

On the equational consistency of order-theoretic models of the λ -calculus*

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Abstract

Answering a question by Honsell and Plotkin, we show that there are two equations between λ -terms, the so-called *subtractive equations*, consistent with λ -calculus but not satisfied in any partially ordered model with bottom element. We also relate the subtractive equations to the open problem of the order-incompleteness of λ -calculus.

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1 Introduction

Lambda theories are congruences on the set of λ -terms, which contain β -conversion. They arise by syntactical or semantic considerations. Indeed, a λ -theory may correspond to a possible operational semantics of the lambda calculus, as well as it may be induced by a model of lambda calculus through the congruence relation associated with the interpretation function. The set of λ -theories is naturally equipped with a structure of complete lattice, whose bottom element is the least λ -theory $\lambda\beta$, and whose top element is the inconsistent λ -theory. The lattice of λ -theories is a very rich and complex structure of cardinality 2^{\aleph_0} (see, for example, [1, 9, 10]). Syntactical techniques are usually difficult to apply in the study of λ -theories. Therefore, semantic methods have been extensively investigated.

One of the most important contributions in the area of mathematical programming semantics was the discovery by D. Scott in the late 1960s, that complete partial orders, having their own function space as a retract, are models for the untyped lambda calculus. On the other hand, there are results that indicate that Scott's methods, based on a combination of order-theory and topology, may not in general be exhaustive: Honsell and Ronchi Della Rocca [8] have shown that there exists a λ -theory that does not arise as the theory of a Scott model. A natural completeness problem then arises for Scott semantics: whether any two λ -terms equal in all Scott models are β -convertible. This equational completeness problem is one of most outstanding open problems of λ -calculus and it seems to have appeared first in the literature in [6]. There is also an analogous consistency problem, raised by Honsell and Plotkin in [5]: whether every finite number of equations between λ -terms, consistent with the λ -calculus, has a Scott model. In this paper we answer negatively to this second question. We provide two equations (called the *subtractive equations*) consistent with λ -calculus, which have no partially ordered model with bottom element.

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Although many familiar models are constructed by order-theoretic methods, it is also known that there are some models of the lambda calculus that cannot be non-trivially ordered (see [11, 12, 13]). In general, we define a combinatory algebra \mathbf{A} to be *unorderable* if there does not exist a non-trivial partial order on A for which the application operation is monotone. Of course, an unorderable model can still arise from an order-theoretic construction, for instance as a subalgebra of some orderable model. The most interesting result has been obtained by Selinger [13], who, enough surprising, has shown that the standard open and closed term models of $\lambda\beta$ and $\lambda\beta\eta$ are unorderable. As a consequence of this result, it follows that if $\lambda\beta$ or $\lambda\beta\eta$ is the theory of a partially ordered model, then the denotations of closed terms in that model are pairwise incomparable, i.e. the term denotations form an anti-chain. This led Selinger [13] to study the related question of absolute unorderability: a model is absolutely unorderable if it cannot be embedded in an orderable one. Plotkin conjectures in [11] that an absolutely unorderable combinatory algebra exists, but the question is still open whether this is so. Selinger has given in [13] a syntactic characterisation of the absolutely unorderable algebras in any algebraic variety (equational class) in terms of the existence of a family of Mal'cev operators. Plotkin's conjecture is thus reduced to the question whether Mal'cev operators are consistent with the lambda calculus or combinatory logic. The question of absolute unorderability can also be formulated in terms of theories, rather than models. In this form, Selinger [13] refers to it as the *order-incompleteness* question: does there exist a λ -theory which does not arise as the theory of a non-trivial partially ordered model? Such a problem can be also characterised in terms of connected components of a partial ordering (minimal subsets which are both upward and downward closed): a λ -theory \mathcal{T} is order-incomplete if, and only if, every partially ordered model, having \mathcal{T} as equational theory, is partitioned in an infinite number of connected components, each one containing exactly one element. In other words, the partial order is the equality.

Toward an answer to the order-incompleteness problem, we find a strengthening \mathcal{T} of the subtractive equations having the following property: every partially ordered model \mathcal{M} satisfying \mathcal{T} has an infinite number of connected components among which that of the looping term Ω is a singleton set. Moreover, each connected component of \mathcal{M} contains the denotation of at most one $\beta\eta$ -normal form. Compared to absolute unorderability, the above situation still has some missing bits. For example we are not in the position to tell where the denotations of all unsolvable λ -terms other than Ω are placed in the model. Same thing for all the solvable λ -terms which do not have a $\beta\eta$ -normal form.

The inspiration for the subtractive equations comes from the notion of *subtractive variety* of algebras introduced by Ursini in [14]. A variety \mathcal{V} of algebras is subtractive if it satisfies the following identities:

$$s(x, x) = 0; \quad s(x, 0) = x$$

for some binary term s and constant 0. Subtractive algebras abound in classical algebras. If we interpret the binary operator “ s ” as subtraction, and we use the infix notation, then we can rewrite the above identities as $x - x = 0$ and $x - 0 = x$. In the context of λ -calculus, the subtractive equations make a certain λ -term behave like a binary subtraction operator (in curried form) whose “zero” is the looping λ -term Ω .

In the last section of this paper we relativize to an element the notion of absolute unorderability. We say that an algebra \mathbf{A} is 0-unorderable if, for every compatible partial order on \mathbf{A} , 0 is not comparable with any other element of the algebra. An algebra \mathbf{A} in a variety \mathcal{V} is absolutely 0-unorderable if, for any $\mathbf{B} \in \mathcal{V}$ and embedding $f : \mathbf{A} \rightarrow \mathbf{B}$, \mathbf{B} is 0-unorderable. Generalising subtractivity to n -subtractivity ($n \geq 2$), we give a syntactic characterisation of the absolutely 0-unorderable algebras with Mal'cev-type conditions. The

consistency of the two subtractive equations with λ -calculus implies the existence of absolutely Ω -unorderable combinatory algebras.

2 Preliminaries

2.1 Partial Orderings

Let (A, \leq) be a partially ordered set (poset). Two elements a, b of A are: (1) *comparable* if either $a \leq b$ or $b \leq a$. A set $B \subseteq A$ is an *upward* (*downward*) closed set if $b \in B$, $a \in A$ and $b \leq a$ ($a \leq b$) imply $a \in B$.

We denote by \approx_{\leq} the least equivalence relation on A containing \leq . A *connected component* of (A, \leq) is an equivalence class of \approx_{\leq} . A connected component can be also characterised as a minimal subset of A which is both upward closed and downward closed. The poset (A, \leq) is called *connected* if \approx_{\leq} determines a unique equivalence class.

2.2 Lambda calculus

With regard to the λ -calculus we follow the notation and terminology of [1]. By Λ and Λ^o , respectively, we indicate the set of λ -terms and of closed λ -terms. By convention application associates to the left. The symbol \equiv denotes syntactical equality. The following are some notable λ -terms: $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$; $\mathbf{I} \equiv \lambda x.x$; $\mathbf{K} \equiv \lambda xy.x$; $\mathbf{S} \equiv \lambda xyz.xz(yz)$.

If M is a λ -term and $\vec{P} \equiv P_1 \dots P_n$ is a sequence of λ -terms, we write $M\vec{P}$ for the application $MP_1 \dots P_n$.

The β -reduction will be denoted by \rightarrow_{β} , while the η -reduction by \rightarrow_{η} . One step of either β -reduction or η -reduction will be denoted by $\rightarrow_{\beta\eta}$.

The letters ξ_1, ξ_2, \dots denote algebraic variables (holes in Barendregt's terminology [1]). Contexts are built up as λ -terms but also allowing occurrences of algebraic variables. Substitution for algebraic variables is made without α -conversion. For example, $(\lambda x.x\xi)[xy/\xi] = \lambda x.x(xy)$.

A λ -term M is *solvable* if it has a *head normal form*, i.e., M is β -convertible to a term of the form $\lambda \vec{x}.y\vec{N}$. A λ -term M is *unsolvable* if it is not solvable. Among unsolvables we distinguish the *zero terms*, which never reduce to an abstraction.

