

# Term Rewriting Systems as Topological Dynamical Systems

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## Abstract

Topological dynamics is, roughly, the study of phenomena related to iterations of continuous maps from a metric space to itself. We show how the rewrite relation in term rewriting gives rise to dynamical systems in two distinct, natural ways: (A) One in which any deterministic rewriting strategy induces a dynamical system on the set of finite and infinite terms endowed with the usual metric, and (B) one in which the unconstrained rewriting relation induces a dynamical system on sets of sets of terms, specifically the set of compact subsets of the set of finite and infinite terms endowed with the Hausdorff metric.

For both approaches, we give sufficient criteria for the induced systems to be well-defined dynamical systems and for (A) we demonstrate how the classic topological invariant called topological entropy turns out to be much less useful in the setting of term rewriting systems than in symbolic dynamics.

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## 1 Topological dynamics and rewriting

A *topological dynamical system* is a pair  $(X, f)$  where  $X$  is a metric space and  $f : X \rightarrow X$  is a continuous map. The primary object of study is the long-term behaviour of iterations of the map  $f$ . Such systems are widely studied in pure and applied mathematics, for example in mathematical physics, in fractal geometry, and in combinatorial number theory [12]. A well-studied subclass of dynamical systems is that of *symbolic dynamical systems* [25] where  $X$  is a topologically closed set of infinite strings and  $f$  is the *shift map*, either chopping off the first symbol of a right-infinite string or shifting a bi-infinite string one position to the left. Such systems with right-infinite strings can be easily modelled by string rewriting systems with very simple rules ( $a \rightarrow \epsilon$  for right-infinite strings) and endowed with a rewrite strategy. Equivalently, they can be modelled by term rewriting systems over unary signatures (with rules  $a(x) \rightarrow x$ ) endowed with a rewrite strategy. The observation that existing dynamical systems can be modelled by almost trivial rules immediately raises the twin questions of whether more general term rewriting systems on well-behaved sets of terms can be viewed as topological dynamical systems.



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We answer this question in the positive by defining two distinct notions of dynamical systems that we believe are quite natural: (A) One in which any (suitably continuous) deterministic rewriting strategy induces a dynamical system on the set of finite and infinite ground terms endowed with the usual metric, and (B) one in which the unconstrained rewriting relation induces a dynamical system on sets of sets of terms, specifically the set of compact subsets of the set of finite and infinite ground terms endowed with the Hausdorff metric. We also provide a starting point for investigating the dynamical properties of these systems by considering continuity, conjugacy and topological entropy.

We employ *infinite*, *ground* terms for two reasons: (1) that they are needed to generalize the classes of dynamical systems that already exist in the literature, (2) that the set of finite and infinite ground terms is topologically much more well-behaved than the set of finite terms, hence allow stronger results. We stress that the *rewriting relation* itself is ordinary one-step rewriting; thus, we have no infinite reductions, and we do *not* consider infinitary rewriting [18]. However, as topological dynamics considers the long-term behaviour of iterated maps, we do expect that connections with this area will be found in the future.

**A clarification from the start:** We do not have any practical applications of the present work, nor have we actively sought out such applications. We believe the topological dynamics of term rewriting to be mathematically interesting *per se*, even more so as the systems we consider properly generalize existing classes of dynamical systems.

## 1.1 Related work

The class of (one-dimensional) symbolic dynamical systems (SDSs) is a special case of the class of systems we define in this paper, and is the main in inspiration for our work; there is a vast literature on SDSs, and we refer the reader to the excellent textbooks and handbook chapters in the area [25, 22, 6]. Kitchens defines three classes of dynamical systems associated to a class of maps on finite trees [23]; the trees are not in general trees of terms and the maps themselves do not correspond to term rewriting. The thematically closest research to ours are the *tree-shifts* of Aubrun and Béal [2, 3], introduced as an intermediate notion between one-sided shifts of dimension one, resp. higher dimensions. In contrast to the systems we consider, tree-shifts pertain to infinite, labelled trees with a fixed arity  $n$  (each node in the tree has  $i$  children), and the *set* of maps associated to the set of such trees are the  $n$  maps  $\sigma_i$  such that  $\sigma_i$  cuts off the root node and returns the  $i$ th child of the root. Aubrun and Béal successfully generalize a number of concepts and results from SDSs to the class of tree-shifts, including the concepts of system of finite type and sofic systems, and prove the highly interesting result that conjugacy of tree-shifts of finite type is decidable. We firmly believe, but have not proved, that these results can be extended to many of the systems we consider in this paper.

## 2 Preliminaries

### 2.1 Term rewriting on finite and infinite terms

We refer to standard textbooks [4, 27] for basics on term rewriting; to fix notation, we give the most necessary definitions below.

A *signature* is a set of *function symbols*, each endowed with a non-negative integer *arity*; if  $f$  is a function symbol of arity  $n$ , we invariably write  $f/n$ . If  $\Sigma$  is a signature and  $V$  is a set of variables, we denote by  $T(\Sigma, V)$  the set of *terms over  $\Sigma$  and  $V$* ; the elements of the set are all variables, all nullary function symbols, and all objects on the form  $f(s_1, \dots, s_n)$

where  $f \in \Sigma$  has arity  $n$  and  $s_1, \dots, s_n$  are terms over  $\Sigma$  and  $V$ . The *root symbol* of a term  $s = f(s_1, \dots, s_n)$  (for  $n \geq 0$ ), denoted  $\text{root}(s)$ , is  $f$ . The set of *ground terms* over  $\Sigma$  is  $T(\Sigma, \emptyset)$ . A (one-hole) *context* is a term in  $T(\Sigma, V \cup \{\square\})$  that contains exactly one occurrence of  $\square$ , where  $\square \notin \Sigma \cup V$ . A *substitution* is a map  $\theta : V \rightarrow T(\Sigma, V)$  where we usually assume that only a finite number of elements  $x \in V$  satisfy  $\theta(x) \neq x$ . A substitution extends homomorphically to a map  $\theta : T(\Sigma, V) \rightarrow T(\Sigma, V)$  in the obvious way. If  $C$  is a context and  $s$  is a term, we define  $C[s] = \theta(C)$  where  $\theta$  is the substitution defined by  $\theta(\square) = s$  and  $\theta(x) = x$  for  $x \neq \square$ .

The *set of positions* of a term  $s$ , denoted  $\text{Pos}(s)$  is the set of finite sequences of positive integers defined inductively as follows: If  $s \in V$  or  $s$  is a nullary function symbol, then  $\text{Pos}(s) = \{\epsilon\}$  where  $\epsilon$  is the empty string. If  $s = f(s_1, \dots, s_n)$ , then  $\text{Pos}(s) = \{\epsilon\} \cup \{ip : p \in \text{Pos}(s_i) \wedge 1 \leq i \leq n\}$ . The *length* of a position is its length as a string. The *subterm* of term  $s$  at position  $p \in \text{Pos}(s)$ , denoted  $s|_p$ , is defined inductively by  $s|_\epsilon = s$  and otherwise (in which case  $s = f(s_1, \dots, s_n)$  and  $p = ip'$  for some  $p' \in \text{Pos}(s_i)$ ) by  $s|_p = s_i|_{p'}$ . For notational convenience, we set  $\text{POS} = \mathbb{N}^{<\omega}$  (“the set of all possible positions”), with the convention that  $\epsilon \in \text{POS}$ . If  $p \in \text{POS}$  and  $j$  is a non-negative integer, we set  $p^0 = \epsilon$  and  $p^j = p^{j-1}p$  (i.e., the sequence consisting of  $j$  copies of  $p$ ). If  $C$  is a context, we occasionally write  $C$  as  $C|_p$  where  $p$  is the unique position of  $\square$  in  $C$ .

