The Determinacy of Context-Free Games

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Abstract

We prove that the determinacy of Gale-Stewart games whose winning sets are accepted by real-time 1-counter Büchi automata is equivalent to the determinacy of (effective) analytic Gale-Stewart games which is known to be a large cardinal assumption. We show also that the determinacy of Wadge games between two players in charge of ω-languages accepted by 1-counter Büchi automata is equivalent to the (effective) analytic Wadge determinacy. Using some results of set theory we prove that one can effectively construct a 1-counter Büchi automaton $\mathcal{A}$ and a Büchi automaton $\mathcal{B}$ such that: (1) There exists a model of ZFC in which Player 2 has a winning strategy in the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$; (2) There exists a model of ZFC in which the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$ is not determined. Moreover these are the only two possibilities, i.e. there are no models of ZFC in which Player 1 has a winning strategy in the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$.

1   Introduction

Two-players infinite games have been much studied in Set Theory and in Descriptive Set Theory, see [9, 8]. In particular, if X is a (countable) alphabet having at least two letters and $A \subseteq X^\omega$, then the Gale-Stewart game $G(A)$ is an infinite game with perfect information between two players. Player 1 first writes a letter $a_1 \in X$, then Player 2 writes a letter $b_1 \in X$, then Player 1 writes $a_2 \in X$, and so on ... After ω steps, the two players have composed an infinite word $x = a_1 b_1 a_2 b_2 \ldots$ of X^ω. Player 1 wins the play iff $x \in A$, otherwise Player 2 wins the play. The game $G(A)$ is said to be determined iff one of the two players has a winning strategy. A fundamental result of Descriptive Set Theory is Martin's Theorem which states that every Gale-Stewart game $G(A)$, where A is a Borel set, is determined [9].

On the other hand, in Computer Science, the conditions of a Gale Stewart game may be seen as a specification of a reactive system, where the two players are respectively a non terminating reactive program and the "environment". Then the problem of the synthesis of winning strategies is of great practical interest for the problem of program synthesis in reactive systems. In particular, if $A \subseteq X^\omega$, where X is here a finite alphabet, and A is effectively presented, i.e. accepted by a given finite machine or defined by a given logical formula, the following questions naturally arise, see [15, 10]: (1) Is the game $G(A)$ determined?   (2) If Player 1 has a winning strategy, is it effective, i.e. computable? (3) What are the amounts of space and time necessary to compute such a winning strategy? Büchi and Landweber gave a solution to the famous Church's Problem, posed in 1957, by stating that in a Gale Stewart game $G(A)$, where A is a regular ω-language, one can decide who the winner is and compute

a winning strategy given by a finite state transducer, see [16] for more information on this subject. In [15, 10] Thomas and Lescow asked for an extension of this result where A is no longer regular but deterministic context-free, i.e. accepted by some deterministic pushdown automaton. Walukiewicz extended Büchi and Landweber's Theorem to this case by showing first in [18] that that one can effectively construct winning strategies in parity games played on pushdown graphs and that these strategies can be computed by pushdown transducers. Notice that later some extensions to the case of higher-order pushdown automata have been established [1].

In this paper, we first address the question (1) of the determinacy of Gale-Stewart games $G(A)$, where A is a context-free ω-language accepted by a (non-deterministic) pushdown automaton, or even by a 1-counter automaton. Notice that there are some context-free ω-languages which are (effective) analytic but non-Borel and thus the determinacy of these games cannot be deduced from Martin's Theorem of Borel determinacy. On the other hand, Martin's Theorem is provable in ZFC, the commonly accepted axiomatic framework for Set Theory in which all usual mathematics can be developed. But the determinacy of Gale-Stewart games $G(A)$, where A is an (effective) analytic set, is not provable in ZFC; Martin and Harrington have proved that it is a large cardinal assumption equivalent to the existence of a particular real, called the real $0^\sharp$, see [8, page 637]. We prove here that the determinacy of Gale-Stewart games $G(A)$, whose winning sets A are accepted by real-time 1-counter Büchi automata, is equivalent to the determinacy of (effective) analytic Gale-Stewart games and thus also equivalent to the existence of the real $0^\sharp$.

Next we consider Wadge games which were firstly studied by Wadge in [17] where he determined a great refinement of the Borel hierarchy defined via the notion of reduction by continuous functions. These games are closely related to the notion of reducibility by continuous functions. For $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$, L is said to be Wadge reducible to L' iff there exists a continuous function $f : X^\omega \to Y^\omega$, such that $L = f^{-1}(L')$; this is then denoted by $L \leq_W L'$. On the other hand, the Wadge game $W(L, L')$ is an infinite game with perfect information between two players, Player 1 who is in charge of L and Player 2 who is in charge of L'. And it turned out that Player 2 has a winning strategy in the Wadge game $W(L, L')$ iff $L \leq_W L'$. It is easy to see that the determinacy of Borel Gale-Stewart games implies the determinacy of Borel Wadge games. On the other hand, Louveau and Saint-Raymond have proved that this latter one is weaker than the first one, since it is already provable in second-order arithmetic, while the first one is not. It is also known that the determinacy of (effective) analytic Gale-Stewart games is equivalent to the determinacy of (effective) analytic Wadge games, see [11]. We prove in this paper that the determinacy of Wadge games between two players in charge of ω-languages accepted by 1-counter Büchi automata is equivalent to the (effective) analytic Wadge determinacy, and thus also equivalent to the existence of the real $0^\sharp$.

Then, using some recent results from [4] and some results of Set Theory, we prove that, (assuming ZFC is consistent), one can effectively construct a 1-counter Büchi automaton $\mathcal{A}$ and a Büchi automaton $\mathcal{B}$ such that: (1) There exists a model of ZFC in which Player 2 has a winning strategy in the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$; (2) There exists a model of ZFC in which the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$ is not determined. Moreover these are the only two possibilities, i.e. there are no models of ZFC in which Player 1 has a winning strategy in the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$.

The paper is organized as follows. We recall some known notions in Section 2. We study context-free Gale-Stewart games in Section 3 and context-free Wadge games in Section 4. Some concluding remarks are given in Section 5.