A λ -theory is a congruence on Λ (with respect to the operators of abstraction and application) which contains $\alpha\beta$ -conversion. We denote by $\lambda\beta$ the least λ -theory. The least extensional λ -theory $\lambda\beta\eta$ is axiomatised over $\lambda\beta$ by the equation $\lambda x.Mx = M$, where $M \in \Lambda$ and x is not free in M .

A λ -theory is *consistent* if it does not equate all λ -terms, *inconsistent* otherwise. A λ -theory is *semisensible* if it does not equate solvable and unsolvable λ -terms, so that semisensible λ -theories are consistent by definition. The set of λ -theories constitutes a complete lattice w.r.t. inclusion, whose top is the inconsistent λ -theory and whose bottom is the theory $\lambda\beta$. The λ -theory generated by a set X of identities is the intersection of all λ -theories containing X .

Although all unsolvable terms have the same Böhm tree, they do not have the same *infinite* normal form. In [2] Berarducci isolates a particular subset of the unsolvable terms, which turn out to have completely undefined behavior even in the context of infinite λ -calculus. Those terms, called *mute* terms, are defined as those unsolvables which are zero-terms and furthermore never reduce to an application whose left side is a zero-term. For example, $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$ is mute. Another mute term that will be used in the rest of the paper, is defined as follows. Let $A \equiv \lambda x.x(\lambda y.yx)$, $B \equiv \lambda y.yA$ and $\Theta \equiv AB$. By a direct computation

we see that the only possible reduction path starting with Θ is the following:

$$\Theta \rightarrow_{\beta} B(\lambda y.yB) \rightarrow_{\beta} (\lambda y.yB)A \rightarrow_{\beta} AB \equiv \Theta$$

Then Θ is a closed mute term. Throughout the paper we consider different reductions. If \rightarrow_{γ} is a reduction, then we denote by $\twoheadrightarrow_{\gamma}$ the reflexive transitive closure of \rightarrow_{γ} , and we write $=_{\gamma}$ to denote the reflexive, symmetric and transitive closure of \rightarrow_{γ} . Finally we define, as usual, the γ -reduction graph of a term M as the set $\mathcal{G}_{\gamma}(M) = \{N \in \Lambda : M \twoheadrightarrow_{\gamma} N\}$.

2.3 Models of λ -calculus

It took some time, after Scott gave his model construction, for consensus to arise on the general notion of a model of the λ -calculus. There are mainly two descriptions that one can give: the category-theoretical and the algebraic one. Besides the different languages in which they are formulated, the two approaches are intimately connected (see [1]). The categorical notion of model is well-suited for constructing concrete models, while the algebraic one is rather used to understand global properties of models (constructions of new models out of existing ones, closure properties, etc.) and to obtain results about the structure of the lattice of λ -theories.

The algebraic description of models of λ -calculus proposes two kinds of structures, viz. the λ -algebras and the λ -models, both based on the notion of *combinatory algebra*. We will focus on λ -models. A *combinatory algebra* $\mathbf{A} = (A, \cdot, K, S)$ is a structure with a binary operation called *application* and two distinguished elements K and S called *basic combinators*. The symbol “ \cdot ” is usually omitted from expressions and by convention application associates to the left, allowing to leave out superfluous parentheses. The class of combinatory algebras is axiomatized by the equations $Kxy = x$ and $Sxyz = xz(yz)$. Intuitively elements on the left-hand side of an application are to be seen as functions operating on arguments, placed on the right-hand side. Hence it is natural to say that a function $f : A^n \rightarrow A$ is *representable (in \mathbf{A})* if there exists an element $a \in A$ such that $f(b_1, \dots, b_n) = ab_1 \dots b_n$ for all $b_1, \dots, b_n \in A$. For example the identity function is represented by the combinator $I \equiv SKK$ and the projection on the first argument by the combinator K .

The axioms of an elementary subclass of combinatory algebras, called λ -models, were expressly chosen to make coherent the definition of interpretation of λ -terms. In addition to the axioms of combinatory algebra, we have:

$$\begin{aligned} \forall xy. (\forall z. xz = yz) &\Rightarrow \mathbf{1}x = \mathbf{1}y \\ \mathbf{1}_2K &= K \\ \mathbf{1}_3S &= S, \end{aligned}$$

where $\mathbf{1}_1 \equiv \mathbf{1} \equiv S(KI)$ and $\mathbf{1}_{n+1} \equiv S(K\mathbf{1})(S(K\mathbf{1}_n))$. The combinators $\mathbf{1}_n$ are made into inner choice operators. Indeed, given any $a \in A$, the element $\mathbf{1}_n a$ represents the same n -ary function as a and $\mathbf{1}_n c = \mathbf{1}_n d$ for all c, d representing the same n -ary function.

Let Env_A be the set of A -environments, i.e., the functions from the set Var of λ -calculus variables to A . For every $x \in Var$ and $a \in A$ we denote by $\rho[x := a]$ the environment ρ' which coincides with ρ everywhere except on x , where ρ' takes the value a .

When \mathbf{A} is a λ -model it is possible to define the following interpretation:

$$\begin{aligned} |x|_{\rho}^{\mathbf{A}} &= \rho(x); \\ |MN|_{\rho}^{\mathbf{A}} &= |M|_{\rho}^{\mathbf{A}} |N|_{\rho}^{\mathbf{A}}; \\ |\lambda x.M|_{\rho}^{\mathbf{A}} &= \mathbf{1}a, \text{ where } a \in A \text{ is any element representing the function } b \in A \mapsto |M|_{\rho[x:=b]}^{\mathbf{A}}. \end{aligned}$$

Note that $|\lambda x.M|_\rho^{\mathbf{A}}$ is well-defined, since each function $b \in A \mapsto |M|_{\rho[x:=b]}^{\mathbf{A}}$ is representable under the hypothesis that \mathbf{A} is a λ -model. This is the kind of interpretation we will refer to.

By the way when M is a closed λ -term and there is no worry of confusion about the model being considered, we write $|M|$ for $|M|_\rho^{\mathbf{A}}$.

Each λ -model \mathbf{A} induces a λ -theory, denoted here by $Th(\mathbf{A})$, and called *the equational theory of \mathbf{A}* . Thus, $M = N \in Th(\mathbf{A})$ if, and only if, M and N have the same interpretation in \mathbf{A} . A *partially ordered λ -model*, a *po-model* for short, is a pair (\mathbf{A}, \leq) , where \mathbf{A} is a λ -model and \leq is a partial order on A which makes the application operator of \mathbf{A} monotone in both arguments. A po-model (\mathbf{A}, \leq) is *non-trivial* if the partial order is not discrete, i.e., $a < b$ for some $a, b \in A$ (thus A is not a singleton).

2.4 The Jacopini–Kuper technique

The Jacopini–Kuper technique, introduced by Jacopini in [7] and generalized by Kuper in [8], can be used to tackle questions of consistency of equational extensions of lambda calculus. In this section we review this technique.

Let \mathcal{T} be an arbitrary consistent λ -theory, $\vec{P} = \vec{Q}$ be a set of identities $P_i = Q_i$ ($i = 1, \dots, n$) between closed λ -terms, and \mathcal{T}' be the λ -theory generated by $\mathcal{T} \cup (\vec{P} = \vec{Q})$. The idea is to reduce inconsistency of \mathcal{T}' to that of \mathcal{T} . If \mathcal{T}' is inconsistent, then there exists a finite equational proof of $\mathbf{K} =_{\mathcal{T}'} \mathbf{S}$, (where $\mathbf{K} \equiv \lambda xy.x$ and $\mathbf{S} \equiv \lambda xyz.xz(yz)$) and such a proof contains a finite number of applications of equations which are in $\vec{P} = \vec{Q}$. The Jacopini–Kuper technique, when applicable, consists in checking two conditions on the sequences \vec{P} and \vec{Q} , namely that \vec{P} is \mathcal{T} -operationally less defined than \vec{Q} (see Definition 2) and that \vec{P} is \mathcal{T} -proof-substitutable by \vec{Q} (see Definition 3). Under these two conditions, it is possible to remove from the proof of $\mathbf{K} =_{\mathcal{T}'} \mathbf{S}$ all occurrences of equations in $\vec{P} = \vec{Q}$, thus yielding a proof of $\mathbf{K} =_{\mathcal{T}} \mathbf{S}$. This is the end of the method, because \mathcal{T} is supposed to be a consistent λ -theory.