A *term rewriting system* (abbreviated TRS) over  $\Sigma$  and  $V$  is a set of pairs, invariably written  $l \rightarrow r$ , such that (i)  $l, r \in T(\Sigma, V)$ , (ii)  $l \notin V$ , and (iii) every variable that occurs in  $r$  also occurs in  $l$ . The *rewrite relation induced by  $R$*  is the binary relation  $\rightarrow_R \subseteq T(\Sigma, V) \times T(\Sigma, V)$  defined by  $s \rightarrow_R t$  if there exist a rule  $l \rightarrow r \in R$ , a context  $C|_p$  and a substitution  $\theta$  such that  $s = C[\theta(l)]_p$  and  $C[\theta(r)]_p = t$  (in which case we say that the pair  $(p, l \rightarrow r)$  is a *redex* in  $s$ ). We invariably drop the subscript  $R$ , writing  $\rightarrow$ , when there is no risk of confusion. A rule  $l \rightarrow r$  is called *collapsing* if  $r = x \in V$ .

The standard *metric*  $d : T(\Sigma, V) \times T(\Sigma, V) \rightarrow \mathbb{R}_+ \cup \{0\}$  is defined by  $d(s, s) = 0$  and  $d(s, t) = 2^{-|p|}$  where  $p$  is a position of minimal length such that  $\text{root}(s|_p) \neq \text{root}(t|_p)$ .

The set of *finite and infinite terms* over  $\Sigma$  and  $V$ , denoted  $T^\infty(\Sigma, V)$  is the (necessarily unique) metric completion of  $T(\Sigma, V)$ . The set of *infinite terms* over  $\Sigma$  and  $V$  is  $T^\infty(\Sigma, V) \setminus T(\Sigma, V)$ ; note that an infinite term may have subterms that are finite. The set of finite and infinite *ground terms* over  $\Sigma$  is  $T^\infty(\Sigma, \emptyset)$ .

An equivalent definition of  $T^\infty(\Sigma, V)$  is obtained by interpreting the term formation rules coinductively instead of inductively, and still other equivalent methods exist in the form of ideal completion and partial functions (see, e.g., [20]) If in doubt: *infinite terms are exactly what you think they are.*

Let  $R$  be a TRS over  $\Sigma$  and  $V$ ; the rewrite relation  $\rightarrow_R \subseteq T^\infty(\Sigma, V) \times T^\infty(\Sigma, V)$  is defined exactly as in the finite case (note that in all rules  $l \rightarrow r \in R$ , both  $l$  and  $r$  are still finite terms). We denote by  $\text{NF}(R)$  the set of  $R$ -normal forms of  $T^\infty(\Sigma, \emptyset)$  (i.e. the set of possibly infinite ground terms  $s$  such that there exists no  $t$  with  $s \rightarrow t$ ).

## 2.2 Topological dynamics

We refer to standard textbooks on topological dynamics such as [12, 28], giving only the most basic definitions below.

A *topological dynamical system* is a pair  $(X, f)$  where  $X$  is a topological space and  $f : X \rightarrow X$  is a continuous map. It is usually assumed that  $X$  is locally compact, metrizable and second countable.

A standard notion of *subsystems* exists for dynamical systems  $(A, f)$ : A dynamical system  $(B, f)$  is a subsystem of  $(A, f)$  if  $B$  is a topologically closed subset of  $A$  that is

$f$ -(forward-)closed, that is  $f(B) \subseteq B$  [28, Ch. 5].

The following two examples introduce *one-* and *two-sided shifts* that are the primary objects of study in the field of *symbolic dynamics*; for more information, consult textbooks specifically on symbolic dynamics, for example [25, 22].

► **Example 2.1** (One-sided shifts). Let  $\tilde{\Sigma}$  be a finite string alphabet, and let  $F \subseteq \tilde{\Sigma}^*$  be a (finite or infinite) set (called the set of *forbidden words*). Let  $\tilde{A}_F$  be the set of all right-infinite strings over  $\tilde{\Sigma}$  that do not contain any element of  $F$  as a substring. Then by standard results  $\tilde{A}_F$  is a topologically closed subset of  $\tilde{\Sigma}^\omega$  equipped with the usual Cantor metric on sequences [25]. The *shift map*  $\sigma$  on  $\tilde{A}_F$  is defined by  $\sigma(b_1b_2b_3\cdots) = b_2b_3\cdots$ , and it is easily seen that  $(\tilde{A}_F, \sigma)$  is a topological dynamical system.

► **Example 2.2** (Two-sided shifts). Proceed as in Example 2.1, but define instead  $\tilde{B}_F$  as the set of all *bi*-infinite strings over  $\tilde{\Sigma}$  that do not contain any element of  $F$  as a substring, and let the shift map  $\sigma$  be given by  $\sigma(\cdots b_{-2}b_{-1}b_0b_1b_2\cdots) = (c_n)_{n \in \mathbb{Z}}$  where  $c_n = b_{n+1}$  for all  $n \in \mathbb{Z}$ . Then,  $(\tilde{B}_F, \sigma)$  is a topological dynamical system.

In both of the above examples we may have  $F = \emptyset$  in which case  $\tilde{A}_F$  (resp.  $\tilde{B}_F$ ) is the set of all right-infinite (resp. bi-infinite) strings over  $\tilde{\Sigma}$ .

Finally, recall that a metric space  $(M, d)$  is an *ultrametric space* if  $d(x, y) \leq \max\{d(x, z), d(z, y)\}$  for arbitrary  $x, y, z \in M$ . The standard metric  $d$  on  $T^\infty(\Sigma, \emptyset)$  is an ultrametric.

► **Lemma 2.3.** *Let  $(M, d)$  be an ultrametric space. If  $B(x, \epsilon)$  and  $B(z, \delta)$  are open balls with  $\delta \leq \epsilon$  and  $B(x, \epsilon) \cap B(z, \delta) \neq \emptyset$  then  $B(z, \delta) \subseteq B(x, \epsilon)$ . In particular, if  $\delta = \epsilon$ ,  $B(x, \epsilon) \cap B(z, \delta) \neq \emptyset$  implies  $B(x, \epsilon) = B(z, \delta)$ .*

*Let  $A \subseteq M$ . Then  $A$  has a cover of cardinality  $k$  of sets of diameter at most  $\epsilon$  iff it has a cover of cardinality  $k$  of open balls of radius  $\epsilon$ .*

*Lastly,  $(M, d_n)$  is an ultrametric space.*

### 3 Topological dynamics and term rewriting

Many results in topological dynamics are contingent on the underlying topological space being locally compact (and, in symbolic dynamics, compact). It is easy to see that  $(T(\Sigma, V), d)$  is locally compact: For any term  $s$ , let  $m$  be a positive integer that is strictly greater than the length of the longest position in  $s$ ; then the open ball with radius  $2^{-m}$  centered on  $s$  is exactly the (necessarily compact) singleton set  $\{s\}$ . However, in many cases we would like  $(T(\Sigma, V), d)$  to be complete—and preferably compact—which it is not in general:

► **Lemma 3.1.** *Let  $\Sigma$  contain at least one symbol with arity  $\geq 1$ . If  $T(\Sigma, \emptyset)$  is not empty, then it is not complete (hence is not compact).*

**Proof.** Let  $f \in \Sigma$  have arity  $m \geq 1$ . If  $T(\Sigma, \emptyset) \neq \emptyset$ , there must be  $a \in \Sigma$  with arity 0. Define the sequence of terms  $(s_n)$  by  $s_0 = a$  and  $s_{n+1} = f(s_n, \dots, s_n)$ . Then,  $(s_n)$  is Cauchy, but has no limit in  $T(\Sigma, \emptyset)$ . ◀

If we extend our attention to  $T^\infty(\Sigma, V)$ , however, the topological properties are much better. The following results hold for the metric space  $(T^\infty(\Sigma, V), d)$ :

► **Lemma 3.2** (See [1]). *Let  $\Sigma$  be a (possibly infinite) signature. Then,  $T^\infty(\Sigma, V)$  is complete. In addition,  $T^\infty(\Sigma, V)$  is compact iff both  $\Sigma$  and  $V$  are finite (in particular, if  $\Sigma$  is finite, then  $T^\infty(\Sigma, \emptyset)$  is compact).*

**An important convention:** In view of Lemma 3.2, we restrict our attention to  $T^\infty(\Sigma, \emptyset)$ —the set of finite and infinite *ground* terms over a *finite* signature, and hence the compact metric space  $(T^\infty(\Sigma, \emptyset), d)$ . In addition, unless explicitly otherwise stated,  $R$  will always denote a TRS with finitely many rules.