2 Recall of some known notions

We assume the reader to be familiar with the theory of formal (ω-)languages [14, 13]. We recall the usual notations of formal language theory.

If Σ is a finite alphabet, a *non-empty finite word* over Σ is any sequence $x = a_1 \ldots a_k$, where $a_i \in \Sigma$ for $i = 1, \ldots, k$, and k is an integer ≥ 1. The *length* of x is k, denoted by $|x|$. The *empty word* has is denoted by λ; its length is 0. $\Sigma^\star$ is the *set of finite words* (including the empty word) over Σ. A (finitary) *language* V over an alphabet Σ is a subset of $\Sigma^\star$.

The *first infinite ordinal* is ω. An *ω-word* over Σ is an ω-sequence $a_1 \ldots a_n \ldots$, where for all integers $i \geq 1$, $a_i \in \Sigma$. When $\sigma = a_1 \ldots a_n \ldots$ is an ω-word over Σ, we write $\sigma(n) = a_n$, $\sigma[n] = \sigma(1)\sigma(2) \ldots \sigma(n)$ for all $n \geq 1$ and $\sigma[0] = \lambda$.

The usual concatenation product of two finite words u and v is denoted $u.v$ (and sometimes just uv). This product is extended to the product of a finite word u and an ω-word v: the infinite word $u.v$ is then the ω-word such that:

$(u.v)(k) = u(k)$ if $k \leq |u|$, and $(u.v)(k) = v(k - |u|)$ if $k > |u|$.

The *set of ω-words* over the alphabet Σ is denoted by Σ^ω. An *ω-language* V over an alphabet Σ is a subset of Σ^ω, and its complement (in Σ^ω) is $\Sigma^\omega - V$, denoted V^-.

The *prefix relation* is denoted $\sqsubseteq$: a finite word u is a *prefix* of a finite word v (respectively, an infinite word v), denoted $u \sqsubseteq v$, if and only if there exists a finite word w (respectively, an infinite word w), such that $v = u.w$.

If L is a finitary language (respectively, an ω-language) over the alphabet Σ then the set $\text{Pref}(L)$ of prefixes of elements of L is defined by $\text{Pref}(L) = \{u \in \Sigma^\star \mid \exists v \in L \ \ u \sqsubseteq v\}$.

We now recall the definition of k-counter Büchi automata which will be useful in the sequel.

Let k be an integer ≥ 1. A *k-counter machine* has k *counters*, each of which containing a non-negative integer. The machine can test whether the content of a given counter is zero or not. And transitions depend on the letter read by the machine, the current state of the finite control, and the tests about the values of the counters. Notice that in this model some λ-transitions are allowed. During these transitions the reading head of the machine does not move to the right, i.e. the machine does not read any more letter.

Formally a k-counter machine is a 4-tuple $\mathcal{M} = (K, \Sigma, \Delta, q_0)$, where K is a finite set of states, Σ is a finite input alphabet, $q_0 \in K$ is the initial state, and $\Delta \subseteq K \times (\Sigma \cup \{\lambda\}) \times \{0,1\}^k \times K \times \{0,1,-1\}^k$ is the transition relation. The k-counter machine $\mathcal{M}$ is said to be *real time* iff: $\Delta \subseteq K \times \Sigma \times \{0,1\}^k \times K \times \{0,1,-1\}^k$, i.e. iff there are no λ-transitions.

If the machine $\mathcal{M}$ is in state q and $c_i \in \mathbf{N}$ is the content of the ith counter $\mathcal{C}_i$ then the configuration (or global state) of $\mathcal{M}$ is the $(k+1)$-tuple $(q, c_1, \ldots, c_k)$.

For $a \in \Sigma \cup \{\lambda\}$, $q, q' \in K$ and $(c_1, \ldots, c_k) \in \mathbf{N}^k$ such that $c_j = 0$ for $j \in E \subseteq \{1, \ldots, k\}$ and $c_j > 0$ for $j \notin E$, if $(q, a, i_1, \ldots, i_k, q', j_1, \ldots, j_k) \in \Delta$ where $i_j = 0$ for $j \in E$ and $i_j = 1$ for $j \notin E$, then we write:

$a : (q, c_1, \ldots, c_k) \mapsto_\mathcal{M} (q', c_1 + j_1, \ldots, c_k + j_k)$.

Thus the transition relation must obviously satisfy: if $(q, a, i_1, \ldots, i_k, q', j_1, \ldots, j_k) \in \Delta$ and $i_m = 0$ for some $m \in \{1, \ldots, k\}$ then $j_m = 0$ or $j_m = 1$ (but j_m may not be equal to -1).

Let $\sigma = a_1 a_2 \ldots a_n \ldots$ be an ω-word over Σ. An ω-sequence of configurations $r = (q_i, c_1^i, \ldots c_k^i)_{i \geq 1}$ is called a run of $\mathcal{M}$ on σ iff:

(1) $(q_1, c_1^1, \ldots c_k^1) = (q_0, 0, \ldots, 0)$

(2) for each $i \geq 1$, there exists $b_i \in \Sigma \cup \{\lambda\}$ such that $b_i : (q_i, c_1^i, \ldots c_k^i) \mapsto_\mathcal{M} (q_{i+1}, c_1^{i+1}, \ldots c_k^{i+1})$ and such that $a_1 a_2 \ldots a_n \ldots = b_1 b_2 \ldots b_n \ldots$

For every such run r, $\text{In}(r)$ is the set of all states entered infinitely often during r.

▶ **Definition 1.** A Büchi k-counter automaton is a 5-tuple $\mathcal{M}=(K,\Sigma,\Delta,q_0,F)$, where $\mathcal{M}'=(K,\Sigma,\Delta,q_0)$ is a k-counter machine and $F \subseteq K$ is the set of accepting states. The ω-language accepted by $\mathcal{M}$ is:

$$L(\mathcal{M}) = \{\sigma \in \Sigma^\omega \mid \text{there exists a run r of } \mathcal{M} \text{ on } \sigma \text{ such that } \text{In}(r) \cap F \neq \emptyset\}$$

The class of ω-languages accepted by Büchi k-counter automata is denoted $\mathbf{BCL}(k)_\omega$. The class of ω-languages accepted by *real time* Büchi k-counter automata will be denoted $\mathbf{r\text{-}BCL}(k)_\omega$. The class $\mathbf{BCL}(1)_\omega$ is a strict subclass of the class $\mathbf{CFL}_\omega$ of context free ω-languages accepted by Büchi pushdown automata.