A very useful property for the application of Jacopini-Kuper method, and in particular for proving \mathcal{T} -proof-substitutability, is the existence of a Church-Rosser reduction, whose induced conversion coincides with the equality induced by \mathcal{T} on λ -terms. This is not evident from the abstract formulation given in this section, but will be clear in the next one, when we will apply the technique.

► **Lemma 1.** *We have that $\mathcal{T} \cup (\vec{P} = \vec{Q}) \vdash M = N$ if, and only if, there exist closed terms F_1, \dots, F_n such that*

$$\begin{aligned} M &=_{\mathcal{T}} F_1 \vec{P} \vec{Q} \\ F_j \vec{Q} \vec{P} &=_{\mathcal{T}} F_{j+1} \vec{P} \vec{Q} \quad \text{for } 1 \leq j \leq n-1 \\ F_n \vec{Q} \vec{P} &=_{\mathcal{T}} N. \end{aligned}$$

Proof. By [4, Theorem 1] there exist binary contexts $C_1(\xi_1, \xi_2), \dots, C_n(\xi_1, \xi_2)$ and identities $P_{i_j} = Q_{i_j}$ in $\vec{P} = \vec{Q}$ such that

$$\begin{aligned} M &=_{\mathcal{T}} C_1(P_{i_1}, Q_{i_1}) \\ C_j(Q_{i_j}, P_{i_j}) &=_{\mathcal{T}} C_{j+1}(P_{i_{j+1}}, Q_{i_{j+1}}) \quad \text{for } 1 \leq j \leq n-1 \\ C_n(Q_{i_n}, P_{i_n}) &=_{\mathcal{T}} N. \end{aligned}$$

It is sufficient to define $F_j \equiv \lambda \vec{x} \vec{y}. C_j(x_{i_j}, y_{i_j})$, where \vec{x} and \vec{y} are sequences of length k of fresh variables. ◀

► **Definition 2** (Operational definiteness). We say that \vec{P} is \mathcal{T} -operationally less defined than \vec{Q} if, for every $\beta\eta$ -normal form N and every term F , we have that

$$F\vec{P} =_{\mathcal{T}} N \Rightarrow F\vec{Q} =_{\mathcal{T}} N.$$

► **Definition 3** (Proof-substitutability). We say that \vec{P} is \mathcal{T} -proof-substitutable by \vec{Q} if

$$\forall F, F' \in \Lambda^{\circ}(F\vec{P} =_{\mathcal{T}} F'\vec{P} \Rightarrow \exists G \in \Lambda^{\circ}(G\vec{P}\vec{Q} =_{\mathcal{T}} F\vec{Q} \text{ and } G\vec{Q}\vec{P} =_{\mathcal{T}} F'\vec{Q})).$$

► **Theorem 4.** *If \vec{P} is \mathcal{T} -operationally less defined than \vec{Q} and \vec{P} is \mathcal{T} -proof-substitutable by \vec{Q} , then the λ -theory \mathcal{T}' generated by $\mathcal{T} \cup (\vec{P} = \vec{Q})$ is consistent.*

Proof. Assume \mathcal{T}' is inconsistent, so that $\mathbf{K} =_{\mathcal{T}'} \mathbf{S}$. Then by Lemma 1 there exists an equational proof of this identity of the form

$$\begin{array}{lcl} \mathbf{K} & =_{\mathcal{T}} & F_1\vec{P}\vec{Q} \\ F_j\vec{Q}\vec{P} & =_{\mathcal{T}} & F_{j+1}\vec{P}\vec{Q} \quad \text{for } 1 \leq j \leq n-1 \\ F_n\vec{Q}\vec{P} & =_{\mathcal{T}} & \mathbf{S}. \end{array}$$

Now we show how to iteratively transform the above proof of $\mathcal{T}' \vdash \mathbf{K} = \mathbf{S}$ in a proof of $\mathcal{T} \vdash \mathbf{K} = \mathbf{S}$.

Suppose $n = 1$, i.e., we have $\mathbf{K} =_{\mathcal{T}} F_1\vec{P}\vec{Q}$ and $F_1\vec{Q}\vec{P} =_{\mathcal{T}} \mathbf{S}$. Since \vec{P} is \mathcal{T} -operationally less defined than \vec{Q} , from $\mathbf{K} =_{\mathcal{T}} F_1\vec{P}\vec{Q} =_{\mathcal{T}} (\lambda\vec{y}.F_1\vec{y}\vec{Q})\vec{P}$ and $F_1\vec{Q}\vec{P} =_{\mathcal{T}} \mathbf{S}$, we get $\mathbf{K} =_{\mathcal{T}} F_1\vec{Q}\vec{Q} =_{\mathcal{T}} \mathbf{S}$.

Suppose $n > 1$. As before, by the hypothesis we get $\mathbf{K} =_{\mathcal{T}} F_1\vec{Q}\vec{Q}$ and $F_n\vec{Q}\vec{Q} =_{\mathcal{T}} \mathbf{S}$. Let \vec{y} be a sequence of fresh variables. For each $j = 1, \dots, n-1$, define

$$H_j \equiv \lambda\vec{y}.F_j\vec{Q}\vec{y}; \quad H'_{j+1} \equiv \lambda\vec{y}.F_{j+1}\vec{y}\vec{Q}.$$

By $F_j\vec{Q}\vec{P} =_{\mathcal{T}} F_{j+1}\vec{P}\vec{Q}$ ($j = 1, \dots, n-1$) we have that

$$H_j\vec{P} =_{\mathcal{T}} H'_{j+1}\vec{P}.$$

Since \vec{P} is \mathcal{T} -proof-substitutable by \vec{Q} , then there exist terms G_j ($j = 1, \dots, n-1$) such that $G_j\vec{P}\vec{Q} =_{\mathcal{T}} H_j\vec{Q} =_{\mathcal{T}} F_j\vec{Q}\vec{Q}$ and $G_j\vec{Q}\vec{P} =_{\mathcal{T}} H'_{j+1}\vec{Q} =_{\mathcal{T}} F_{j+1}\vec{Q}\vec{Q}$. Therefore we obtain that

$$\begin{array}{lclcl} \mathbf{K} & =_{\mathcal{T}} & F_1\vec{Q}\vec{Q} & =_{\mathcal{T}} & G_1\vec{P}\vec{Q} \\ G_1\vec{Q}\vec{P} & =_{\mathcal{T}} & F_2\vec{Q}\vec{Q} & =_{\mathcal{T}} & G_2\vec{P}\vec{Q} \\ & & \vdots & & \vdots \\ G_{n-1}\vec{Q}\vec{P} & =_{\mathcal{T}} & F_n\vec{Q}\vec{Q} & =_{\mathcal{T}} & \mathbf{S} \end{array}$$

Therefore one can iterate the argument and get a proof of $\mathcal{T} \vdash \mathbf{K} = \mathbf{S}$. ◀

3 On a question by Honsell and Plotkin

In this section we turn to a question posed in [5] by Honsell and Plotkin. The problem is whether or not there exists a formula φ of first-order logic written as a possibly empty list of universal quantifiers followed by a conjunction of equalities between λ -terms such that φ does not admit po-models with bottom element. According to Honsell and Plotkin, this problem falls under the name of Π_1 -consistency of the class of po-models with bottom element. We observe that in the context of λ -calculus the Π_1 -consistency is equivalent to the equational consistency, which is the particular case of Π_1 -consistency in which the formula φ is quantifier-free. In this section we find a counterexample to the equational consistency of the class of po-models with bottom element.