#### 4 Take A: Dynamics on sets of terms with a rewriting strategy

We consider deterministic strategies on the set of finite and infinite terms; intuitively, a strategy picks one among the (possibly infinite number of) redexes in a term to contract:

► **Definition 4.1.** Let  $R$  be a TRS on  $T^\infty(\Sigma, \emptyset)$ . A (deterministic) strategy  $S$  for  $R$  is a (total) map  $S : (T^\infty(\Sigma, \emptyset) \setminus \text{NF}(R)) \rightarrow \text{POS} \times R$  such that  $S(t) = (p, l \rightarrow r)$  is a redex in  $t$ .

If  $S$  is a strategy for  $R$ , the *map induced by  $S$* , denoted  $F_{S,R} : T^\infty(\Sigma, \emptyset) \rightarrow T^\infty(\Sigma, \emptyset)$  (invariably abbreviated  $F_S$  when  $R$  is clear from the context), is defined by  $F_S(t) = t$  if  $t$  is a normal form, and otherwise  $F_S(t) = s$  if  $t \rightarrow s$  by contraction of the redex of rule  $l \rightarrow r$  at position  $p$  where  $S(t) = (p, l \rightarrow r)$ .

There are two potential pitfalls in the above definition: (i) as terms may be infinite, the set  $\text{NF}(R)$  of normal forms of  $R$  is not in general decidable—even if  $R$  is finite—whence computability of  $F_S$  is not immediate, (ii) the map  $F_S$  need not necessarily be continuous, which is required of a topological dynamical system.

► **Definition 4.2.** Let  $S$  be a strategy for the TRS  $R$  on  $T^\infty(\Sigma, \emptyset)$  such that  $F_S : T^\infty(\Sigma, \emptyset) \rightarrow T^\infty(\Sigma, \emptyset)$  is continuous. Then  $(T^\infty(\Sigma, \emptyset), F_S)$  is a topological dynamical system and is called the  *$S$ -induced dynamical system on  $T^\infty(\Sigma, \emptyset)$* .

Thus, the main question of interest is under which circumstances the map  $F_S$  is continuous.

Note that if  $S$  is *not* continuous as a map  $T^\infty(\Sigma, \emptyset) \setminus \text{NF}(R) \rightarrow \text{POS} \times R$  when  $\text{POS}$  and  $R$  are endowed with the discrete topology and  $\text{POS} \times R$  has the product topology, then in general  $F_S$  will not be continuous either. For example, consider the signature  $\{a/1, b/1\}$ , the TRS  $\{a(x) \rightarrow b(x), b(x) \rightarrow a(x)\}$  and the pathological strategy  $S$  defined by  $S(b^k(a(t))) = (1^k, a(x) \rightarrow b(x))$  (for any  $k \geq 0$  and term  $t$ ), and  $S(b^\omega) = (\epsilon, b(x) \rightarrow a(x))$ . Then,  $S$  is not continuous, as witnessed by the fact that  $(b^n(a^\omega))_n$  is sequence converging to  $b^\omega$ , but  $S(b^n(a^\omega)) = (1^n, a(x) \rightarrow b(x))$  and  $S(b^\omega) = (\epsilon, b(x) \rightarrow a(x))$ , whereas the sequence  $(S(b^n(a^\omega)))$  does not converge to  $S(b^\omega)$ . Hence,  $F_S$  is not continuous.

We have the following useful lemma:

► **Lemma 4.3.** *Let  $R$  be a (finite!) TRS and let  $S : (T^\infty(\Sigma, \emptyset) \setminus \text{NF}(R)) \rightarrow \text{POS} \times R$  be a strategy for  $R$ . Equip  $R$  and  $\text{POS}$  with the discrete topology, and let  $\text{POS} \times R$  be equipped with the product topology. If  $S$  is continuous, then  $F_S : T^\infty(\Sigma, \emptyset) \rightarrow T^\infty(\Sigma, \emptyset)$  is continuous.*

**Proof.** We show pointwise continuity at  $s \in T^\infty(\Sigma, \emptyset)$ . Let  $k$  be any non-negative integer and split on cases according to  $s$ :

- If  $s$  is not a normal form, let  $S(s) = (p, l \rightarrow r)$ . By continuity of  $S$ , there is a non-negative integer  $n_1$  such that if  $d(s, t) < 2^{-n_1}$ , and  $S(t) = (p', l' \rightarrow r')$ , then  $p = p'$ ,  $l = l'$  and  $r = r'$ . Let  $w$  be the length of the longest position in  $l$  and set  $m_1 = \max\{k, n_1, |p| + w\}$ . If  $d(s, t) < 2^{-m_1}$ , we hence have  $d(F_S(s), F_S(t)) < 2^{-k}$ .
- If  $s$  is a normal form, let  $w$  be the length of a maximally long position occurring in a left-hand side of a rule in  $R$ , and set  $m_2 = k + w$ . Let  $t \in T^\infty(\Sigma, \emptyset)$  such that  $d(s, t) < 2^{-m_2}$ . If  $t$  is not a normal form, consider  $S(t) = (q, l \rightarrow r)$ ; if  $|q| \leq k$ , then there is a redex in  $s$  at  $q$ , a contradiction. Hence,  $d(F_S(s), F_S(t)) < 2^{-k}$ . If  $t$  is a normal form, then  $d(F_S(s), F_S(t)) = d(s, t) < 2^{-m_2} < 2^{-k}$ . ◀

Thus, continuity of  $F_S$  merely requires us to prove that  $S$  is continuous.

► **Corollary 4.4.** *Let  $R$  be any (finite) TRS, let  $S$  be a leftmost-outermost strategy (in case more than one redex is present at a position, assume that  $S$  deterministically picks one of them)<sup>1</sup>. Then,  $(T^\infty(\Sigma, \emptyset), F_S)$  is a topological dynamical system.*

**Proof.** Let  $s \in T^\infty(\Sigma, \emptyset)$  and let  $k$  be a non-negative integer. Let  $w$  be the length of the longest position occurring in a left-hand side of a rule in  $R$ . Set  $m = k + w$ . If  $s, t \in T^\infty(\Sigma, \emptyset)$  are both non-normal forms with  $d(s, t) < 2^{-m}$ , we reason as follows: For the leftmost-outermost strategy, if  $S$  picks a redex in  $s$  at position  $p$  with  $|p| \leq k$ , then that redex is present in  $t$  as well, and must be leftmost and outermost in  $t$  (otherwise another redex present at depth at most  $k$  is leftmost-outermost, but then it would also be leftmost-outermost in  $s$ , a contradiction). Contracting both redexes yields terms identical up to depth  $k$  (as the length of any position across a rewrite step can be *shortened* by at most  $w$ ). Alternatively,  $S$  picks a redex at a position with length strictly greater than  $k$ ; in this case,  $S$  cannot pick a redex at a position of length  $\leq k$  in  $t$ , as this redex would also be present in  $s$  and thus at a position of strictly shorter length than the redex picked for  $s$ , contradicting outermostness. In both cases,  $d(F_S(s), F_S(t)) < 2^{-k}$ , establishing continuity. ◀

Perhaps surprisingly, *innermost* strategies are in general not continuous: Consider the signature  $\Sigma = \{a/1, b/1, \triangleright/0\}$  and the TRS  $R = \{a(x) \rightarrow \triangleright\}$ . Consider terms on the form  $a(b^k(a(\triangleright)))$  for  $k \geq 1$ . Then, any innermost strategy will always pick the redex resulting in the step  $a(b^k(a(\triangleright))) \rightarrow a(b^k(\triangleright))$ . But for any term on the form  $a(b^k(\triangleright))$ , any innermost strategy will always pick the redex at the root of the term, resulting in the step  $a(b^k(\triangleright)) \rightarrow \triangleright$ . If the innermost strategy were continuous, then there would be an integer  $m$  such that  $d(a(b^k(a(\triangleright))), a(b^k(\triangleright))) < 2^{-m}$  implies that  $S$  chooses the same redex in both terms; but  $S$  chooses different redexes in the terms  $a(b^m(a(\triangleright)))$  and  $a(b^m(\triangleright))$ , a contradiction.