We assume the reader to be familiar with basic notions of topology which may be found in [9, 10, 14, 13]. There is a natural metric on the set Σ^ω of infinite words over a finite alphabet Σ containing at least two letters which is called the *prefix metric* and is defined as follows. For $u, v \in \Sigma^\omega$ and $u \neq v$ let $\delta(u,v) = 2^{-l_{\text{pref}(u,v)}}$ where $l_{\text{pref}(u,v)}$ is the first integer n such that the $(n+1)^{st}$ letter of u is different from the $(n+1)^{st}$ letter of v. This metric induces on Σ^ω the usual Cantor topology in which the *open subsets* of Σ^ω are of the form $W.\Sigma^\omega$, for $W \subseteq \Sigma^\star$. A set $L \subseteq \Sigma^\omega$ is a *closed set* iff its complement $\Sigma^\omega - L$ is an open set.

For $V \subseteq \Sigma^\star$ we denote $\text{Lim}(V) = \{x \in \Sigma^\omega \mid \exists^\infty n \geq 1 \ \ x[n] \in V\}$ the set of infinite words over Σ having infinitely many prefixes in V. Then the topological closure $\text{Cl}(L)$ of a set $L \subseteq \Sigma^\omega$ is equal to $\text{Lim}(\text{Pref}(L))$. Thus we have also the following characterization of closed subsets of Σ^ω: a set $L \subseteq \Sigma^\omega$ is a closed subset of the Cantor space Σ^ω iff $L = \text{Lim}(\text{Pref}(L))$.

We now recall the definition of the *Borel Hierarchy* of subsets of X^ω.

▶ **Definition 2.** For a non-null countable ordinal α, the classes $\mathbf{\Sigma}_\alpha^0$ and $\mathbf{\Pi}_\alpha^0$ of the Borel Hierarchy on the topological space X^ω are defined as follows: $\mathbf{\Sigma}_1^0$ is the class of open subsets of X^ω, $\mathbf{\Pi}_1^0$ is the class of closed subsets of X^ω, and for any countable ordinal $\alpha \geq 2$:
$\mathbf{\Sigma}_\alpha^0$ is the class of countable unions of subsets of X^ω in $\bigcup_{\gamma < \alpha} \mathbf{\Pi}_\gamma^0$.
$\mathbf{\Pi}_\alpha^0$ is the class of countable intersections of subsets of X^ω in $\bigcup_{\gamma < \alpha} \mathbf{\Sigma}_\gamma^0$.

A set $L \subseteq X^\omega$ is Borel iff it is in the union $\bigcup_{\alpha < \omega_1} \mathbf{\Sigma}_\alpha^0 = \bigcup_{\alpha < \omega_1} \mathbf{\Pi}_\alpha^0$, where ω_1 is the first uncountable ordinal.

There are also some subsets of X^ω which are not Borel. In particular the class of Borel subsets of X^ω is strictly included into the class $\mathbf{\Sigma}_1^1$ of *analytic sets* which are obtained by projection of Borel sets. The *co-analytic sets* are the complements of analytic sets.

▶ **Definition 3.** A subset A of X^ω is in the class $\mathbf{\Sigma}_1^1$ of *analytic* sets iff there exist a finite alphabet Y and a Borel subset B of $(X \times Y)^\omega$ such that $x \in A \leftrightarrow \exists y \in Y^\omega$ such that $(x,y) \in B$, where (x,y) is the infinite word over the alphabet $X \times Y$ such that $(x,y)(i) = (x(i), y(i))$ for each integer $i \geq 1$.

We now recall the notion of completeness with regard to reduction by continuous functions. For a countable ordinal $\alpha \geq 1$, a set $F \subseteq X^\omega$ is said to be a $\mathbf{\Sigma}_\alpha^0$ (respectively, $\mathbf{\Pi}_\alpha^0$, $\mathbf{\Sigma}_1^1$)-*complete set* iff for any set $E \subseteq Y^\omega$ (with Y a finite alphabet): $E \in \mathbf{\Sigma}_\alpha^0$ (respectively, $E \in \mathbf{\Pi}_\alpha^0$, $E \in \mathbf{\Sigma}_1^1$) iff there exists a continuous function $f : Y^\omega \to X^\omega$ such that $E = f^{-1}(F)$.

We now recall the definition of classes of the arithmetical hierarchy of ω-languages, see [14]. Let X be a finite alphabet. An ω-language $L \subseteq X^\omega$ belongs to the class Σ_n if and only if there exists a recursive relation $R_L \subseteq (\mathbb{N})^{n-1} \times X^\star$ such that:
$L = \{\sigma \in X^\omega \mid \exists a_1 \ldots Q_n a_n \quad (a_1, \ldots, a_{n-1}, \sigma[a_n + 1]) \in R_L\},$

where Q_i is one of the quantifiers $\forall$ or $\exists$ (not necessarily in an alternating order). An ω-language $L \subseteq X^\omega$ belongs to the class Π_n if and only if its complement $X^\omega - L$ belongs to the class Σ_n. The class Σ_1^1 is the class of *effective analytic sets* which are obtained by projection of arithmetical sets. An ω-language $L \subseteq X^\omega$ belongs to the class Σ_1^1 if and only if there exists a recursive relation $R_L \subseteq \mathbb{N} \times \{0,1\}^\star \times X^\star$ such that:

$$L = \{\sigma \in X^\omega \mid \exists \tau (\tau \in \{0,1\}^\omega \wedge \forall n \exists m((n, \tau[m], \sigma[m]) \in R_L))\}.$$

Then an ω-language $L \subseteq X^\omega$ is in the class Σ_1^1 iff it is the projection of an ω-language over the alphabet $X \times \{0,1\}$ which is in the class Π_2. The class Π_1^1 of *effective co-analytic sets* is simply the class of complements of effective analytic sets.

Recall that the (lightface) class Σ_1^1 of effective analytic sets is strictly included into the (boldface) class $\mathbf{\Sigma}_1^1$ of analytic sets.

Recall that a Büchi Turing machine is just a Turing machine working on infinite inputs with a Büchi-like acceptance condition, and that the class of ω-languages accepted by Büchi Turing machines is the class Σ_1^1 of effective analytic sets [2, 14]. On the other hand, one can construct, using a classical construction (see for instance [7]), from a Büchi Turing machine $\mathcal{T}$, a 2-counter Büchi automaton $\mathcal{A}$ accepting the same ω-language. Thus one can state the following proposition.

▶ **Proposition 4.** *An ω-language $L \subseteq X^\omega$ is in the class Σ_1^1 iff it is accepted by a non deterministic Büchi Turing machine, hence iff it is in the class $\mathbf{BCL}(2)_\omega$.*

3 Context-free Gale-Stewart games

We first recall the definition of Gale-Stewart games.

▶ **Definition 5** ([8]). Let $A \subseteq X^\omega$, where X is a finite alphabet. The Gale-Stewart game $G(A)$ is a game with perfect information between two players. Player 1 first writes a letter $a_1 \in X$, then Player 2 writes a letter $b_1 \in X$, then Player 1 writes $a_2 \in X$, and so on ... After ω steps, the two players have composed a word $x = a_1 b_1 a_2 b_2 \ldots$ of X^ω. Player 1 wins the play iff $x \in A$, otherwise Player 2 wins the play.

Let $A \subseteq X^\omega$ and $G(A)$ be the associated Gale-Stewart game. A strategy for Player 1 is a function $F_1 : (X^2)^\star \to X$ and a strategy for Player 2 is a function $F_2 : (X^2)^\star X \to X$. Player 1 follows the strategy F_1 in a play if for each integer $n \geq 1$ $a_n = F_1(a_1 b_1 a_2 b_2 \cdots a_{n-1} b_{n-1})$. If Player 1 wins every play in which she has followed the strategy F_1, then we say that the strategy F_1 is a winning strategy (w.s.) for Player 1. The notion of winning strategy for Player 2 is defined in a similar manner.

The game $G(A)$ is said to be determined if one of the two players has a winning strategy.

We shall denote $\mathbf{Det}(\mathcal{C})$, where $\mathcal{C}$ is a class of ω-languages, the sentence : "Every Gale-Stewart game $G(A)$, where $A \subseteq X^\omega$ is an ω-language in the class $\mathcal{C}$, is determined".

Notice that, in the whole paper, we assume that ZFC is consistent, and all results, lemmas, propositions, theorems, are stated in ZFC unless we explicitly give another axiomatic framework.

We can now state our first result.

▶ **Proposition 6.** $\mathbf{Det}(\Sigma_1^1) \iff \mathbf{Det}(\mathbf{r}\text{-}\mathbf{BCL}(8)_\omega)$.

Proof. The implication $\mathbf{Det}(\Sigma_1^1) \implies \mathbf{Det}(\mathbf{r}\text{-}\mathbf{BCL}(8)_\omega)$ is obvious since $\mathbf{r}\text{-}\mathbf{BCL}(8)_\omega \subseteq \Sigma_1^1$.

To prove the reverse implication, we assume that $\mathbf{Det}(\mathbf{r}\text{-}\mathbf{BCL}(8)_\omega)$ holds and we show that every Gale-Stewart game $G(A)$, where $A \subseteq X^\omega$ is an ω-language in the class Σ_1^1, or equivalently in the class $\mathbf{BCL}(2)_\omega$ by Proposition 4, is determined.

Let then $L \subseteq \Sigma^\omega$, where Σ is a finite alphabet, be an ω-language in the class $\mathbf{BCL}(2)_\omega$.

Let E be a new letter not in Σ, S be an integer ≥ 1, and $\theta_S : \Sigma^\omega \to (\Sigma \cup \{E\})^\omega$ be the function defined, for all $x \in \Sigma^\omega$, by:

$$\theta_S(x) = x(1).E^S.x(2).E^{S^2}.x(3).E^{S^3}.x(4)\ldots x(n).E^{S^n}.x(n+1).E^{S^{n+1}}\ldots$$

We proved in [3] that if $k = cardinal(\Sigma) + 2$, $S \geq (3k)^3$ is an integer, then one can effectively construct from a Büchi 2-counter automaton $\mathcal{A}_1$ accepting L a real time Büchi 8-counter automaton $\mathcal{A}_2$ such that $L(\mathcal{A}_2) = \theta_S(L)$. In the sequel we assume that we have fixed an integer $S \geq (3k)^3$ which is *even*.

Notice that the set $\theta_S(\Sigma^\omega)$ is a closed subset of the Cantor space Σ^ω. An ω-word $x \in (\Sigma \cup \{E\})^\omega$ is in $\theta_S(\Sigma^\omega)^-$ iff it has one prefix which is not in $\text{Pref}(\theta_S(\Sigma^\omega))$. Let $L' \subseteq (\Sigma \cup \{E\})^\omega$ be the set of ω-words $y \in (\Sigma \cup \{E\})^\omega$ for which there is an integer $n \geq 1$ such that $y[2n-1] \in \text{Pref}(\theta_S(\Sigma^\omega))$ and $y[2n] \notin \text{Pref}(\theta_S(\Sigma^\omega))$. It is easy to see that L' is accepted by a real time Büchi 2-counter automaton.

The class $\mathbf{r}\text{-}\mathbf{BCL}(8)_\omega \supseteq \mathbf{r}\text{-}\mathbf{BCL}(2)_\omega$ is closed under finite union in an effective way, so $\theta_S(L) \cup L'$ is accepted by a real time Büchi 8-counter automaton $\mathcal{A}_3$ which can be effectively constructed from $\mathcal{A}_2$.

As we have assumed that $\mathbf{Det}(\mathbf{r}\text{-}\mathbf{BCL}(8)_\omega)$ holds, the game $G(\theta_S(L) \cup L')$ is determined, i.e. one of the two players has a w.s. in the game $G(\theta_S(L) \cup L')$. We now show that the game $G(L)$ is itself determined.

We shall say that, during an infinite play, Player 1 "goes out" of the *closed* set $\theta_S(\Sigma^\omega)$ if the final play y composed by the two players has a prefix $y[2n] \in \text{Pref}(\theta_S(\Sigma^\omega))$ such that $y[2n+1] \notin \text{Pref}(\theta_S(\Sigma^\omega))$. We define in a similar way the sentence "Player 2 goes out of the *closed* set $\theta_S(\Sigma^\omega)$".

Assume first that Player 1 has a w.s. F_1 in the game $G(\theta_S(L) \cup L')$. Then Player 1 never "goes out" of the set $\theta_S(\Sigma^\omega)$ when she follows this w.s. because otherwise the final play y composed by the two players has a prefix $y[2n] \in \text{Pref}(\theta_S(\Sigma^\omega))$ such that $y[2n+1] \notin \text{Pref}(\theta_S(\Sigma^\omega))$ and thus $y \notin \theta_S(L) \cup L'$. Consider now a play in which Player 2 does not go out of $\theta_S(\Sigma^\omega)$. If player 1 follows her w.s. F_1 then the two players remain in the set $\theta_S(\Sigma^\omega)$. But we have fixed S to be an **even** integer. So the two players compose an ω-word

$$\theta_S(x) = x(1).E^S.x(2).E^{S^2}.x(3).E^{S^3}.x(4)\ldots x(n).E^{S^n}.x(n+1).E^{S^{n+1}}\ldots$$

and the letters $x(k)$ are written by player 1 for k an odd integer and by Player 2 for k an even integer because S is even. Moreover Player 1 wins the play iff the ω-word $x(1)x(2)x(3)\ldots x(n)\ldots$ is in L. This implies that Player 1 has also a w.s. in the game $G(L)$.

Assume now that Player 2 has a w.s. F_2 in the game $G(\theta_S(L) \cup L')$. Then Player 2 never "goes out" of the set $\theta_S(\Sigma^\omega)$ when he follows this w.s. because otherwise the final play y composed by the two players has a prefix $y[2n-1] \in \text{Pref}(\theta_S(\Sigma^\omega))$ such that $y[2n] \notin \text{Pref}(\theta_S(\Sigma^\omega))$ and thus $y \in L'$ hence also $y \in \theta_S(L) \cup L'$. Consider now a play in which Player 1 does not go out of $\theta_S(\Sigma^\omega)$. If player 2 follows his w.s. F_2 then the two players remain in the set $\theta_S(\Sigma^\omega)$. So the two players compose an ω-word

$$\theta_S(x) = x(1).E^S.x(2).E^{S^2}.x(3).E^{S^3}.x(4)\ldots x(n).E^{S^n}.x(n+1).E^{S^{n+1}}\ldots$$

where the letters $x(k)$ are written by player 1 for k an odd integer and by Player 2 for k an even integer. Moreover Player 2 wins the play iff the ω-word $x(1)x(2)x(3)\ldots x(n)\ldots$ is not in L. This implies that Player 2 has also a w.s. in the game $G(L)$. ◄

▶ **Theorem 7.** $\mathbf{Det}(\Sigma^1_1) \iff \mathbf{Det}(\mathbf{CFL}_\omega) \iff \mathbf{Det}(\mathbf{BCL}(1)_\omega)$.

Proof. The implications $\mathbf{Det}(\Sigma^1_1) \implies \mathbf{Det}(\mathbf{CFL}_\omega) \implies \mathbf{Det}(\mathbf{BCL}(1)_\omega)$ are obvious since $\mathbf{BCL}(1)_\omega \subseteq \mathbf{CFL}_\omega \subseteq \Sigma^1_1$.

To prove the reverse implication $\mathbf{Det}(\mathbf{BCL}(1)_\omega) \implies \mathbf{Det}(\Sigma^1_1)$, we assume that $\mathbf{Det}(\mathbf{BCL}(1)_\omega)$ holds and we are going to show that then every Gale-Stewart game $G(L)$, where $L \subseteq X^\omega$ is an ω-language in the class $\mathbf{r\text{-}BCL}(8)_\omega$ is determined. Then Proposition 6 will imply that $\mathbf{Det}(\Sigma^1_1)$ also holds. Let then $L(\mathcal{A}) \subseteq \Gamma^\omega$, where Γ is a finite alphabet and $\mathcal{A}$ is a real time Büchi 8-counter automaton.

We now recall the following coding which was used in the paper [3].

Let K be the product of the eight first prime numbers. An ω-word $x \in \Gamma^\omega$ was coded by the ω-word

$$h_K(x) = A.C^K.x(1).B.C^{K^2}.A.C^{K^2}.x(2).B.C^{K^3}.A.C^{K^3}.x(3).B\ldots B.C^{K^n}.A.C^{K^n}.x(n).B\ldots$$

over the alphabet $\Gamma_1 = \Gamma \cup \{A, B, C\}$, where A, B, C are new letters not in Γ. We are going to use here a slightly different coding which we now define. Let then

$$h(x) = C^K.C.A.x(1).C^{K^2}.A.C^{K^2}.C.x(2).B.C^{K^3}.A.C^{K^3}.C.A.x(3)\ldots$$

$$\ldots C^{K^{2n}}.A.C^{K^{2n}}.C.x(2n).B.C^{K^{2n+1}}.A.C^{K^{2n+1}}.C.A.x(2n+1)\ldots$$

We now explain the rules used to obtain the ω-word $h(x)$ from the ω-word $h_K(x)$.

(1) The first letter A of the word $h_K(x)$ has been suppressed.

(2) The letters B following a letter $x(2n+1)$, for $n \geq 1$, have been suppressed.

(3) A letter C has been added before each letter $x(2n)$, for $n \geq 1$.

(4) A block of two letters $C.A$ has been added before each letter $x(2n+1)$, for $n \geq 1$.

The reasons behind this changes are the following ones. Assume that two players alternatively write letters from the alphabet $\Gamma_1 = \Gamma \cup \{A, B, C\}$ and that they finally produce an ω-word in the form $h(x)$. Due to the above changes we have now the two following properties which will be useful in the sequel.

(1) The letters $x(2n+1)$, for $n \geq 0$, have been written by Player 1, and the letters $x(2n)$, for $n \geq 1$, have been written by Player 2.

(2) After a sequence of consecutive letters C, the first letter which is not a C has always been written by Player 2.

We proved in [3] that, from a real time Büchi 8-counter automaton $\mathcal{A}$ accepting $L(\mathcal{A}) \subseteq \Gamma^\omega$, one can effectively construct a Büchi 1-counter automaton $\mathcal{A}_1$ accepting the ω-language $h_K(L(\mathcal{A})) \cup h_K(\Gamma^\omega)^-$. We can easily check that the changes in $h_K(x)$ leading to the coding $h(x)$ have no influence with regard to the proof of this result in [3] and thus one can also effectively construct a Büchi 1-counter automaton $\mathcal{A}_2$ accepting the ω-language $h(L(\mathcal{A})) \cup h(\Gamma^\omega)^-$.

On the other hand we can remark that all ω-words in the form $h(x)$ belong to the ω-language $H \subseteq (\Gamma_1)^\omega$ of ω-words y of the following form:

$$y = C^{n_1}.C.A.x(1).C^{n_2}.A.C^{n'_2}.C.x(2).B.C^{n_3}.A.C^{n'_3}.C.A.x(3)\ldots$$
$$\ldots C^{n_{2n}}.A.C^{n'_{2n}}.C.x(2n).B.C^{n_{2n+1}}.A.C^{n'_{2n+1}}.C.A.x(2n+1)\ldots$$

where for all integers $i \geq 1$ the letters $x(i)$ belong to Γ and the n_i, n_i', are even non-null integers.

An important fact is the following property of H which extends the same property of the set $h(\Gamma^\omega)$. Assume that two players alternatively write letters from the alphabet $\Gamma_1 = \Gamma \cup \{A, B, C\}$ and that they finally produce an ω-word y in H in the above form. Then we have the two following facts:

(1) The letters $x(2n+1)$, for $n \geq 0$, have been written by Player 1, and the letters $x(2n)$, for $n \geq 1$, have been written by Player 2.

(2) After a sequence of consecutive letters C, the first letter which is not a C has always been written by Player 2.

Let now $V = \mathrm{Pref}(H) \cap (\Gamma_1)^\star.C$. So a finite word over the alphabet Γ_1 is in V iff it is a prefix of some word in H and its last letter is a C. It is easy to see that the topological closure of H is

$$\mathrm{Cl}(H) = H \ \cup \ V.C^\omega.$$

Notice that an ω-word in $\mathrm{Cl}(H)$ is not in $h(\Gamma^\omega)$ iff a sequence of consecutive letters C has not the good length. Thus if two players alternatively write letters from the alphabet Γ_1 and produce an ω-word $y \in \mathrm{Cl}(H) - h(\Gamma^\omega)$ then it is Player 2 who has gone out of the set $h(\Gamma^\omega)$ at some step of the play. This will be important in the sequel.

It is very easy to see that the ω-language H is regular and to construct a Büchi automaton $\mathcal{H}$ accepting it. Moreover it is known that the class $\mathbf{BCL}(1)_\omega$ is effectively closed under intersection with regular ω-languages (this can be seen using a classical construction of a product automaton). Thus one can also construct a Büchi 1-counter automaton $\mathcal{A}_3$ accepting the ω-language $h(L(\mathcal{A})) \cup [h(\Gamma^\omega)^- \cap H]$.

We denote also U the set of finite words u over Γ_1 such that $|u| = 2n$ for some integer $n \geq 1$ and $u[2n-1] \in \mathrm{Pref}(H)$ and $u = u[2n] \notin \mathrm{Pref}(H)$.

Now we set:

$$\mathcal{L} \ = \ h(L(\mathcal{A})) \ \cup \ [h(\Gamma^\omega)^- \cap H] \ \cup \ V.C^\omega \ \cup \ U.(\Gamma_1)^\omega$$

We have already seen that the ω-language $h(L(\mathcal{A})) \cup [h(\Gamma^\omega)^- \cap H]$ is accepted by a Büchi 1-counter automaton $\mathcal{A}_3$. On the other hand the ω-language H is regular and it is accepted by a Büchi automaton $\mathcal{H}$. Thus the finitary language $\mathrm{Pref}(H)$ is also regular, the languages U and V are also regular, and the ω-languages $V.C^\omega$ and $U.(\Gamma_1)^\omega$ are regular. This implies that one can construct a Büchi 1-counter automaton $\mathcal{A}_4$ accepting the language $\mathcal{L}$.

By hypothesis we assume that $\mathbf{Det}(\mathbf{BCL}(1)_\omega)$ holds and thus the game $G(\mathcal{L})$ is determined. We are going to show that this implies that the game $G(L(\mathcal{A}))$ itself is determined.

Assume firstly that Player 1 has a winning strategy F_1 in the game $G(\mathcal{L})$.

If during an infinite play, the two players compose an infinite word z, and Player 2 "does not go out of the set $h(\Gamma^\omega)$" then we claim that also Player 1, following her strategy F_1, "does not go out of the set $h(\Gamma^\omega)$". Indeed if Player 1 goes out of the set $h(\Gamma^\omega)$ then due to the above remark this would imply that Player 1 also goes out of the set $\mathrm{Cl}(H)$: there is an integer $n \geq 0$ such that $z[2n] \in \mathrm{Pref}(H)$ but $z[2n+1] \notin \mathrm{Pref}(H)$. So $z \notin h(L(\mathcal{A})) \ \cup \ [h(\Gamma^\omega)^- \cap H] \cup V.C^\omega$. Moreover it follows from the definition of U that $z \notin U.(\Gamma_1)^\omega$. Thus If Player 1 goes out of the set $h(\Gamma^\omega)$ then she looses the game.