3.1 The λ -theory $\lambda\pi\phi$

We introduce two equations between λ -terms, whose models will be shown to have strong properties with respect to the possible partial orderings they can be endowed with. Of course we have to prove that the λ -theory $\lambda\pi\phi$ generated by these equations is consistent and this will be done in Section 3.2.

The two equations we are going to introduce represent within λ -calculus the notion of subtractivity, which has been introduced in Universal Algebra by Ursini [14].

► **Definition 5.** An algebra \mathbf{A} is *subtractive* if there exist a binary term $s(x, y)$ and a constant 0 in the algebraic similarity type of \mathbf{A} such that

$$s(x, x) = 0; \quad s(x, 0) = x.$$

Subtractive algebras abound in classical algebras and in algebraic logic since term s simulates part of subtraction. If we interpret the binary operator “ s ” as subtraction “ $-$ ” and we use the infix notation, then we can rewrite the above identities as $x - x = 0$ and $x - 0 = x$.

Let Θ be the closed mute term defined in Section 2.2. We define $s(x, y) \equiv \Theta xy$ and $0 \equiv \Omega$. Then the λ -theory $\lambda\pi\phi$ is defined as the least extensional λ -theory generated by the following two equations, called the *subtractive equations*:

$$(\pi) \quad \Theta xx = \Omega; \quad (\phi) \quad \Theta x\Omega = x.$$

The intuitive meaning of the equations (π) and (ϕ) is that they make the term Θ behave like a binary subtraction operator (in curried form) whose “zero” is the term Ω . This intuition will be treated precisely in Section 5. The following theorem illustrates a curious aspect of the equations (π) and (ϕ) : the choice of Ω is the right one.

► **Theorem 6.** *Let O be a λ -term such that $x \notin FV(O)$, and let \mathcal{T} be any λ -theory including the identities $\Theta xO = x$ and $\Theta xx = O$. Then $\mathcal{T} \vdash O = \Omega$.*

Proof. We apply a technique introduced by Gordon Plotkin and Alex Simpson (see [13]). Let $Y \equiv \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$ be the Curry fixpoint combinator. Then, for any λ -term M , define $\mu x.M \equiv Y(\lambda x.M)$. Now let $D \equiv \mu y.\mu x.\Theta xy$. Then we have $D =_{\beta} \Theta DD =_{\mathcal{T}} O$ and $D =_{\beta} \mu x.\Theta xD =_{\mathcal{T}} \mu x.\Theta xO =_{\mathcal{T}} \mu x.x =_{\beta} \Omega$, therefore $\mathcal{T} \vdash O = \Omega$. ◀

3.1.1 The λ -theory $\lambda\pi$

The extensional λ -theory $\lambda\pi$ is axiomatised over $\lambda\beta\eta$ by the equation (π) . It is consistent because semisensible. We will show the consistency of $\lambda\pi\phi$ relying on the consistency of $\lambda\pi$.

We remark that the λ -theory axiomatised by $\Omega xx = \Omega$ was introduced in [12]. Here we use $\Theta xx = \Omega$ for technical reason.

The following notion of reduction will be useful in the next sections (recall from Section 2.2 the definition of reduction graph $\mathcal{G}_{\beta}(\Theta)$ of Θ).

► **Definition 7** ($\lambda\pi$ -reduction). We formally introduce here $\lambda\pi$ -reduction, notation $\rightarrow_{\lambda\pi}$, as the contextual closure of $\rightarrow_{\beta\eta} \cup \rightarrow_{\pi}$, where

$$\Psi MN \rightarrow_{\pi} \Omega \quad \text{if } \Psi \in \mathcal{G}_{\beta}(\Theta) \text{ and } \lambda\pi \vdash M = N.$$

Of course the conversion $=_{\lambda\pi}$ coincides with the equality induced by $\lambda\pi$.

► **Theorem 8.** ■ *The reduction $\rightarrow_{\lambda\pi}$ is Church-Rosser;*
 ■ *For all terms M and N , we have $\Theta MN =_{\lambda\pi} \Omega$ iff $M =_{\lambda\pi} N$.*

Proof. As in the proof of [12, Lemma 3.1], it is sufficient to verify that \rightarrow_π satisfies the diamond property (see [1, Lemma 3.2.2]) and that the relations $\rightarrow_{\beta\eta}$ and \rightarrow_π commute (see [1, Def. 3.3.4]). The conclusion follows from the Hindley–Rosen Lemma (see [1, Prop. 3.3.5]). ◀

Another useful result is the forthcoming lemma, which says that all $\lambda\pi$ -reduction paths may be “simulated” by a reduction path which allows π -steps only at the end.

► **Lemma 9 (Factorization).** *If $M \rightarrow_{\lambda\pi} N$, then there exists P such that $M \rightarrow_{\beta\eta} P \rightarrow_\pi N$.*

Proof. Use iteratively the fact that whenever $M \rightarrow_\pi N \rightarrow_{\beta\eta} Q$, then there exists N' such that $M \rightarrow_{\beta\eta} N' \rightarrow_\pi Q$. ◀

► **Lemma 10.** *The terms Θ and Ω are not $\lambda\pi$ -convertible.*

Proof. By the reduction graph of Θ and the confluence of $\rightarrow_{\lambda\pi}$. ◀

We remark that $\rightarrow_{\lambda\pi}$ -residuals do not create new \rightarrow_π -redexes. For example, if M is a \rightarrow_π -redex, then the reduction $MNZ \rightarrow_\pi \Omega NZ$ only contains \rightarrow_π -redexes already present in N or Z . Any further reduction can only duplicate, erase or contract those redexes. This would not have been true if $\Omega \in \mathcal{G}_\beta(\Theta)$.

3.2 Jacopini–Kuper technique for $\lambda\pi\phi$

In this section we apply the Jacopini–Kuper technique explained in Section 2.4 to prove the consistency of the theory $\lambda\pi\phi$. More precisely, the results presented here show that the closure $\lambda x.\Theta x\Omega = \mathbf{I}$ of the equation (ϕ) , that axiomatizes $\lambda\pi\phi$ over $\lambda\pi$, satisfies the hypotheses of Theorem 4.

► **Lemma 11.** *The term $\lambda x.\Theta x\Omega$ is $\lambda\pi$ -operationally less defined than \mathbf{I} .*

Proof. Let F be a λ -term and N be a $\beta\eta$ -normal form, and suppose $F(\lambda x.\Theta x\Omega) =_{\lambda\pi} N$. Since $\lambda\pi$ -reduction is confluent and N is $\beta\eta$ -normal, we have that $F(\lambda x.\Theta x\Omega) \rightarrow_{\lambda\pi} N$. By Lemma 9 there exists a term M such that

$$F(\lambda x.\Theta x\Omega) \rightarrow_{\beta\eta} M \rightarrow_\pi N.$$

Since N is a $\beta\eta$ -normal form, we must have that $M \equiv N$. Therefore we have $\lambda\beta\eta \vdash F(\lambda x.\Theta x\Omega) = N$ and, since $\lambda x.\Theta x\Omega$ is unsolvable, we can apply the Genericity Lemma of lambda calculus to obtain $\lambda\beta\eta \vdash F\mathbf{I} = N$, and hence obviously $\lambda\pi \vdash F\mathbf{I} = N$ which is the desired conclusion. ◀

In Lemma 12 below we keep track of the residuals of the λ -term $\lambda x.\Theta x\Omega$ during the reduction of the term $F(\lambda x.\Theta x\Omega)$. We have three kinds of residuals: $\lambda x.\Psi x\Omega$, $\Psi M\Omega$ and Ω (with $\Psi \in \mathcal{G}_\beta(\Theta)$) as the following informal example shows:

$$\begin{array}{llll} F(\lambda x.\Theta x\Omega) & \rightarrow_{\lambda\pi} & \cdots (\lambda x.\Theta x\Omega) \cdots (\lambda x.\Theta x\Omega) \cdots (\lambda x.\Theta x\Omega) \cdots & \\ & \rightarrow_\beta & \cdots (\lambda x.\Theta x\Omega) \cdots (\lambda x.\Psi x\Omega) \cdots \cdots (\lambda x.\Theta x\Omega) \cdots & (\Theta \rightarrow_\beta \Psi) \\ & \rightarrow_{\lambda\pi} & \cdots (\lambda x.\Theta x\Omega) \cdots (\lambda x.\Psi x\Omega)M \cdots (\lambda x.\Theta x\Omega) \cdots & \\ & \rightarrow_\beta & \cdots (\lambda x.\Theta x\Omega) \cdots (\Psi M\Omega) \cdots (\lambda x.\Theta x\Omega) \cdots & (\beta\text{-reduction}) \\ & \rightarrow_{\lambda\pi} & \cdots (\lambda x.\Theta x\Omega) \cdots (\Psi N\Omega) \cdots (\lambda x.\Theta x\Omega) \cdots & (M \rightarrow_{\lambda\pi} N) \\ & \rightarrow_\pi & \cdots (\lambda x.\Theta x\Omega) \cdots (\Omega) \cdots (\lambda x.\Theta x\Omega) \cdots & (N =_{\lambda\pi} \Omega) \\ & \rightarrow_{\lambda\pi} & \cdots \cdots \cdots & \end{array}$$

In order to trace the residuals it is useful to enrich the syntax of λ -terms with labels as follows:

$$M, N ::= x \mid \lambda x.M \mid MN \\ \mid (\lambda x.\Psi x\Omega)^n \mid (\Psi M\Omega)^n \mid (\Omega)^n \quad (n \geq 1 \text{ and } \Psi \in \mathcal{G}_\beta(\Theta))$$

We denote by $\Lambda^{\mathbb{N}}$ the set of labelled terms and we write \underline{M} for the λ -term, called *erasure* of M , obtained by erasing the labels from M .

Since $(\Omega)^n$ and $(\lambda x.\Psi x\Omega)^n$ are closed terms, then it is sufficient to extend substitution to labelled terms by setting $(\Psi M\Omega)^n[N/x] = (\Psi M[N/x]\Omega)^n$, where $\Psi \in \mathcal{G}_\beta(\Theta)$. Then we define a reduction on labelled terms as the smallest contextual reduction \rightarrow_{lab} satisfying the following clauses, for all labelled terms M, N :

$$\begin{array}{ll} (\lambda x.M)N \rightarrow_{\text{lab}} M[N/x] & \\ \lambda x.Mx \rightarrow_{\text{lab}} M & \text{if } x \notin \text{FV}(M) \\ (\lambda x.\Psi x\Omega)^n M \rightarrow_{\text{lab}} (\Psi M\Omega)^n & \text{if } \Psi \in \mathcal{G}_\beta(\Theta) \\ \Psi MN \rightarrow_{\text{lab}} \Omega & \text{if } \Psi \in \mathcal{G}_\beta(\Theta) \text{ and } \underline{M} =_{\lambda\pi} \underline{N}. \end{array}$$

Note that (i) $\rightarrow_{\lambda\pi} \subseteq \rightarrow_{\text{lab}}$; (ii) if $M \rightarrow_{\text{lab}} N$ then $(\Psi M\Omega)^n \rightarrow_{\text{lab}} (\Psi N\Omega)^n$. If σ is a reduction path of labelled terms, then we denote by $\underline{\sigma}$ the corresponding reduction path, where all labels are erased.

We will make use of an additional operation on labelled terms. Given terms $M, N \in \Lambda^{\mathbb{N}}$ such that $\underline{M} \equiv \underline{N}$, we define their *superposition* as the labelled term obtained from the syntax tree of \underline{M} by adding a possible label k to each subtree T of \underline{M} according to the following schema:

- Put $k = m + n$ if T has label m in M and n in N ;
- Put $k = m$ if T has label m in M and no label in N ;
- Put $k = n$ if T has label n in N and no label in M ;
- Put no label otherwise.

► **Lemma 12.** *The term $\lambda x.\Theta x\Omega$ is $\lambda\pi$ -proof-substitutable by \mathbf{I} .*

Proof. In this proof Ψ ranges over $\mathcal{G}_\beta(\Theta)$. Let F_1, F_2 be closed λ -terms and suppose $F_1(\lambda x.\Theta x\Omega) =_{\lambda\pi} F_2(\lambda x.\Theta x\Omega)$. Since the reduction $\rightarrow_{\lambda\pi}$ is confluent, then the two sides of the equality are the beginning of two reduction paths σ_1 and σ_2 that end in a common term R . Consider now the labelled terms $F_i(\lambda x.\Theta x\Omega)^i$ for $i = 1, 2$. Then there exists a labelled reduction path σ'_i starting with $F_i(\lambda x.\Theta x\Omega)^i$ such that $\underline{\sigma'_i} \equiv \sigma_i$. We denote by R_i the last labelled term in the reduction path σ'_i . Then we have that $R \equiv \underline{R_1} \equiv \underline{R_2}$.

Let S be the term obtained by superposition of R_1 and R_2 . Then the labels of S range over the set $\mathcal{L} = \{1, 2, 3\}$. We now describe how to extract a witness of $\lambda\pi$ -proof-substitutability by suitably modifying S . All residuals with label 3 in S are common to the reduction paths σ'_1 and σ'_2 . Then, if we mimic the reduction path σ_i starting from $F_i\mathbf{I}$ ($i = 1, 2$), we will find in place of the residuals with label 3 the term \mathbf{I} for $(\lambda x.\Psi x\Omega)^3$; M for $(\Psi M\Omega)^3$ and a term N ($\lambda\pi$ -convertible with Ω) for $(\Omega)^3$:

$$\begin{array}{llll} F_i(\lambda x.\Theta x\Omega) & \twoheadrightarrow_{\lambda\pi} & (i = 1, 2) & F_i\mathbf{I} & \twoheadrightarrow_{\lambda\pi} & (i = 1, 2) \\ \dots (\lambda x.\Theta x\Omega) \dots & \twoheadrightarrow_{\beta} & & \dots \mathbf{I} \dots & \equiv & \\ \dots (\lambda x.\Psi x\Omega) \dots & \twoheadrightarrow_{\lambda\pi} & (\Theta \twoheadrightarrow_{\beta} \Psi) & \dots \mathbf{I} \dots & \twoheadrightarrow_{\lambda\pi} & \\ \dots (\lambda x.\Psi x\Omega)M \dots & \twoheadrightarrow_{\beta} & & \dots \mathbf{I}M \dots & \twoheadrightarrow_{\beta} & \\ \dots \Psi M\Omega \dots & \twoheadrightarrow_{\lambda\pi} & (M \twoheadrightarrow_{\lambda\pi} N) & \dots M \dots & \twoheadrightarrow_{\lambda\pi} & (M \twoheadrightarrow_{\lambda\pi} N) \\ \dots \Psi N\Omega \dots & \twoheadrightarrow_{\lambda\pi} & (N =_{\lambda\pi} \Omega) & \dots N \dots & \equiv & (N =_{\lambda\pi} \Omega) \\ \dots \Omega \dots & & & \dots N \dots & & \end{array}$$

Then we let $S' \equiv S[\mathbf{I}/(\lambda x.\Psi x\Omega)^3; M/(\Psi M\Omega)^3; \Omega/(\Omega)^3]$. The last substitution Ω for $(\Omega)^3$ is possible because the term N in the above reduction path (right column) is $\lambda\pi$ -convertible with Ω . We see that, by mimicking the steps in the paths σ_1, σ_2 , we have that

$$(*) \quad F_i \mathbf{I} =_{\lambda\pi} L_1, \text{ where } L_1 \text{ is the erasure of } S'[\mathbf{I}/(\lambda x.\Psi x\Omega)^i; M/(\Psi M\Omega)^i] \quad (i = 1, 2)$$

Let x_1, x_2 be fresh variables and let H be the term obtained from S' by replacing bottom-up the subterms (labeled by $i \in \mathcal{L}$)

$$\text{for } i = 1, 2 \quad \begin{cases} (\lambda x.\Psi x\Omega)^i & \text{with } x_i; \\ (\Psi M\Omega)^i & \text{with } x_i M; \\ (\Omega)^i & \text{with } x_i \Omega. \end{cases}$$

Then the following equivalences hold:

$$(a) \quad L_1 =_{\lambda\pi} H[\mathbf{I}/x_1; (\lambda x.\Psi x\Omega)/x_2];$$

$$(b) \quad L_2 =_{\lambda\pi} H[(\lambda x.\Psi x\Omega)/x_1; \mathbf{I}/x_2].$$

Therefore by setting $G \equiv \lambda x_2 x_1. H$, we obtain that

$$\blacksquare \quad G(\lambda x.\Theta x\Omega)\mathbf{I} \rightarrow_{\beta} H[\mathbf{I}/x_1; (\lambda x.\Psi x\Omega)/x_2] =_{\lambda\pi} L_1 =_{\lambda\pi} F_1 \mathbf{I}, \quad \text{by } (*) \text{ and } (a)$$

$$\blacksquare \quad G\mathbf{I}(\lambda x.\Theta x\Omega) \rightarrow_{\beta} H[(\lambda x.\Psi x\Omega)/x_1; \mathbf{I}/x_2] =_{\lambda\pi} L_2 =_{\lambda\pi} F_2 \mathbf{I}, \quad \text{by } (*) \text{ and } (b)$$

This shows that G is the witness term we were looking for. \blacktriangleleft

Now we are ready to give the main theorem of the section.

► **Theorem 13.** *The λ -theory $\lambda\pi\phi$ is consistent.*

Proof. Lemma 12 and Lemma 11 show that the hypotheses of Theorem 4 are satisfied by the equation that axiomatizes $\lambda\pi\phi$ over $\lambda\pi$, and therefore $\lambda\pi\phi$ must be consistent. \blacktriangleleft

3.3 The main theorem

The following results prove that there is no po-model with bottom element satisfying the equations (π) and (ϕ) that define the λ -theory $\lambda\pi\phi$.

► **Lemma 14.** *Let \mathcal{M} be a po-model such that $\mathcal{M} \models \Theta x x = \Omega \wedge \Theta x \Omega = x$ (i.e., $Th(\mathcal{M}) \supseteq \lambda\pi\phi$). Then for all closed λ -terms P and Q we have:*

(i) *If $\mathcal{M} \not\models \Theta P Q = \Omega$, then the interpretations of P and Q are in distinct connected components of \mathcal{M} .*

(ii) *The connected component of the interpretation of Ω is a singleton set.*

Proof. (i) Following [12, Section 4] we define the subtraction sequence of the pair (P, Q) :

$$s_1 \equiv \Theta P Q; \quad s_{n+1} \equiv \Theta s_n \Omega.$$

By hypothesis $\mathcal{M} \models s_1 \neq \Omega$ and by subtractivity $\mathcal{M} \models s_n = s_1$ for all n . Then the conclusion follows from [12, Corollary 4.6].

(ii) Let $a \in \mathcal{M}$ and $a \neq |\Omega|$. Consider the subtraction sequence of the pair (a, Ω) : $s_1 = \Theta a \Omega$ and $s_{n+1} = \Theta s_n \Omega$. Since $\mathcal{M} \models s_n = a$ for all n , again an application of [12, Corollary 4.6] implies that a and the interpretation of Ω are in distinct connected components. \blacktriangleleft

The situation described by Lemma 14 can be regarded to as a relativized version of absolute unorderability to one fixed element. In particular the interpretation of Ω is isolated in every model. This property will be studied in Section 5 in the framework of Universal Algebra.

We recall from [5, Theorem 7] that consistency fails for quantifier-free sentences and po-models with bottom element. The sentence $\lambda x. \Omega x x = \lambda x. \Omega \wedge \Omega \neq \Omega \Omega (\Omega \mathbf{KI})$ is consistent with the extensional λ -calculus but no po-model with bottom element satisfies it. The following theorem improves this result by Honsell and Plotkin.

► **Theorem 15.** *For every non-trivial po-model \mathcal{M} with bottom element, we have $\mathcal{M} \not\models \Theta x x = \Omega \wedge \Theta x \Omega = x$.*

Proof. Suppose, by contradiction, that $\mathcal{M} \models \Theta x x = \Omega \wedge \Theta x \Omega = x$. Since $|\Omega|$ is comparable with \perp , then by Lemma 14(ii) Ω is interpreted as the bottom element \perp . The bottom element is comparable with all other elements of the model. This contradicts Lemma 14(ii). ◀

Therefore the equations $\lambda x. \Theta x x = \lambda x. \Omega$ and $\lambda x. \Theta x \Omega = \lambda x. x$ are indeed a counterexample to the equational consistency for the class of po-models with bottom element. We can also get a stronger result.

► **Theorem 16.** *If \mathcal{M} is a po-model such that $Th(\mathcal{M}) \supseteq \lambda\pi\phi$, then the partial ordering of \mathcal{M} is not connected.*

4 On the order-incompleteness of λ -calculus

The open problem of the *order-incompleteness* of λ -calculus was raised by Selinger in [13]: does there exist a λ -theory which does not arise as the theory of a non-trivial po-model? Such a problem can be also characterised in terms of connected components of a partial ordering (minimal subsets which are both upward and downward closed): a λ -theory \mathcal{T} is order-incomplete if, and only if, every po-model, having \mathcal{T} as equational theory, is partitioned in an infinite number of connected components, each one containing exactly one element. In other words, the partial order is the equality.

So far we have shown that the subtractive equations force their models not to be connected as partial orders. However, the order-incompleteness is far more distant to connectedness. Toward order-incompleteness, we propose a strengthening \mathcal{T} of the λ -theory $\lambda\pi\phi$ having the following property: every po-model \mathcal{M} such that $Th(\mathcal{M}) \supseteq \mathcal{T}$ has an infinite number of connected components among which that of $|\Omega|$ is a singleton set. Moreover each connected component contains the denotation of at most one $\beta\eta$ -normal form.

We are now going to introduce the above-mentioned strengthening of $\lambda\pi\phi$. It will make use of another mute term, that we will call Θ_2 , obtained as follows:

- Define inductively $A_0 \equiv x$ and $A_{n+1} \equiv \lambda y. y A_n$, where $y \neq x$. Note that $FV(A_n) = \{x\}$, for each $n \in \mathbb{N}$.
- Now set $B_2 \equiv \lambda x. x A_2$, $C_2 \equiv (\lambda z. z B_2)$ and $\Theta_2 \equiv B_2 C_2$.

It is not difficult to check that Θ_2 is a mute closed term. Moreover Ω , Θ and Θ_2 are pairwise non- $\lambda\pi$ -convertible: this is an immediate consequence of the confluence of $\rightarrow_{\lambda\pi}$ and of the form of the reduction graphs of the terms in question.

Let \mathcal{T} be the theory axiomatized over $\lambda\pi\phi$ by the following equations:

$$\begin{aligned} \Theta_2 \Omega &= \mathbf{K}; \\ \Theta_2(\Theta M N) &= \mathbf{S}, \quad M \text{ and } N \text{ distinct closed } \beta\eta\text{-normal forms.} \end{aligned}$$

Next we show that \mathcal{T} is consistent. In order to do that it suffices, by compactness reasons, to prove that any finite subset of the above equations is eliminable from a proof of $\mathcal{T} \vdash \mathbf{K} = \mathbf{S}$ via the Jacopini–Kuper technique. The proof of this fact closely resembles the consistency proof given for $\lambda\pi\phi$ (see Section 3.2), so we will just sketch it, only considering the extension of $\lambda\pi$ by three equations $\Theta_2(\Theta MN) = \mathbf{S}$, $\Theta_2\Omega = \mathbf{K}$ and $\lambda x.\Theta x\Omega = \mathbf{I}$, where (M, N) is an arbitrary but fixed pair of closed distinct $\beta\eta$ -normal forms.

