A Direct Connection Approach to Integrated Line Planning and Passenger Routing*

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Abstract

The treatment of transfers is a major challenge in line planning. Existing models either route passengers and lines sequentially, and hence disregard essential degrees of freedom, or they are of extremely large scale, and seem to be computationally intractable. We propose a novel direct connection approach that allows an integrated optimization of line and passenger routing, including accurate estimates of the number of direct travelers, for large-scale real-world instances.

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1 Introduction

Line planning is a classical optimization problem in the design of a public transportation system: Find, in an infrastructure network, a set of lines with corresponding operation frequencies, such that a given travel demand can be satisfied. There are two main objectives, namely, minimization of operation costs (the operator's point of view) and minimization of travel and transfer times (the passengers' point of view).

Since the late nineteen-nineties, the line planning literature has developed a series of integer programming approaches that try to capture these objectives better and better, see Odoni, Rousseau, and Wilson [15] and Bussieck, Winter, and Zimmermann [9] for an overview. A detailed treatment of *operation costs* is given in the articles of Claessens, van Dijk, and Zwaneveld [10], Bussieck, Lindner, and Lübbecke [8], and Goossens, van Hoesel, and Kroon [12, 13]; in this article, however, we focus on *travel and transfer times*. A first approach in this direction was taken by Bussieck, Kreuzer, and Zimmermann [7] (see also the thesis of Bussieck [6]), who proposed an integer programming model that maximizes the number of *direct travelers*, i.e., travelers that do zero transfers, on the basis of a "system split" of the demand, i.e., an a priori distribution of the passenger flow on the arcs of the transportation network. The direct travelers approach is therefore a sequential passengers-first lines-second routing method. However, the passenger flow strongly depends on the line plan which is to be computed. Hence, a number of approaches that integrate line planning and passenger routing have been developed. Schöbel and Scholl [16, 17] model travel and transfer times explicitly in terms of a "change-and-go graph" that is constructed on the basis of all potential lines. This model allows a complete and accurate formulation of travel and transfer time objectives; its only drawback is its enormous size, which seems to make this

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model computationally intractable. Nachtigall and Jerosch [14] achieve a graph reduction with a column generation approach in terms of partial passenger paths between two transfers; however, the associated integer programming formulation still requires a capacity constraint for each edge in each line. Borndörfer, Grötschel, and Pfetsch [3, 4] propose an integrated line planning and passenger routing model with a polynomial number of constraints. This model ignores transfers between lines of the same mode (transfers between, e.g., bus and tram lines are considered).

We propose in this paper a novel *direct connection* approach that encourages direct connections depending on the line plan to be computed, i.e., an integrated line planning and passenger routing approach that penalizes non-direct connections. The model can be interpreted as an advancement of Bussieck, Kreuzer, and Zimmermann's direct travelers approach that overcomes the system split. It can also be seen as a computationally tractable "first order approximation" to the change-and-go approach of Schöbel and Scholl, or as a "transfer improvement" of the model of Borndörfer, Grötschel, and Pfetsch. As far as we know, our direct connection approach is currently the only computationally tractable line planning method that provides good estimates of transfer times.

The paper is structured as follows. Sect. 2.1 starts by deriving a direct connection model in an extended variable space, that correctly accounts for all direct travelers in the same way as the change-and-go approach of Schöbel and Scholl. All other passenger paths, however, receive a uniform penalty, independent of the number of transfers. This model is reduced in Sect. 2.2 to a much smaller space of purely spatial variables via projection, in fact, via a partial projection that uses only a small, explicit subset of combinatorially meaningful inequalities. The resulting direct connection model can overestimate the number of direct travelers. Our computational results in Section 3, however, show that the model works well in practice and estimates the number of direct travelers in a surprisingly accurate way.

2 Modeling Direct Connections

We consider a public transportation network with lines of different modes, e.g., bus, tram, and subway. Passengers travel along these lines from the origins of their trips to their destinations with or without transfers. The *direct connection model* distinguishes between *direct connections*, i.e., passenger paths without transfers, and passenger paths with one or more transfers, with which a *transfer penalty* will be associated. Because of this penalty, passengers will prefer direct connections, unless routes with transfers are forced by a lack of transportation capacity. The task is to design a system of lines with associated operation frequencies such that a weighted sum of operation costs and total traveling time, including transfer penalties, is minimized. A formal description of our approach is as follows.

