

We consider now any ordinal α. Let n be the smallest value such that $\mathsf{dom}(\alpha), \mathsf{inc}(\alpha)$ and $\mathsf{dec}(\alpha)$ are all in $Reg(Ops_{n-1})$.

Let $\mathsf{tail}(\alpha) := \mathsf{copy}_n.(\mathsf{id} + \mathsf{dec}(\alpha))$. Informally, each ordinal $\gamma < \omega^\alpha$ is either 0 or may be written as $\gamma = \omega^{\gamma_0} + \cdots + \omega^{\gamma_k}$ with $\gamma_i < \alpha$; so we code γ as a sequence of stacks respectively coding $\gamma_0 \ldots \gamma_k$. The tail operation takes the last stack (representing γ_k) and adds a stack coding an ordinal $\leq \gamma_k$, so that the CNF constraint is respected. For the relation $<$, inc

either adds a decreasing sequence (by tail), or it first pops stacks, then increases a given one before adding a tail.

$$
\begin{aligned}
\mathsf{dom}(\omega^\alpha) &:= \mathsf{dom}(\alpha).\mathsf{tail}(\alpha)^* \\
\mathsf{inc}(\omega^\alpha) &:= [\mathsf{pop}_n^*.\mathsf{inc}(\alpha) + \mathsf{tail}(\alpha)].\mathsf{tail}(\alpha)^* \\
\mathsf{dec}(\omega^\alpha) &:= \mathsf{pop}_n^*.[\mathsf{pop}_n + \mathsf{dec}(\alpha).\mathsf{tail}(\alpha)^*]
\end{aligned}
$$

We get this version of Theorem 11 restricted to towers of ω.

Theorem 17 *The graph of $\omega \Uparrow n$ is isomorphic to the prefix-recognizable graph of order n with support $S(\textit{dom}(\omega \Uparrow n))$ and one relation $R(\textit{inc}(\omega \Uparrow n))$.*

The proof of this proposition encodes exponentiation of ω, so the case of all ordinals smaller than ε_0 can be obtained by encoding also addition. This can be done with a greater starting alphabet and using markers to differentiate each part of the addition.

6   Perspectives

We have defined covering graphs as graphs of fundamental sequence and successor relations and shown the existence of a formula identifying a covering graph among others, via the degree word. Then, the covering graphs and the corresponding ordinals have been located in the pushdown hierarchy according to the size in terms of tower of ω, in a strict way for the covering graph case.

Theorem 11 raises the question of the strictness of the classification of ordinals in the hierarchy. Theorem 15 naturally suggests that if $\alpha \geq \omega \Uparrow n$, then α does not belong to $\mathcal{H}_{n-1}$, and therefore ε_0 is banned from the hierarchy.

Conjecture 1 *ε_0 does not belong to the hierarchy.*

If this were proved, ε_0 would actually be a good candidate for extending the hierarchy above the $\mathcal{H}_n$. Indeed, a current field of research is to capture as many structures with decidable monadic theory as possible. A way to do so would be to find an operation extending those used in this paper — interpretation and treegraph.

One can find definitions [16] of a canonical fundamental sequence for ordinals greater than ε_0 and therefore define covering graphs outside of the hierarchy. For instance, one can take $\varepsilon_0[n] = \omega \Uparrow (n + 1)$. In this way, covering graphs may be defined for a large number of ordinals; but we conjecture that the Theorem 7 does not stand any more, i.e. for any definition of fundamental sequence, there are two ordinals whose covering graphs have the same monadic theories.

Also, the ability to differentiate covering graphs smaller than ε_0 leads to check this robustness for more difficult questions. One of them is selection in monadic theory, which is negative for ordinals greater than ω^ω [12].

In another direction, it would be interesting to remove the well-ordering property and to consider more general linear orderings. The orders of $\mathbb{Q}$ and $\mathbb{Z}$ are obviously prefix-recognizable. We would like to reach structures of more complex orders.

Acknowledgments

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