► **Remark.** It turns out that every *computable* strategy  $S$  is continuous, but due to space constraints we omit the lengthy definitions needed to define computable strategies on infinite terms. After the definitions have been made, continuity follows immediately from the Kreisel-Lacombe-Shoenfield Theorem (“every computable map on a computably complete computable metric space is continuous”) [24]. Note that the leftmost-outermost strategy is computable, but the leftmost-innermost is *not*.

## 4.1 Examples

Let  $S$  be a strategy for TRS  $R$  such that the  $S$ -induced system  $(T^\infty(\Sigma, \emptyset), F_S)$  exists. A *subsystem* of  $(T^\infty(\Sigma, \emptyset), F_S)$  is a dynamical system  $(T', F_S)$  such that (i)  $T'$  is a topologically closed subset of  $T^\infty(\Sigma, \emptyset)$ , and (ii)  $F_S(T') \subseteq T'$

► **Example 4.5.** We can easily model the one- and two-sided shifts of Examples 2.1 and 2.2:

Let  $\Sigma = \{a_1/1, \dots, a_k/1\}$ . For the one-sided shift, set  $R = \{a_1(x) \rightarrow x, \dots, a_k(x) \rightarrow x\}$  and let  $S$  be the (necessarily unique, as all symbols are unary) outermost strategy for  $R$ . For a set of “forbidden words”  $F$ , all on the form  $a_{i_1}(\dots(a_{i_n}(x)))$ , we may consider the set  $A_F$  of infinite, ground terms  $s$  such that no element of  $F$  occurs in  $s$ ; formally: there is no context

<sup>1</sup> For the purposes of this paper, a leftmost-outermost strategy is a strategy that picks a redex that is (i) at a position of minimal length, and (ii) is minimal in the lexicographic order on positions among redexes satisfying (i). For further discussion of the difficulties of generalizing leftmost-outermost reduction to infinitary rewriting, see [21].

$C[]$ , no element  $a_{i_1}(\dots(a_{i_n}(x))) \in F$ , and no  $s' \in T^\infty(\Sigma, \emptyset)$  such that  $s = C[a_{i_1}(\dots(a_{i_n}(s')))]$ . Then,  $(A_F, F_S)$  is a subsystem of  $(T^\infty(\Sigma, \emptyset), F_S)$ .

Correspondingly, for the two-sided shift, let  $u \notin \Sigma$  be a fresh *binary* symbol and define  $B_F$  to be the set of all terms on the form  $u(s_1, s_2)$  where  $s_1, s_2 \in T^\infty(\Sigma, \emptyset)$  such that no string of symbols corresponding to  $w \in F$  occurs in  $s$ , and no string of symbols corresponding to the *reverse* of  $w \in F$  occurs in  $s_1$ . Define  $R = \{u(x, a_1(y)) \rightarrow u(a_1(x), y), \dots, u(x, a_k(y)) \rightarrow u(a_k(x), y)\}$  and let  $S$  be an outermost strategy for  $R$ . Then,  $(B_F, F_S)$  is a subsystem of  $(T^\infty(\Sigma, \emptyset), F_S)$ .

► **Example 4.6.** Let  $\Sigma = \{:/2, \text{nil}/0, +/1, 0/0, f/1\}$ , write  $:$  (“cons”) as in infix operator as usual, and set  $R = \{f(\text{nil}) \rightarrow \text{nil}, f(x : y) \rightarrow +(x) : f(y)\}$ .

Let  $N$  be the set of finite and infinite ground terms built solely from  $+$  and  $0$  (i.e.,  $N$  models the set  $\mathbb{N} \cup \{\infty\}$ ), and let  $L$  be the least subset of  $T^\infty(\Sigma, \emptyset)$  such that  $\text{nil} \in L$ , and such that  $s \in N$  and  $t \in L$  implies  $s : t \in L$  (i.e.  $L$  models the set of finite and infinite lists of natural numbers). Finally, let  $K \subseteq T^\infty(\Sigma, V)$  be the subset of terms such that  $s \in K$  implies either  $s \in L$ , or  $s = t_1 : (t_2 : \dots f(t_n : \dots))$  where all  $t_i$  are elements of  $N$ . Then,  $K$  is topologically closed, and if  $F$  is any outermost strategy,  $F_S(s) = s$  for any  $s \in K$  and  $F_S(s) \in L$  if  $s \in K \setminus L$ . Finally, as there is at most one redex in any element of  $K$ ,  $F_S$  is continuous as  $S$  is an outermost strategy.

The map  $F_S$  contracts the unique redex, if any, in a term  $s \in K$ . By inspection, we see that iteration of the map corresponds to “adding one” to each natural number in successive elements of the list  $s$ . If  $s = f(s')$  where  $s' \in L$ , the sequence  $F^n(s)$  converges to the (finite or infinite) term  $t \in K$  with one added to every natural number in  $s$ .

## 5 Take B: Dynamics on sets of sets of terms

We now consider the action of maps  $F_R$  induced by  $R$  where the objects being “moved” by  $F_R$  are *set* of terms. Hence, the natural dynamical system is a map on a set of sets of terms.

► **Definition 5.1.** Let  $(X, d)$  be a metric space and let  $\mathcal{H}(X)$  be the set of non-empty compact subsets of  $X$ . The *Hausdorff metric* on  $\mathcal{H}(X)$  is then the metric  $d_H$  defined by

$$d_H(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b) \right\}$$

A proof that  $d_H$  is indeed a metric on  $\mathcal{H}(X)$  can be found for example in [5]. It is not hard to see that if  $(X, d)$  is an ultrametric space, then so is  $(\mathcal{H}(X), d_H)$ , and that completeness of  $(X, d)$  implies completeness of  $(\mathcal{H}(X), d_H)$ . For the standard metric on terms, there is a simple, neat characterization of the Hausdorff metric:

► **Theorem 5.2** (See e.g. Ch. 1 of [13]). *Let  $x$  be a variable and define, for any non-negative integer  $n$  and any  $s \in T^\infty(\Sigma, \emptyset)$ , the term  $\alpha_n(s)$  as the term  $s$  where we have replaced any subterm of  $s$  at positions  $p$  with  $|p| > n$  by  $x$ . For  $A \in \mathcal{H}(T^\infty(\Sigma, \emptyset))$ , define  $\alpha_n(A) = \{\alpha_n(s) : s \in A\}$ .*

*The Hausdorff metric on  $T^\infty(\Sigma, \emptyset)$  is then given by*

$$d_H(A, B) = 2^{-\inf\{n : \alpha_n(A) \neq \alpha_n(B)\}}$$

► **Definition 5.3.** Let  $R$  be a TRS. We then define the map  $F_R : \mathcal{H}(T^\infty(\Sigma, \emptyset)) \rightarrow \mathcal{H}(T^\infty(\Sigma, \emptyset))$  by

$$F_R(A) = \{t \in T^\infty(\Sigma, \emptyset) : s \in A \wedge s \rightarrow_R t\} \cup \{t \in A : t \text{ is a normal form}\}$$

If the map  $F_R$  is continuous, then  $(\mathcal{H}(T^\infty(\Sigma, \emptyset)), F_R)$  is a dynamical system, which we denote the  $R$ -induced system.

Thus,  $F_R(A)$  consists of (i) all reducts of elements of  $A$ , and (ii) all normal forms in  $A$ . The reason for including the normal forms in  $A$  is to ensure that  $F_R$  is well-defined (as, otherwise, any set consisting solely of normal forms would be mapped into the empty set, which is not an element of  $\mathcal{H}(X)$ ).