We consider a *multi-modal transportation network* with M *modes* in terms of an undirected graph $N = (V, E)$. The nodes consist of $M + 1$ disjoint sets $V_0 \cup V_1 \cup \ldots \cup V_M$, the edges of $M + 2$ disjoint sets $E_0 \cup \ldots \cup E_{M+1}$. The *OD-nodes* V_0 are the *origins* and *destinations* of passenger trips. Nodes V_i represent stations of lines of transport mode $i = 1, \ldots, M$. The *OD-edges* $E_0 \subseteq V_0 \times (V_1 \cup \ldots \cup V_M)$ mark beginnings and ends of trips. The *infrastructure edges* E_i denote streets and tracks on which lines of mode $i = 1, \ldots, M$ can be established. The *transfer edges* $E_{M+1} \subseteq \bigcup_{1 \leq i,j \leq M} V_i \times V_j$ are walking connections between stations of different or equal modes. Each edge $e \in E$ has a *travel time* $\tau_e \in \mathbb{Q}_{\geq 0}$, including a *transfer penalty* $\sigma \in \mathbb{Q}_{\geq 0}$ for each transfer edge $e \in E_{M+1}$, and each infrastructure edge $e \in \bigcup E_i$ has a *cost* $c_e \in \mathbb{Q}_{\geq 0}$. Figure 1 shows the infrastructure networks of the public transportation system of the city of Potsdam in Germany.

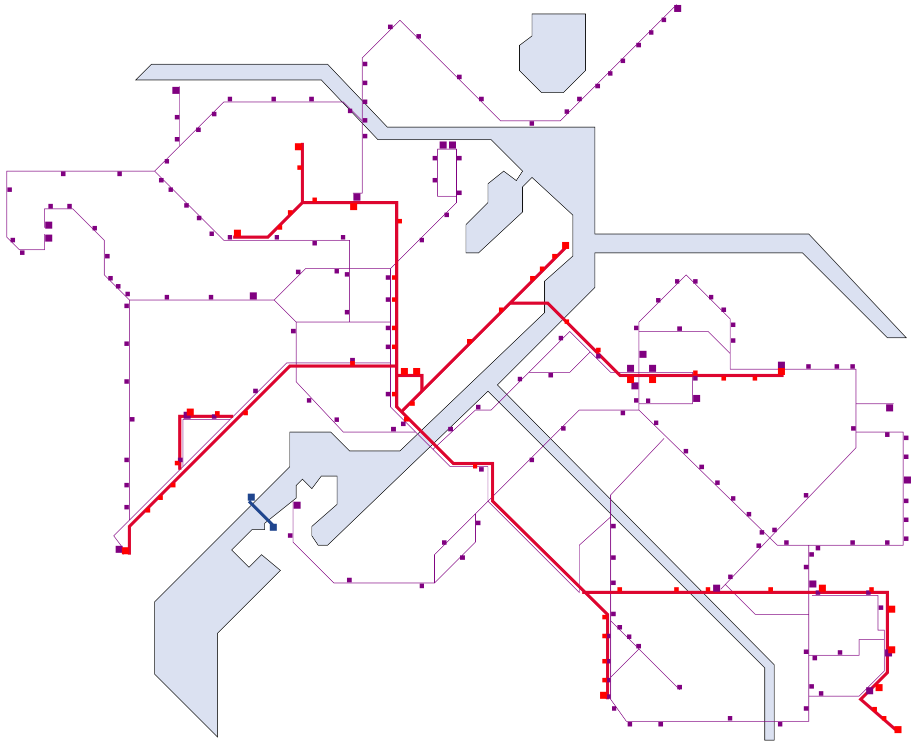

Figure 1 Multi-modal transportation network in Potsdam. Red: tram, violet: bus, blue: ferry, large nodes: terminals, small nodes: stations, light blue: rivers and lakes.

A *line* ℓ of mode i is a (not necessarily simple) path in the mode infrastructure network $N_i = (V_i, E_i)$ that starts and ends in a set of *terminal nodes* $T_i \subseteq V_i$, $i = 1, \ldots, M$. It is operated at a *frequency* f out of a finite set $\mathcal{F} \subseteq \mathbb{N}$. Line ℓ at frequency f has *transportation capacity* $\kappa_{\ell,f} = \kappa_i \cdot f_\ell$, where κ_i is a *standard capacity* of a line of mode i, e.g., the size of a standard bus, and *operation cost* $c_{\ell,f} = c_i + f_\ell \cdot \sum_{e \in \ell} c_e$, where c_i is a *standard fixed cost* of a line of mode i. Working with standard capacities and costs is a simplification. Note, however, that a more detailed treatment of different capacities, e.g., depending on bus types or numbers of vehicles, can be handled by introducing additional modes. We denote by $\mathcal{L}$ the set of all lines which we assume to be given in this paper.

The *travel demand* is given by an *OD-matrix* $d \in \mathbb{Q}_{\geq 0}^{V_0 \times V_0}$, i.e., d_{st} is the number of passengers that want to travel from origin $s \in V_0$ to destination $t \in V_0$; note that d does not have to be symmetric. We denote by $D = \{(s,t) \in V_0^2 \,|\, d_{st} > 0\}$ the set of all *OD-pairs* with positive travel demand. Passengers travel along routes in a directed *passenger routing graph* $G = (V, A)$ that arises from the transportation network $N = (V, E)$ by replacing each edge $e \in E$ by two antiparallel arcs $a(e)$ and $\bar{a}(e)$; let conversely $e(a)$ be the undirected edge corresponding to such an arc $a \in A$. Travel times and lengths of the undirected edges carry over to their directed counterparts. