Quantum State Description Complexity

Umesh V. Vazirani[1]

1 **University of California, Berkeley,** `vazirani@eecs.berkeley.edu`

Abstract

Quantum states generally require exponential sized classical descriptions, but the long conjectured area law provides hope that a large class of natural quantum states can be described succinctly. Recent progress in formally proving the area law is described.

1998 ACM Subject Classification F.1.1 Models of Computation

Keywords and phrases area law, Hamiltonian, description complexity, detectability lemma, entanglement

Digital Object Identifier 10.4230/LIPIcs.FSTTCS.2011.26

1 Summary

Arguably the most fundamental difference between quantum and classical systems is the exponential difference in the number of parameters required to describe them - whereas a classical system of n particles can be described by $\mathcal{O}(n)$ parameters, an *arbitrary* state of a similar quantum system would in general require $2^{\Omega n}$ parameters. The reason for this discrepancy is quantum entanglement, which is a fundamental resource in quantum information processing, is also the principal obstacle in the efficient simulation of quantum systems on a classical computer. But is there a significant class of quantum states which can be described succinctly? More formally is there a set S of states such that for every state $|\psi\rangle \in S$ there is a classical description $w \in \{0,1\}^{\text{poly}(n)}$ that "describes" $|\psi\rangle$ — in the sense that it is possible to efficiently compute interesting quantities about $|\psi\rangle$, such as energy or two point correlations, from the classical description w. This may be formalized by saying that the result of any k-local measurement, for some constant k, on $|\psi\rangle$ can be efficiently computed from w.

A possible candidate for S is the set of ground states of gapped local Hamiltonians. Local Hamiltonians are quantum analogs of CSPs, and ground states are quantum analogs of satisfying assignments or configurations that minimize the number of unsatisfied constraints. More formally, consider a set of n (d dimensional) spins arranged in a lattice, with nearest-neighbor interactions described by a local Hamiltonian $H = \sum_{i=1}^{n-1} H_i$. H is a $d^n \times d^n$ Hermitian matrix, whose eigenvectors are the states of the system with definite energy equal to the corresponding eigenvalue. We assume that H has a unique ground state $|\Omega\rangle$, the lowest energy eigenvector. The spectral gap of H, denoted by ϵ, is the difference between the two lowest eigenvalues. We will assume that each term H_i has unit norm and say that H is gapped if the spectral gap ϵ is bounded below by some constant independent of n.

A remarkable conjecture in condensed matter physics dating back about a half century is the Area Law, which strongly bounds the entanglement in ground states of gapped local Hamiltonians. More specifically, for any contiguous region of the grid L, the entanglement entropy between the particles inside L and the particles outside L is trivially bounded by ($\log d$ times) the number of particles in L, which we may think of as the volume of L. The surface area of L is the number of grid edges crossing between L and $\overline{L}$ or equivalently

31st Int'l Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS 2011).
Editors: Supratik Chakraborty, Amit Kumar; pp. 26–27
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

thenumber of terms of the Hamiltonian describing interactions between particles in L with particles in $\overline{L}$. Then a state $|\psi\rangle$ obeys an area law if the *entanglement entropy* between L and $\overline{L}$ is upper-bounded by a quantity proportional to the surface area of L, rather than its volume.

A few years ago, in a seminal paper [5] Hastings proved that ground states of gapped 1D systems obey an area law. More specifically, for such systems, he proved that the entanglement entropy across any cut in the chain is bounded by $S_{1D} \leq e^{\mathcal{O}(X)}$, where $X = \frac{\log d}{\epsilon}$. As a consequence he showed that the ground state of such systems has a polynomial classical description by a Matrix Product State of size $ne^{S_{1D}}$.

This left the area law for two (and higher) dimensional systems as one of the most important open questions in Quantum Hamiltonian Complexity. One way to tackle this question is to improve the bound on the entanglement entropy, since a bound of $\mathcal{O}(\log d)$ would automatically imply area laws for higher dimensional systems — simply treat the system as a 1D system by grouping all the particles at the boundary as a huge particle of dimension d^B, where B is the number of particles on the boundary. A logarithmic bound on entanglement entropy now yields a bound of $\mathcal{O}(B \log d))$.

A combinatorial approach to proving the area law for 1D frustration-free systems (i.e., systems where the ground state $|\Omega\rangle$ is also the common ground state of all local terms) was introduced in [2]. The new proof replaced Hastings' analytical machinery, including the Lieb-Robinson bound and spectral Fourier analysis, with the Detectability Lemma [1], a combinatorial lemma about local Hamiltonians. However, the resulting bound was no better than Hastings' bound because, at its heart, the argument followed the same outline as Hastings', including the use of a "monogamy of entanglement"-type argument that leads to an exponential slack.

A completely new approach to proving the 1D area law (based on the detectability lemma) was pursued in [3] and [4]. This led to an exponential improvement in the bound on the entanglement entropy. Formally, in [4] it was proved that for 1D frustration-free systems, the entanglement entropy of every cut in the chain is upper bounded by $S_{1D} \leq \mathcal{O}(X^3 \log^8 X)$. This also improves by an exponential factor the bound on the classical description of such states by a Matrix Product State description. Moreover, by using the locality properties of the Hamiltonian, for the case of 2D systems the entanglement bound can be improved to $S \leq \mathcal{O}(B^2 \cdot X^3 \log^8(B \cdot X))$. This leaves the 2D area law poised at a very interesting point — any non-trivial improvement in the existing bounds would result in the first sub-volume law for 2D systems.

References

1 Dorit Aharonov, Itai Arad, Zeph Landau, and Umesh Vazirani. The detectability lemma and quantum gap amplification. In *STOC '09: Proc. 41st Annual ACM Symposium on Theory of Computing*, `arXiv:0811.3412`, pages 417–426, New York, NY, USA, 2009. ACM.

2 Dorit Aharonov, Itai Arad, Zeph Landau, and Umesh Vazirani. Quantum Hamiltonian complexity and the detectability lemma. *arXiv:1011.3445*, November 2010.

3 Dorit Aharonov, Itai Arad, Zeph Landau, and Umesh Vazirani. The 1d area law and the complexity of quantum states: A combinatorial approach. In *FOCS '11: Proceedings of the 52nd Annual IEEE Symposium on Foundations of Computer Science (FOCS 2011)*, 2011.

4 Itai Arad, Zeph Landau, and Umesh Vazirani. An improved 1d area law for frustration-free systems. 2011. arXiv:1111.2970v1 [quant-ph].

5 MB Hastings. An Area Law for One Dimensional Quantum Systems. *JSTAT, P*, 8024, 2007.