Fortunately,  $F_R$  is continuous for all ordinary TRSs  $R$ :

► **Theorem 5.4.** *If  $R$  has finitely many rules, then  $F_R$  is continuous.*

**Proof.** Observe that  $d_H(A, B) < \epsilon$  iff

$$(\forall s \in A \exists t \in B. d(s, t) < \epsilon) \wedge (\forall t \in B \exists s \in A. d(s, t) < \epsilon)$$

Let  $m$  be the length of a position of maximal length occurring in the left-hand side of a rule of  $R$ , and let  $k$  be an arbitrary non-negative integer. To show that  $F_R$  is continuous, it suffices to show that:

1.  $(\forall s \in A \exists t \in B. d(s, t) < 2^{-(2k+m)}) \Rightarrow (\forall s' \in F_R(A) \exists t' \in F_R(B). d(s', t') < 2^{-k})$ , and
2.  $(\forall t \in B \exists s \in A. d(s, t) < 2^{-(2k+m)}) \Rightarrow (\forall t' \in F_R(B) \exists s' \in F_R(A). d(t', s') < 2^{-k})$ .

Note that the two conditions above are completely symmetrical in  $A$  and  $B$ ; we hence prove only the first.

Assume that  $\forall s \in A \exists t \in B. d(s, t) < 2^{-(2k+m)}$  and let  $s' \in F_R(A)$  be arbitrary. Then there exists  $s \in A$  such that either (i)  $s = s'$  and  $s$  is a normal form, or (ii)  $s \rightarrow_R s'$ .

In case (i), consider  $t \in B$ : If  $t$  is a normal form, then  $t \in F_R(B)$  and we may set  $t' = t$ , and we have  $d(s', t') = d(s, t) < 2^{-(2k+m)} \leq 2^{-k}$ , as desired. If  $t$  is not a normal form, then  $t \rightarrow t'$  for some term  $t'$  by a rewrite step at some position  $p$ ; note that  $|p| \geq k$  as  $s$  is a normal form and  $s$  and  $t$  are identical up to depth  $2k + m$ . Hence,  $d(s', t') = d(s, t') < 2^{-k}$ , as desired.

In case (ii), let  $p$  be the position of the redex contracted in the step  $s \rightarrow s'$ . Split on cases as follows:

- If  $|p| < k$ , note that  $d(s, t) < 2^{-(2k+m)}$ , whence there is a redex of the same rule at  $p$  in  $t$ ; let  $t'$  be the term such that  $t \rightarrow t'$  by contraction of this redex, and observe that  $t' \in F_R(B)$ . Observe that the rule employed in this redex can *shorten* positions by at most  $m$ ; hence, if the redex is contracted at depth  $|p| < k$ , the symbols occurring at depth  $\leq k$  in  $t'$  are either created by the right-hand side of the rule, or are descendants of positions occurring at depth at most  $2k + m$ . As  $s$  and  $t$  were identical up to depth  $2k + m$ ,  $t'$  and  $s'$  are thus identical up to depth  $k$ , whence  $d(s', t') < 2^{-k}$ , as desired.
- If  $|p| \geq k$ , then as  $d$  is an ultrametric, we have  $d(s', t') \leq \max\{d(s', s), d(s, t')\} \leq \max\{d(s', s), \max\{d(s, t), d(t, t')\}\} < 2^{-k}$ . ◀

The assumption that  $R$  has a finite number of rules cannot be omitted from the statement of Theorem 5.4 as there are systems  $R$  with infinitely many rules for which  $F_R$  is not continuous, cf. the following example.

► **Example 5.5.** Consider the signature  $\Sigma = \{a/1, b/1\}$  and  $R = \{a^n(b(x)) \rightarrow b(x) : n \in \mathbb{N}_0\}$ . Define  $A = \{a^\omega\}$  and, for each nonnegative integer  $k$ ,  $B_k = \{a^k(b^\omega)\}$ . All sets  $A$  and  $B_k$  are singletons, hence non-empty and compact, whence  $A, B_k \in \mathcal{H}(T^\infty(\Sigma, \emptyset))$  for all  $k$ . For arbitrary  $k$ , we have  $d_H(A, B_k) = 2^{-k}$ , and  $(B_k)_k$  thus converges to  $A$ . If  $F_R$

were continuous, the sequence  $(F_R(B_k))_k$  would converge to  $F(A)$ . But  $F_R(A) = A$  and  $F_R(B_k) = \{b^\omega, a(b^\omega), \dots, a^k(b^\omega)\}$  for each  $k$ , whence  $d_H(F_R(A), F_R(B_k)) = 1$  for all  $k$ , disproving continuity.

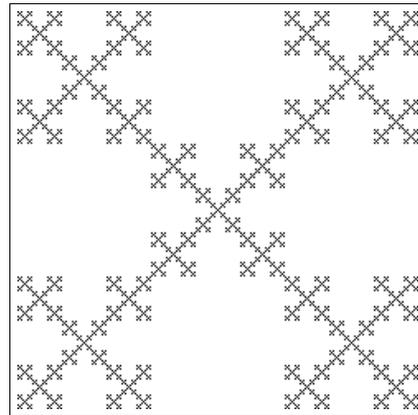
### 5.1 Examples

As the dynamical system  $(\mathcal{H}(T^\infty(\Sigma, \emptyset)), F_R)$  tracks the evolution of sets of terms while encompassing all possible rewrite steps in the terms, it is natural to consider the dynamical system applied to processes usually described by non-deterministic evolution where the set of *all* possible trajectories is the point of interest. One such area is fractals; we show a single example below.

► **Example 5.6.** Let  $\Sigma = \{0/0, 1/1, f/9\}$ , and let  $R$  consist of the single rule  $1 \rightarrow f(1, 0, 1, 0, 1, 0, 1, 0, 1)$ . Consider the set  $T' = T^\infty(\Sigma, \emptyset) \setminus \{0\}$ .  $T'$  is a closed subset of the compact set  $T^\infty(\Sigma, \emptyset)$ , hence compact. Then  $(\mathcal{H}(T'), F_R)$  is a topological dynamical system (indeed, a subsystem of  $(\mathcal{H}(T^\infty(\Sigma, \emptyset)), F_R)$ ).

It is easy to see that  $F_R|_{\mathcal{H}(T')} : \mathcal{H}(T') \rightarrow \mathcal{H}(T')$  satisfies  $d_H(F_R(A), F_R(B)) = d_H(A, B)/2$  for arbitrary non-empty compact subsets of  $T'$ . Hence, by the Banach Fixed Point Theorem,  $F_R$  has a unique fixed point given by  $\lim_{n \rightarrow \infty} F_R^n(A)$  where  $A$  is any element of  $\mathcal{H}(T')$ . In particular, we may choose  $A = \{1\}$  and hence see that the fixed point is the term  $t_f = \lim_{n \rightarrow \infty} t_n$  where  $t_0 = 1$  and  $t_n = f(t_{n-1}, 0, t_{n-1}, 0, t_{n-1}, 0, t_{n-1}, 0, t_{n-1})$ .

It is convenient to visualize  $t_f$  by letting the root represent a square with edges of unit length. If this square is partitioned into 9 smaller squares with edge length  $1/3$  in the obvious manner, the 9 subterms of the root are then represented by one of the smaller squares, and so forth. This generates the box fractal as shown to the right. (More precisely, the box fractal can be drawn as follows: Define  $f : \{1, \dots, 9\} \rightarrow [0, 1] \times [0, 1]$  by  $f(i) = \frac{1}{3}(\lfloor \frac{i-1}{3} \rfloor, (i-1) \bmod 3)$ . For each  $p = \epsilon p_1 \dots p_n \in Pos(t_n)$  with  $t_n|_p = 1$  compute  $c_p = \sum_{i=1}^n f(p_i)/(3^i)$  and draw a square with lower left corner  $c_p$  and side length  $1/(3^{n+1})$ .)