Consider now an infinite play in which Player 2 "does not go out of the set $h(\Gamma^\omega)$". Then Player 1, following her strategy F_1, "does not go out of the set $h(\Gamma^\omega)$". Thus the two players write an infinite word $z = h(x)$ for some infinite word $x \in \Gamma^\omega$. But the letters $x(2n + 1)$, for $n \geq 0$, have been written by Player 1, and the letters $x(2n)$, for $n \geq 1$, have been written by Player 2. Player 1 wins the play iff $x \in L(\mathcal{A})$ and Player 1 wins always the play when she uses her strategy F_1. This implies that Player 1 has also a w.s. in the game $G(L(\mathcal{A}))$.

Assume now that Player 2 has a winning strategy F_2 in the game $G(\mathcal{L})$.

If during an infinite play, the two players compose an infinite word z, and Player 1 "does not go out of the set $h(\Gamma^\omega)$" then we claim that also Player 2, following his strategy F_2, "does not go out of the set $h(\Gamma^\omega)$". Indeed if Player 2 goes out of the set $h(\Gamma^\omega)$ and the final play z remains in $\mathrm{Cl}(H)$ then $z \in [h(\Gamma^\omega)^- \cap H] \cup V.C^\omega \subseteq \mathcal{L}$ and Player 2 looses. If Player 1 does not go out of the set $\mathrm{Cl}(H)$ and at some step of the play, Player 2 goes out of $\mathrm{Pref}(H)$, i.e. there is an integer $n \geq 1$ such that $z[2n - 1] \in \mathrm{Pref}(H)$ and $z[2n] \notin \mathrm{Pref}(H)$, then $z \in U.(\Gamma_1)^\omega \subseteq \mathcal{L}$ and Player 2 looses.

Assume now that Player 1 "does not go out of the set $h(\Gamma^\omega)$". Then Player 2 follows his w. s. F_2, and then "never goes out of the set $h(\Gamma^\omega)$". Thus the two players write an infinite word $z = h(x)$ for some infinite word $x \in \Gamma^\omega$. But the letters $x(2n + 1)$, for $n \geq 0$, have been written by Player 1, and the letters $x(2n)$, for $n \geq 1$, have been written by Player 2. Player 2 wins the play iff $x \notin L(\mathcal{A})$ and Player 2 wins always the play when he uses his strategy F_2. This implies that Player 2 has also a w.s. in the game $G(L(\mathcal{A}))$. ◀

Looking carefully at the above proof, we can obtain a stronger result:

▶ **Theorem 8.** $\mathbf{Det}(\Sigma_1^1) \Longleftrightarrow \mathbf{Det}(\mathbf{CFL}_\omega) \Longleftrightarrow \mathbf{Det}(\mathbf{r\text{-}BCL}(1)_\omega).$

4 Context-free Wadge games

We first recall the notion of Wadge games.

▶ **Definition 9** (Wadge [17]). Let $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$. The Wadge game $W(L, L')$ is a game with perfect information between two players, Player 1 who is in charge of L and Player 2 who is in charge of L'. Player 1 first writes a letter $a_1 \in X$, then Player 2 writes a letter $b_1 \in Y$, then Player 1 writes a letter $a_2 \in X$, and so on. The two players alternatively write letters a_n of X for Player 1 and b_n of Y for Player 2. After ω steps, Player 1 has written an ω-word $a \in X^\omega$ and Player 2 has written an ω-word $b \in Y^\omega$. Player 2 is allowed to skip, even infinitely often, provided he really writes an ω-word in ω steps. Player 2 wins the play iff $[a \in L \leftrightarrow b \in L']$, i.e. iff: $\quad [(a \in L$ and $b \in L')$ or $(a \notin L$ and $b \notin L'$ and b is infinite$)]$.

Recall that a strategy for Player 1 is a function $\sigma : (Y \cup \{s\})^\star \to X$. And a strategy for Player 2 is a function $f : X^+ \to Y \cup \{s\}$. The strategy σ is a winning strategy for Player 1 iff she always wins a play when she uses the strategy σ, i.e. when the nth letter she writes is given by $a_n = \sigma(b_1 \ldots b_{n-1})$, where b_i is the letter written by Player 2 at step i and $b_i = s$ if Player 2 skips at step i. A winning strategy for Player 2 is defined in a similar manner.

The game $W(L, L')$ is said to be determined if one of the two players has a winning strategy. In the sequel we shall denote $\mathbf{W\text{-}Det}(\mathcal{C})$, where $\mathcal{C}$ is a class of ω-languages, the sentence: "All Wadge games $W(L, L')$, where $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$ are ω-languages in the class $\mathcal{C}$, are determined".

There is a close relationship between Wadge reducibility and games.

▶ **Definition 10** (Wadge [17])**.** Let X, Y be two finite alphabets. For $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$, L is said to be Wadge reducible to L' ($L \leq_W L'$) iff there exists a continuous function $f : X^\omega \to Y^\omega$, such that $L = f^{-1}(L')$. L and L' are Wadge equivalent iff $L \leq_W L'$ and $L' \leq_W L$. This will be denoted by $L \equiv_W L'$. And we shall say that $L <_W L'$ iff $L \leq_W L'$ but not $L' \leq_W L$.

The relation $\leq_W$ is reflexive and transitive, and $\equiv_W$ is an equivalence relation.

The *equivalence classes* of $\equiv_W$ are called *Wadge degrees*.

▶ **Theorem 11** (Wadge)**.** *Let $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$ where X and Y are finite alphabets. Then $L \leq_W L'$ if and only if Player 2 has a winning strategy in the Wadge game $W(L, L')$.*

The Wadge hierarchy WH is the class of Borel subsets of a set X^ω, where X is a finite set, equipped with $\leq_W$ and with $\equiv_W$. Using Wadge games, Wadge proved that, up to the complement and $\equiv_W$, it is a well ordered hierarchy which provides a great refinement of the Borel hierarchy.

We can now state the following result on determinacy of context-free Wadge games.

▶ **Theorem 12. Det**$(\Sigma^1_1) \iff$ **W-Det**$(\mathbf{CFL}_\omega) \iff$ **W-Det**$(\mathbf{BCL}(1)_\omega) \iff$ **W-Det**$(\mathbf{r}\text{-}\mathbf{BCL}(1)_\omega)$.