Define the two sequences $\vec{P} = \Theta_2(\Theta MN), \Theta_2\Omega, \lambda x.\Theta x\Omega$ and $\vec{Q} = \mathbf{S}, \mathbf{K}, \mathbf{I}$.

We observe that it follows directly from Lemma 11 that \vec{P} is $\lambda\pi$ -operationally less defined than \vec{Q} .

► **Lemma 17.** *\vec{P} is $\lambda\pi$ -proof-substitutable by \vec{Q} .*

Proof. In this proof Ψ and Ψ_2 range, respectively, over $\mathcal{G}_\beta(\Theta)$ and $\mathcal{G}_\beta(\Theta_2)$. Let F_1, F_2 be closed λ -terms and suppose $F_1\vec{P} =_{\lambda\pi} F_2\vec{P}$. Since the reduction $\rightarrow_{\lambda\pi}$ is confluent, then the two sides of the equality are the beginning of two reduction paths σ_1 and σ_2 that end in a common term R .

Consider now the labelled term

$$\blacksquare A_1 \equiv F_1(\Theta_2(\Theta MN))^1(\Theta_2\Omega)^4(\lambda x.\Theta x\Omega)^{10}$$

$$\blacksquare A_2 \equiv F_2(\Theta_2(\Theta MN))^2(\Theta_2\Omega)^5(\lambda x.\Theta x\Omega)^{11}$$

Then there exist labelled reduction paths σ'_i starting with A_i ($i = 1, 2$) such that $\sigma'_i \equiv \sigma_i$. We denote by R_i the last labelled term in the reduction path σ'_i . Then we have $\bar{R} \equiv \underline{R}_i$ ($i = 1, 2$). Let S be the term obtained by superposition of R_1 and R_2 . Then the labels of S range over the set $\mathcal{L} = \{1, 2, 3, 4, 5, 9, 10, 11\}$. Note that if S has a labelled subterm of the shape $(\Theta_2\Omega)^l$, then $l \in \{4, 5, 9\}$ because the contrary would require $\Theta MN \rightarrow_{\lambda\pi} \Omega$ (by Theorem 8(ii)), which is impossible because it would imply $\lambda\pi \vdash M = N$, contradicting the consistency of $\lambda\pi$ (as a consequence of Böhm’s Theorem [1, Thm. 10.4.2]).

We now describe how to extract a witness of $\lambda\pi$ -proof-substitutability by suitably modifying S . All residuals with label 3, 9, or 21 in S are common to the reduction paths σ'_1 and σ'_2 . Then, if we mimic the reduction path σ_i starting from $F_i\mathbf{I}$ ($i = 1, 2$), we will find in place of the residuals with label 21 the term \mathbf{I} for $(\lambda x.\Psi x\Omega)^{21}$; M for $(\Psi M\Omega)^{21}$ and a term N ($\lambda\pi$ -convertible with Ω) for $(\Omega)^{21}$. Similarly those residuals with labels 3 and 9 are replaced by \mathbf{S} and \mathbf{K} , respectively. Then we let

$$S' \equiv S[\mathbf{I}/(\lambda x.\Psi x\Omega)^{21}; M/(\Psi M\Omega)^{21}; \Omega/(\Omega)^{21}; \mathbf{S}/(\Theta_2(\Theta MN))^3; \mathbf{K}/(\Theta_2\Omega)^9]$$

and we define a term H out of S' by replacing bottom-up some subterms (labeled by $i \in \mathcal{L}$), using fresh variables $x_1, x_2, x_3, x_4, x_5, x_{10}, x_{11}$ as follows

$$\text{for } i = 10, 11 \quad \left\{ \begin{array}{ll} (\lambda x.\Psi x\Omega)^i & \text{with } x_i; \\ (\Psi M\Omega)^i & \text{with } x_i M; \\ (\Omega)^i & \text{with } x_i \Omega. \end{array} \right. \quad \text{for } i = 4, 5 \text{ and } j = 1, 2 \quad \left\{ \begin{array}{ll} (\Psi_2\Omega)^i & \text{with } x_i; \\ (\Psi_2(\Psi MN))^j & \text{with } x_j. \end{array} \right.$$

Finally, as in the proof of Lemma 12, it is possible to find a term G such that:

$$G\vec{P}\vec{Q} \rightarrow_\beta H[(\Psi_2(\Psi MN))/x_1; \mathbf{S}/x_2; (\Psi_2\Omega)/x_4; \mathbf{K}/x_5; \mathbf{I}/x_{10}; (\lambda x.\Psi x\Omega)/x_{11}] =_{\lambda\pi} F_1\vec{Q}$$

$$G\vec{Q}\vec{P} \rightarrow_\beta H[\mathbf{S}/x_1; (\Psi_2(\Psi MN))/x_2; \mathbf{K}/x_4; (\Psi_2\Omega)/x_5; (\lambda x.\Psi x\Omega)/x_{10}; \mathbf{I}/x_{11}] =_{\lambda\pi} F_2\vec{Q}$$

◀

The following theorem, which relies on Lemma 17, it is analogous to Theorem 13.

► **Theorem 18.** *The λ -theory \mathcal{T} is consistent.*

We conclude the section with a theorem that improves a result in [12], where it is shown that every po-model \mathcal{M} such that $Th(\mathcal{M}) = \lambda\pi$ has an infinite number of connected components.

► **Theorem 19.** *Let \mathcal{M} be a po-model such that $Th(\mathcal{M}) \supseteq \mathcal{T}$. Then \mathcal{M} has an infinite number of connected components and it has the following properties:*

1. *The interpretation in \mathcal{M} of two distinct $\beta\eta$ -normal forms belongs to different connected components;*
2. *The connected component of $|\Omega|$ is a singleton set.*

Proof. Let M, N be two distinct $\beta\eta$ -normal forms and suppose, by way of contradiction, that $|M|$ and $|N|$ lie in the same connected component of \mathcal{M} . Then $\mathcal{M} \models \Theta MN = \Omega$ by Lemma 14(i). But then from $\mathcal{M} \models \mathbf{S} = \Theta_2(\Theta MN)$ and $\mathcal{M} \models \Theta_2\Omega = \mathbf{K}$ we derive that $\mathcal{M} \models \mathbf{S} = \mathbf{K}$, which contradicts the non-triviality of \mathcal{M} . Hence each denotation of a $\beta\eta$ -normal form belongs to exactly one connected component. The second part of the statement follows directly from Lemma 14(ii). ◀

5 Subtractivity and orderings

The inspiration for the subtractive equations comes from a general algebraic framework, developed by Ursini [14], called *subtractivity*. Salibra in [12] investigated the weaker notion of *semi-subtractivity*, linking it to properties of po-models of λ -calculus. Here we follow that path illustrating the stronger properties of subtractivity.

We start the section briefly reviewing the connection established by Selinger in [13] between the absolute unorderability and the validity of certain Mal'cev-type conditions.

Let \mathbf{A} be an algebra of some variety \mathcal{V} (i.e., equational class). A preorder \leq on \mathbf{A} is *compatible* if it is monotone in each coordinate of every function symbol of \mathcal{V} . Then we have: (i) \mathbf{A} is *unorderable* if it admits only equality as a compatible partial order; (ii) \mathbf{A} is *absolutely unorderable* if, for every algebra $\mathbf{B} \in \mathcal{V}$ and every embedding $f : \mathbf{A} \rightarrow \mathbf{B}$ (i.e., injective homomorphism), the algebra \mathbf{B} is unorderable.