It is tempting to try to derive a general result from Example 5.6. However, the use of the Banach Fixed Point Theorem (which requires the map  $F_R$  to be a contraction, i.e.,  $d(F_R(A), F_R(B)) < d(A, B)$  for all  $A, B$  with  $A \neq B$ ), is problematic from the vantage point of term rewriting: If  $R$  has at least two distinct normal forms  $t \neq t'$ , then  $F_R$  is not a contraction, due to  $F_R(\{t\}) = \{t\}$  and  $F_R(\{t'\}) = \{t'\}$ , whence  $d(F_R(\{t\}), F_R(\{t'\})) = d(\{t\}, \{t'\})$ . The obvious “fix” for this problem is to consider  $F_R$  as a map on the set of compact subsets on  $T^\infty(\Sigma, \emptyset) \setminus NF(R)$ ; but this attempt fails as removing the set of normal forms in general destroys compactness<sup>2</sup>.

<sup>2</sup> More precisely: If there is a normal form  $t \in T^\infty(\Sigma, \emptyset)$  such that (i)  $t$  is an infinite term, (ii) there is a sequence  $(t_n)$  with  $t_n \in T^\infty(\Sigma, \emptyset) \setminus NF(R)$  for all  $n$  and  $t = \lim_n t_n$ , then  $T^\infty(\Sigma, \emptyset) \setminus NF(R)$  is not complete, hence not compact.

Another example of use is to consider a construction close to the tree-shifts of Aubrun and Béal under the action of all shift maps:

► **Example 5.7.** Let  $\Sigma$  be any finite, non-empty signature where each symbol has the same, positive arity (we choose  $\Sigma = \{f/2, g/2\}$  in this example). Consider the system  $R = \{f(x_1, \dots, x_n) \rightarrow x_i : f \in \Sigma, 1 \leq i \leq n\}$ . Then,  $F_R : \mathcal{H}(T^\infty(\Sigma, \emptyset)) \rightarrow \mathcal{H}(T^\infty(\Sigma, \emptyset))$  is continuous by Theorem 5.4. If  $A = T^\infty(\Sigma, \emptyset)$ , we have  $F_R(A) = A$ ; if  $B$  is the subset of  $T^\infty(\Sigma, \emptyset)$  consisting of all those infinite ground terms that do not contain the pattern  $f(g(\cdot, \cdot), g(\cdot, \cdot))$ , then  $B$  is compact and  $F_B(B) = B$ .

## 6 Conjugacy and topological entropy

Dynamical systems are usually identified up to *topological conjugacy* (roughly: The dynamical properties of two conjugate systems are the same). As topological conjugacy is, in general, undecidable, even for finitely presented dynamical systems, a number of *topological invariants* (quantities known to be identical for conjugate systems) are studied with the purpose of proving that two distinct dynamical systems are *not* conjugate; the most well-known of these being the so-called *topological entropy*. We briefly define these concepts for  $S$ -induced systems and show that the topological entropy is unlikely to be a useful topological invariant in this case.

► **Definition 6.1.** Two dynamical systems  $(X, f)$  and  $(Y, g)$  are said to be *topologically conjugate* (or simply *conjugate*) if there is a homeomorphism  $h : X \rightarrow Y$  such that  $h \circ f = g \circ h$ .

► **Example 6.2.** For  $S$ -induced systems, systems obtained by simple renamings are conjugate; e.g. if  $\Sigma_1 = \{a/1, 0/0\}$ ,  $\Sigma_2 = \{b/1, 1/0\}$ ,  $R_1 = \{0 \rightarrow a(0)\}$  and  $R_2 = \{1 \rightarrow b(1)\}$ , then there is only a single strategy  $S_1$  for  $R_1$  and a single strategy  $S_2$  for  $R_2$ , and  $(T^\infty(\Sigma_1, \emptyset), F_{S_1})$  and  $(T^\infty(\Sigma_2, \emptyset), F_{S_2})$  are conjugate.

Likewise, let  $\Sigma = \{f/2\}$ ,  $R_1 = \{f(x, y) \rightarrow x\}$  and  $R_2 = \{f(x, y) \rightarrow y\}$ , and define, for any  $t \in T^\infty(\Sigma, \emptyset)$ , the term  $h(t) \in T^\infty(\Sigma, \emptyset)$  by swapping left- and right-subterms of each occurrence of  $f$  top-down. Then  $H : \mathcal{H}(T^\infty(\Sigma, \emptyset)) \rightarrow \mathcal{H}(T^\infty(\Sigma, \emptyset))$  defined by  $H(A) = \{h(t) : t \in A\}$  is a homeomorphism and  $F_{R_1}$  and  $F_{R_2}$  are conjugate by  $H$ .

► **Example 6.3.** The one- and two-sided shifts defined in Examples 2.1 and 2.2 are conjugate to their term rewriting counterparts of Example 4.5—for the two-sided shift, define  $h(\dots b_{-2}b_{-1}b_0b_1b_2 \dots) = u(b_{-1}(b_{-2}(\dots)), b_0(b_1(b_2(\dots))))$ .

► **Remark.** Unsurprisingly, the question of whether two, suitably presented, subsystems are conjugate, is undecidable:

Let  $\Sigma$  be a signature with at least two distinct symbols both of which have arity  $\geq 1$ . The following question is undecidable:

- *Given:* (i) Two finite TRSs  $R$  and  $R'$  (it may wlog. be assumed that  $R = R'$ ), (ii) two strategies  $S$  and  $S'$  (it may wlog. be assumed that  $S = S'$ ) for resp.  $R$  and  $R'$ , and (iii) two recursively enumerable, topologically closed subsets  $A, A' \subseteq T^\infty(\Sigma, \emptyset)$  that are resp.  $F_S$  and  $F_{S'}$ -invariant.
- *To decide:* Is  $(A, F_S)$  conjugate to  $(A', F_{S'})$ ?

**Proof.** Let  $\Sigma = \{0/1, \triangleright/0\}$ , set  $R = R' = \{0(x) \rightarrow x\}$  and let  $S = S'$  be the outermost strategy. Let  $M$  be an inputless Turing machine and define the set

$$A_M = \{0^\omega\} \cup \{0^k \triangleright : M \text{ has not halted in the first } k \geq 0 \text{ steps of its execution}\}$$

Note that  $A_M$  is finite iff  $M$  halts after some number of steps and hence that  $A_M$  is closed in this case. If  $M$  does not halt, then  $A_M$  is infinite and  $A_M = \{\triangleright, 0\triangleright, 00\triangleright, 000\triangleright, \dots\} \cup \{0^\omega\}$ , whence  $A_M$  is also closed in this case. Let  $S$  be the (unique) outermost strategy for  $R$ . Then  $F_S(0^\omega) = 0^\omega$  and, for  $k \geq 1$ ,  $F_S(0^k\triangleright) = 0^{k-1}\triangleright$ , whence  $F_S(A_M) \subseteq A_M$  in all cases.

Let  $N$  be any non-halting Turing machine. Then  $A_N = \{0^k\triangleright : k \geq 0\} \cup \{0^\omega\}$ . Note that as  $A_M$  is finite for any halting  $M$  and a conjugacy  $h : A_M \rightarrow A_N$  must be a homeomorphism (in particular must be a bijection), then  $(A_M, F_S)$  and  $(A_N, F_S)$  are conjugate iff  $M$  does *not* halt. Conversely, if  $M$  does not halt, then  $(A_M, F_S)$  and  $(A_N, F_S)$  are clearly conjugate (as  $A_M = A_N$  and the two systems have the same rule set and strategies).

If it were decidable for all  $(A_M, R)$  whether  $(A_M, F_S)$  and  $(A_N, F_S)$  are conjugate, then we could decide whether  $M$  halts as the two systems are conjugate iff  $M$  does not halt, and we obtain a contradiction.  $\blacktriangleleft$

We do not know whether it is decidable in general whether two systems on the full sets of terms over (possibly distinct) signatures are conjugate, whether conjugacy is decidable for subsystems  $(A, F_S)$  with decidable  $A$ . We very strongly suspect that both of these questions are undecidable as well.