Recall that, assuming that ZFC is consistent, there are some models of ZFC in which **Det**(Σ^1_1) does not hold. Therefore there are some models of ZFC in which some Wadge games $W(L(\mathcal{A}), L(\mathcal{B}))$, where $\mathcal{A}$ and $\mathcal{B}$ are Büchi 1-counter automata, are not determined. We are going to prove that this may be also the case when $\mathcal{B}$ is a Büchi automaton (without counter). To prove this, we use a recent result of [4] and some results of set theory, so we now briefly recall some notions of set theory and refer the reader to [4] and to a textbook like [8] for more background on set theory.

The usual axiomatic system ZFC is Zermelo-Fraenkel system ZF plus the axiom of choice AC. The axioms of ZFC express some natural facts that we consider to hold in the universe of sets. A model $(\mathbf{V}, \in)$ of an arbitrary set of axioms $\mathbb{A}$ is a collection $\mathbf{V}$ of sets, equipped with the membership relation $\in$, where "$x \in y$" means that the set x is an element of the set y, which satisfies the axioms of $\mathbb{A}$. We often say " the model $\mathbf{V}$" instead of "the model $(\mathbf{V}, \in)$".

We say that two sets A and B have same cardinality iff there is a bijection from A onto B and we denote this by $A \approx B$. The relation $\approx$ is an equivalence relation. Using the axiom of choice AC, one can prove that any set A can be well-ordered so there is an ordinal γ such that $A \approx \gamma$. In set theory the cardinal of the set A is then formally defined as the smallest such ordinal γ. The infinite cardinals are usually denoted by $\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\alpha, \ldots$ The continuum hypothesis CH says that the first uncountable cardinal $\aleph_1$ is equal to $2^{\aleph_0}$ which is the cardinal of the continuum.

If $\mathbf{V}$ is a model of ZF and $\mathbf{L}$ is the class of *constructible sets* of $\mathbf{V}$, then the class $\mathbf{L}$ is a model of ZFC + CH. Notice that the axiom V=L, which means "every set is constructible", is consistent with ZFC because $\mathbf{L}$ is a model of ZFC + V=L.

Consider now a model $\mathbf{V}$ of ZFC and the class of its constructible sets $\mathbf{L} \subseteq \mathbf{V}$ which is another model of ZFC. It is known that the ordinals of $\mathbf{L}$ are also the ordinals of $\mathbf{V}$, but the cardinals in $\mathbf{V}$ may be different from the cardinals in $\mathbf{L}$. In particular, the first uncountable cardinal in $\mathbf{L}$ is denoted $\aleph_1^{\mathbf{L}}$, and it is in fact an ordinal of $\mathbf{V}$ which is denoted $\omega_1^{\mathbf{L}}$. It is well-known that in general this ordinal satisfies the inequality $\omega_1^{\mathbf{L}} \leq \omega_1$. In a model $\mathbf{V}$ of

the axiomatic system ZFC + V=L the equality $\omega_1^{\mathbf{L}} = \omega_1$ holds, but in some other models of ZFC the inequality may be strict and then $\omega_1^{\mathbf{L}} < \omega_1$.

The following result was proved in [4].

▶ **Theorem 13.** *There exists a real-time 1-counter Büchi automaton $\mathcal{A}$, which can be effectively constructed, such that the topological complexity of the ω-language $L(\mathcal{A})$ is not determined by the axiomatic system ZFC. Indeed it holds that :*

(1) *(ZFC + V=L).* The ω-language $L(\mathcal{A})$ is an analytic but non-Borel set.

(2) *(ZFC + $\omega_1^{\mathbf{L}} < \omega_1$).* The ω-language $L(\mathcal{A})$ is a $\mathbf{\Pi}_2^0$-set.

We now state the following new result. To prove it we use in particular the above Theorem 13, the link between Wadge games and Wadge reducibility, the $\mathbf{\Pi}_2^0$-completeness of the regular ω-language $(0^\star.1)^\omega \subseteq \{0,1\}^\omega$, the Shoenfield's Absoluteness Theorem, and the notion of extensions of a model of ZFC.

▶ **Theorem 14.** [1] *Let $\mathcal{B}$ be a Büchi automaton accepting the regular ω-language $(0^\star.1)^\omega \subseteq \{0,1\}^\omega$. Then one can effectively construct a real-time 1-counter Büchi automaton $\mathcal{A}$ such that:*

(1) *(ZFC + $\omega_1^{\mathbf{L}} < \omega_1$). Player 2 has a winning strategy F in the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$. But F cannot be recursive and not even hyperarithmetical.*

(2) *(ZFC + $\omega_1^{\mathbf{L}} = \omega_1$). The Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$ is not determined.*

▶ Remark 15. Every model of ZFC is either a model of (ZFC + $\omega_1^{\mathbf{L}} < \omega_1$) or a model of (ZFC + $\omega_1^{\mathbf{L}} = \omega_1$). Thus there are no models of ZFC in which Player 1 has a winning strategy in the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$.

▶ Remark 16. In order to prove Theorem 14 we do not need to use any large cardinal axiom or even the consistency of such an axiom, like the axiom of analytic determinacy.

5 Concluding remarks

We have proved that the determinacy of Gale-Stewart games whose winning sets are accepted by (real-time) 1-counter Büchi automata is equivalent to the determinacy of (effective) analytic Gale-Stewart games which is known to be a large cardinal assumption.

On the other hand we have proved a similar result about the determinacy of Wadge games. We have also obtained an amazing result, proving that one can effectively construct a real-time 1-counter Büchi automaton $\mathcal{A}$ and a Büchi automaton $\mathcal{B}$ such that the sentence "the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$ is determined" is actually independent from ZFC.

Notice that it is still unknown whether the determinacy of Wadge games $W(L(\mathcal{A}), L(\mathcal{B}))$, where $\mathcal{A}$ and $\mathcal{B}$ are Muller tree automata (reading infinite labelled trees) , is provable within ZFC or needs some large cardinal assumptions to be proved.

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[1] This result has been recently exposed in the Workshops GASICS 2010, Paris, September 2010, and GAMES 2010, Oxford, September 2010, but it has never been published.

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