Let \mathcal{V} be a variety, $\mathbf{A} \in \mathcal{V}$ and X be a set of indeterminates. We denote by $\mathbf{A}[X]$ the free extension of \mathbf{A} in the variety \mathcal{V} . The algebra $\mathbf{A}[X]$ is defined up to isomorphism by the following universal mapping properties: (1) $A \cup X \subseteq \mathbf{A}[X]$; (2) $\mathbf{A}[X] \in \mathcal{V}$; (3) for every $\mathbf{B} \in \mathcal{V}$, homomorphism $h : \mathbf{A} \rightarrow \mathbf{B}$ and every function $f : X \rightarrow B$, there exists a unique homomorphism $f : \mathbf{A}[X] \rightarrow \mathbf{B}$ extending h and f . When $X = \{x_1, \dots, x_n\}$ is finite, we write $\mathbf{A}[x_1, \dots, x_n]$ for $\mathbf{A}[X]$.

The following result by Selinger [13] characterises those algebras which are absolutely unorderable.

► **Theorem 20.** *Let \mathcal{V} be a variety. An algebra $\mathbf{A} \in \mathcal{V}$ is absolutely unorderable if, and only if, there exist a natural number $n \geq 1$ and ternary terms p_1, \dots, p_n in the type of \mathcal{V} such that the algebra $\mathbf{A}[x, y]$ satisfies the following identities:*

$$\begin{aligned} x &= p_1(x, y, y); \\ p_i(x, x, y) &= p_{i+1}(x, y, y) \quad (i = 1, \dots, n-1); \\ p_n(x, x, y) &= y. \end{aligned}$$

In the case the variety \mathcal{V} has a constant 0, then we can relativise the Mal'cev identities as follows:

$$\begin{aligned} 0 &= p_1(0, y, y); \\ p_i(0, 0, y) &= p_{i+1}(0, y, y) \quad (i = 1, \dots, n-1); \\ p_n(0, 0, y) &= y. \end{aligned}$$

This suggests that the absolute unorderability relative to the element 0 can be expressed by the following identities defining n -subtractivity.

► **Definition 21.** Let \mathcal{V} be a variety of algebras with a constant 0. We say that \mathcal{V} is n -subtractive ($n \geq 2$) if there exist $n - 1$ binary terms $s_1(x, y), \dots, s_{n-1}(x, y)$ such that \mathcal{V} satisfies the following identities:

$$\begin{aligned} 0 &= s_1(x, x) \\ s_i(x, 0) &= s_{i+1}(x, x) \quad (i = 1, \dots, n - 2); \\ s_{n-1}(x, 0) &= x. \end{aligned}$$

Then Ursini's subtractivity (see Definition 5) means 2-subtractivity.

Every model of the two equations (π) and (ϕ) is subtractive, when we define the binary operator $s_1(x, y)$ defining subtractivity as the λ -term Θxy . As a consequence of the consistency of the λ -theory $\lambda\pi\phi$, it follows that there exists a non-trivial subtractive variety of combinatory algebras.

► **Definition 22.** An algebra \mathbf{A} is *0-unorderable* if, for every compatible partial order \leq on A and every $a \neq 0 \in A$, neither $0 \leq a$ nor $a \leq 0$.

► **Definition 23.** Let \mathcal{V} be a variety. An algebra $\mathbf{A} \in \mathcal{V}$ is said to be *absolutely 0-unorderable* if, for any algebra $\mathbf{B} \in \mathcal{V}$ and embedding $f : \mathbf{A} \rightarrow \mathbf{B}$, \mathbf{B} is 0-unorderable.

Let R_1 (resp. R_2) be the smallest compatible preorder on $\mathbf{A}[x]$ such that xR_10 (resp. $0R_2x$).

► **Lemma 24.** Let \mathcal{V} be a variety. An algebra $\mathbf{A} \in \mathcal{V}$ is absolutely 0-unorderable iff $0R_1x$ and xR_20 .

Proof. Assume that \mathbf{A} is not absolutely 0-unorderable. Then there exists an embedding $f : \mathbf{A} \rightarrow \mathbf{B} \in \mathcal{V}$, where \mathbf{B} has a non-trivial partial ordering \leq and there exists an element $b \neq 0 \in B$ such that either $0 \leq b$ or $b \leq 0$. Consider the unique homomorphism $g : \mathbf{A}[x] \rightarrow \mathbf{B}$ extending f such that $g(x) = b$. Define aSc in $\mathbf{A}[x]$ iff $g(a) \leq g(c)$ in \mathbf{B} . We have that S is a compatible preorder on $\mathbf{A}[x]$ such that either $0Sx$ or $xS0$ but not both (!). If $xS0$, then $R_1 \subseteq S$ but not $0R_1x$. If $0Sx$, then $R_2 \subseteq S$ but not xR_20 . ◀

Note that, as a consequence of Lemma 24, if \mathbf{A} is absolutely 0-unorderable, then $R_1 = R_2$.

► **Theorem 25.** Let \mathcal{V} be a variety. An algebra $\mathbf{A} \in \mathcal{V}$ is absolutely 0-unorderable if, and only if, the free extension $\mathbf{A}[x]$ of \mathbf{A} is n -subtractive for some $n \geq 2$.

Proof. The argument is similar to Selinger's proof of [13, Theorem 3.4]. Define a relation \prec on $\mathbf{A}[x]$ as follows: $t \prec u$ iff there exists a polynomial $p(x, y) \in \mathbf{A}[x, y]$ such that $\mathbf{A}[x] \models t = p(x, x)$ and $\mathbf{A}[x] \models p(x, 0) = u$.

We start by showing that tR_1u iff $t \prec^* u$.

(\Rightarrow) The relation \prec is compatible and contains the pair $(x, 0)$ since the polynomial $p(x, y) \equiv y$ witnesses $x \prec 0$. Hence by its minimality, R_1 is contained in \prec^* .

(\Leftarrow) On the other hand suppose $t \prec u$ and let $p(x, y) \in \mathbf{A}[x, y]$ be such that $\mathbf{A}[x] \models t = p(x, x)$ and $\mathbf{A}[x] \models p(x, 0) = u$. Then $t = p(x, x)R_1p(x, 0) = u$ by compatibility and the fact that xR_10 . Finally the transitivity of R_1 implies that $\prec^* \subseteq R_1$.

By Lemma 24 and the above paragraph, if \mathbf{A} is absolutely 0-unorderable, then there are $p_1, \dots, p_{n-1} \in \mathbf{A}[x, y]$ such that $p_1 \prec \dots \prec p_{n-1}$. These polynomials witness n -subtractivity.

Conversely if \mathbf{A} is n -subtractive, then $0 \prec^* x$ and hence \mathbf{A} is absolutely 0-unorderable. ◀

- **Corollary 26.** (i) *The term model of the λ -theory $\lambda\pi\phi$ is absolutely $|\Omega|$ -unorderable in the variety of combinatory algebras.*
- (ii) *If \mathcal{M} is a λ -model such that $\text{Th}(\mathcal{M}) \supseteq \lambda\pi\phi$, then \mathcal{M} is absolutely $|\Omega|$ -unorderable in the variety of combinatory algebras.*

We would like to conclude this paper by remarking that Ursini [14] has shown that subtractive algebras have a good theory of ideals. We recall that ideals in general algebras generalize normal subgroups, ideals in rings, filters in Boolean or Heyting algebras, ideals in Banach algebra, in l -groups, etc. One feature of subtractive varieties is that their ideals are exactly the congruence classes of 0, but one does not have the usual one-one correspondence ideals-congruences: mapping a congruence θ to its equivalence class $0/\theta$ only establishes a lattice homomorphism between the congruence lattice and the ideal lattice. This points to another feature: the join of two congruences is a tricky thing to deal with. The join of two ideals in a subtractive algebra behaves nicely: for I, J ideals, we have that $b \in I \vee J$ iff for some $a \in I$, $s(b, a) \in J$. Thanks to the consistency of the subtractive equations with λ -calculus, the theory of ideals for subtractive varieties can be applied to all λ -theories extending $\lambda\pi\phi$.

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