A similar result holds for subsystems of  $(\mathcal{H}(T^\infty(\Sigma, \emptyset)), F_R)$ ; the proof uses a construction very similar to the one above and is thus omitted:

The following question is undecidable:

- *Given:* (i) Two finite TRSs  $R$  and  $R'$  (it may wlog. be assumed that  $R = R'$ ) and (ii) two recursively enumerable, topologically closed subsets  $\mathcal{A}, \mathcal{A}' \subseteq \mathcal{H}(T^\infty(\Sigma, \emptyset))$  (both consisting of finite subsets) that are resp. closed under  $F_R$  and  $F_{R'}$ .
- *To decide:* Is  $(\mathcal{A}, F_R)$  conjugate to  $(\mathcal{A}', F_{R'})$ ?

## 6.1 Topological entropy

A topological *invariant* is a quantity that is equal for conjugate systems. Such invariants are used to prove that certain systems are *not* conjugate. The topological entropy, defined below, is a common such invariant (see, e.g., [12, Cor. 2.5.4] for a proof its invariance).

► **Definition 6.4.** Let  $(X, d)$  be a compact metric space and  $f : X \rightarrow X$  be continuous. For positive integer  $n$ , define  $d_n(x, y) = \max_{0 \leq j \leq n-1} d(f^j(x), f^j(y))$ . For  $\epsilon > 0$ , let  $\text{cov}(n, \epsilon, f)$  be the *minimum* number of sets of  $d_n$ -diameter at most  $\epsilon$  whose union contains  $X$  (the  $d_n$ -diameter of a set  $A$  is the quantity  $\sup_{x, y \in A} d_n(x, y)$ ).

The *topological entropy* of  $f$ , denoted  $h(f)$ , is then:

$$h(f) = \lim_{\epsilon \rightarrow 0^+} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{cov}(n, \epsilon, f)$$

The definition of topological entropy can be intuitively understood as follows: Imagine a computer screen with a picture of  $A$  as a two-dimensional set; the map  $f$  then defines how the points in  $A$  move in one time step, and  $d_n(x, y)$  measures the maximum distance of the trajectories of the points  $x, y \in A$  after the first  $n - 1$  time steps;  $\epsilon$  can be viewed as the “resolution” of our computer screen (small  $\epsilon$  gives high resolution). The topological entropy is then, roughly, the (exponential) rate of evolution of the number of distinct trajectories we can discern as the number of time steps becomes very large and our resolution becomes very high.

► **Example 6.5.** Let  $\Sigma_1 = \{f_1/2, \dots, f_N/2\}$  where  $N \geq 1$  and  $R_1 = \{f_1(x_1, x_2) \rightarrow f_2(x_1, x_2), \dots, f_N(x_1, x_2) \rightarrow f_1(x_1, x_2)\}$ , and let  $S_1$  be the (necessarily unique) outermost strategy for  $R_1$ . Then,  $F_{S_1}$  is an isometry, whence  $d_n(s, t) = d(s, t)$  for all non-negative integers  $n$  and all  $s, t \in T^\infty(\Sigma, \emptyset)$ . But for each  $\epsilon > 0$ , the minimum number of sets of  $d$ -diameter needed to cover  $X$  is then a number,  $k$ , independent of  $n$ , whence  $h(F_{S_1}) = \lim_{\epsilon \rightarrow 0^+} \limsup_{n \rightarrow \infty} 1/n \log k = 0$ .

Let  $\Sigma_2 = \{f_1/1, \dots, f_N/1\}$  where  $N \geq 1$  and  $R_2 = \{f_1(x) \rightarrow x, \dots, f_N(x) \rightarrow x\}$ , and  $S_2$  be the (again, unique) outermost strategy for  $R_2$ . Then,  $F_{S_2}$  is conjugate to the full one-sided shift on  $N$  symbols, hence has identical entropy. By standard results (see, e.g., [17, p. 120]), the topological entropy of this system is  $\log N$ ; hence  $h(F_{S_2}) = \log N$ .

Let  $\Sigma_3 = \{f_1/n_1, \dots, f_N/N_n\}$ , let  $R_3$  be an orthogonal TRS containing at least one rule, and such that each rule is on the form  $f_i(x_1, \dots, x_{n_i}) \rightarrow r$  where every variable in  $r$  occurs at depth at least 2. Let  $S_3$  be the leftmost-outermost strategy for  $R_3$ . Then, for all  $s, t \in T^\infty(\Sigma, \emptyset)$ , we have  $d(F_{S_3}(s), F_{S_3}(t)) \leq d(s, t)$ , and by arguing as for  $R_1$  above, we obtain, *mutatis mutandis*, that  $h(F_{S_3}) = 0$ .

A moment's thought reveals that when  $f$  is a weak contraction (i.e.,  $d(f(x), f(y)) \leq d(x, y)$  for all  $x, y \in X$ ), then  $h(f) = 0$ , and as Example 6.5 shows, this situation may occur if the depth of any occurrence of a variable is not decreased in any rule. For depth-*decreasing* systems such as  $R_2$  in Example 6.5, the situation at first glance seems more promising as we obtain positive, finite entropy. However, we will momentarily show that this is an artifact of all function symbols in the signature being unary. In general, systems containing at least two symbols of arity at least 2, and just a single collapsing rule with at least two variables in the left-hand side will have infinite entropy (for outermost strategies).

Before we proceed, we need an ancillary notion: If  $A$  is a closed subset of  $(X, d)$  such that  $F(A) \subseteq A$ , the restriction  $f|_A$  of  $f$  to  $A$  induces a topological dynamical system, and we may hence consider the topological entropy  $h(f|_A)$ . The following result is standard (see, e.g., [12, Prop. 2.5.5(3)]):

► **Lemma 6.6.** *Let  $(X, d)$  be compact and let  $f : X \rightarrow X$  be continuous. If  $A \subseteq X$  is closed and satisfies  $F(A) \subseteq A$ , then  $h(f|_A) \leq h(f)$ .*

We then have:

► **Theorem 6.7.** *Let  $\Sigma$  be a signature containing at least two elements  $f, g \in \Sigma$  each of which has arity at least 2. Let  $l \rightarrow x$  be a rule where  $l$  is a linear term (i.e. each variable occurs at most once) containing at least two distinct variables  $x$  and  $y$ .*

*Let  $R \supseteq \{l \rightarrow x\}$  be a TRS where no rule in  $R \setminus \{l \rightarrow x\}$  overlaps  $l \rightarrow x$  at the root, and let  $S$  be any outermost strategy for  $R$ . Then,  $h(F_S) = \infty$ .*

**Proof.** Let  $p_x$  and  $p_y$  be the unique positions of  $x$  and  $y$  in  $l$ . Define the map  $g_l : T(\Sigma, V) \rightarrow T(\Sigma, V)$  by  $g_l(s) = l[s]_{p_x}$ . As  $l$  is not a variable, the sequence  $(g_l^n)_n$  is convergent in  $T^\infty(\Sigma, V)$  for every  $s$  and must converge to an infinite term. Thus, define  $t_l = \lim_{n \rightarrow \infty} g_l^n(l)$ . Note that  $t_l$  is an infinite term that is *not* ground, as  $l$  contains at least one variable distinct from  $x$ .

Let  $A \subseteq T^\infty(\Sigma, \emptyset)$  be the set of infinite ground terms such that  $s \in A$  iff (i) for each position  $p$  of  $t_l$  such that  $t_l|_p \notin V$ , we have  $t_l|_p = s|_p$ , and (ii) if  $q$  is a position in  $s$  not covered by (i), then the root symbol of  $s|_q$  is either  $f$  or  $g$ . (Intuition:  $s$  is obtained by starting with  $t_l$  and then filling in infinite ground terms over the signature  $\{f/2, g/2\}$  at all occurrences of variables in  $t_l$ .) Let  $(s_n)$  be a convergent sequence of elements of  $A$ . Clearly, the limit of the sequence must satisfy (i) and (ii) above, whence  $A$  is a closed set. For  $s \in A$ ,

we permit ourselves to talk about the “stacked copies of  $l$ ” when referring to the copies of  $l$  in the “spine”  $t_l$  that is present  $s$ .

Observe that, as  $l$  was a linear term, each element of  $A$  contains a redex of rule  $l \rightarrow x$  at the root. As no rule in  $R$  distinct from  $l \rightarrow x$  overlaps  $l \rightarrow x$  at the root, any outermost strategy  $S$  must, when applied to  $s \in A$ , always select the redex of rule  $l \rightarrow x$  at the root. By construction of  $A$ , contraction of this redex yields an element of  $A$  (specifically, we have  $F_S(s) = s|_{p_x}$ ), whence  $F_S(A) \subseteq A$ .

By Lemma 6.6 it thus suffices to prove that  $h(F_S|_A) = \infty$ .

Let  $p$  be any position,  $j \geq 0$ , and consider  $p_x^j$ ; by the above observations, we have  $F_S^j(s) = s|_{p_x^j}$ , and hence for any position  $p$  in  $F_S^j(s)$ , we have  $F_S^j(s)|_p = s|_{p_x^j p}$ . Thus, for arbitrary  $s, t \in A$ :

$$d_n(s, t) = \max_{0 \leq j \leq n-1} \max\{2^{-|p|} : \text{root}(s|_{p_x^j p}) \neq \text{root}(t|_{p_x^j p})\}.$$

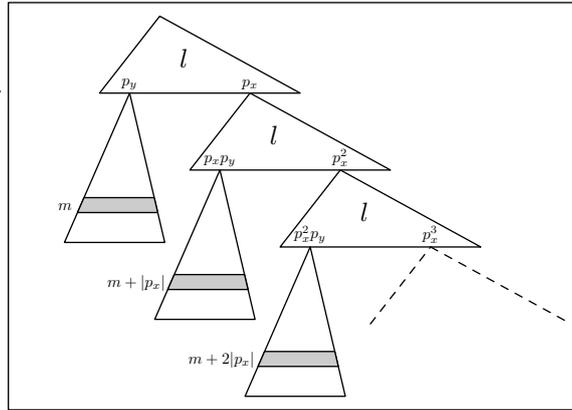
(The second max in the above is due to the fact that the position  $p$  of *minimal* length satisfying the conditions defines the metric; for such a  $p$  the quantity  $2^{-|p|}$  is *maximized*.)

By Lemma 2.3,  $d_n$  is an ultrametric, and we may thus compute the topological entropy by counting balls of radius  $\epsilon$  instead of sets of diameter  $\epsilon$ . Also by Lemma 2.3, two open  $d_n$ -balls  $B_n(s, 2^{-m})$  and  $B_n(t, 2^{-m})$  are either disjoint or equal, and we can thus compute  $\text{cov}(n, 2^{-m}, F_S)$  by counting the number of distinct balls of radius  $2^{-m}$ . We proceed by giving a lower bound on the number of such balls.

Let  $B$  be the set of infinite ground terms built solely of symbols  $f$  and  $g$  such that  $t \in B$  iff  $\text{root}(t|_p) = f$  for  $|p| \neq m - |p_y|$ . (i.e.,  $t$  “consists of  $f$ s at all depths except  $m - |p_y|$  where  $t$  may use both  $f$  and  $g$ ”). Observe that  $|B| = 2^{2^{m-|p_y|}}$  (there are exactly  $2^{m-|p_y|}$  positions of length  $m - |p_y|$ , and each of these can hold either  $f$  or  $g$ ). Let  $s \in A$  be arbitrary and set

$$C_{m,1} = \{s[t]_{p_y} : t \in B\}, \text{ resp. } C_{m,n} = \{s_n[t]_{p_x^{n-1} p_y} : t \in B \wedge s_n \in C_{m,n-1}\}.$$

(Intuition: In the first  $n$  stacked copies of  $l$ , we replace the subterm at  $p_y$  by an element of  $B$ . See the drawing to the right.) Observe that  $|C_{m,1}| = 2^{2^{m-|p_y|}}$ ,  $|C_{m,n}| = 2^{2^{m-|p_y|}} \cdot |C_{m,n-1}|$ , whence  $|C_{m,n}| = 2^{n(2^m - |p_y|)}$ . Let  $s, s' \in C_{m,n}$  with  $t \neq t'$ . Observe that  $\text{root}(s|_p) = \text{root}(s'|_p)$  for all positions  $p$  not on the form  $p = p_x^j p_y q$  where  $|q| = m - |p_y|$ .



Thus, there must be a position  $p = p_x^j p_y q$  with  $|q| = m - |p_y|$  such that  $\text{root}(s|_p) \neq \text{root}(s'|_p)$ . Let  $p$  have minimal length among such positions. Observe that  $F_S(s) = s|_{p_x}$  and  $F_S(s') = s'|_{p_x}$ ; thus,  $d_n(s, s') = 2^{-|p_y q|} = 2^{-m}$ .

All open  $d_n$ -balls  $B_n(s, 2^{-m})$  with  $s \in C_{m,n}$  are disjoint by Lemma 2.3. Note that all of these balls must occur in *any* minimal cover of  $A$  by balls of radius  $2^{-m}$  (for, if some ball  $B_n(t, 2^{-m})$  of radius  $2^{-m}$  contains  $s \in C_{m,n}$ , then  $B_n(t, 2^{-m}) \cap B_n(s, 2^{-m}) \neq \emptyset$ , whence by Lemma 2.3 we have  $B_n(t, 2^{-m}) = B_n(s, 2^{-m})$ ).

Hence,  $\text{cov}(n, 2^{-m}, F_S) \geq |C_{m,n}| = 2^{n2^{m-|p_y|}}$ . Thus:

$$\begin{aligned} h(F_S|_A) &= \lim_{\epsilon \rightarrow 0^+} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{cov}(n, \epsilon, F_S|_A) \geq \lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{n} \log 2^{n2^{m-|p_y|}} \\ &= \lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{n2^{m-|p_y|}}{n} \geq \lim_{m \rightarrow \infty} 2^{m-|p_y|} = \infty, \end{aligned}$$

and the result follows. ◀

Thus, if for example  $\Sigma = \{f/2, g/3, h/0\}$  and  $R = \{f(x, y) \rightarrow y, g(x, y, z) \rightarrow g(f(x, y), h, h)\}$  and  $S$  is the outermost-left strategy, then  $h(F_S) = \infty$ .

Theorem 6.7 and Example 6.5 together show that the problems with collapsing rules already known in *infinitary* rewriting rear their heads in our setting of finite reductions as well; this is not entirely surprising as the topological entropy concerns the limiting behaviour of the one-step rewrite relation. The problems with entropy are reminiscent of the result from infinitary rewriting that an orthogonal system is confluent iff it is *almost non-collapsing*, that is, has at most one collapsing rule, and such that the unique collapsing rule has exactly one variable in the left-hand side [19].

## 7 Conclusion and future work

We have laid the groundwork for the study of topological dynamical systems induced by term rewriting systems; while we have only scratched the surface, we have shown that such systems properly generalize well-known classes of systems such as symbolic dynamical systems, and are equipped with very interesting dynamical properties of their own.

A plethora of open questions remain. We mention a few and give suggestions for future work:

- Is conjugacy undecidable for rewriting systems over the *full* set of terms (i.e., without passing to a closed proper subset of  $T^\infty(\Sigma, \emptyset)$ )? Conjecture: yes.
- There is a rich interaction between topological dynamical systems and measure theory; in the same vein, it is highly conceivable that there are links between our work and probabilistic rewriting [9, 10, 11].
- There are several properties of topological dynamical systems that are typically studied for each class of such systems, for example topological transitivity, mixing and expansiveness [12]; it would be interesting to obtain characterizations of the term rewriting systems whose induced topological dynamical systems have these properties.
- The computability of the dynamics of general dynamical systems is well-studied (see, e.g., [14, 15, 16]), as is complexity issues related to certain aspects of dynamical systems (a few examples: [8, 7, 26]). It would be of interest to perform similar investigations of computability and complexity for the systems defined in this paper.
- There is a need for better topological invariants than the topological entropy; in addition, the entropy of  $F_R$  should be investigated.
- What are the connections to the field of infinitary rewriting?

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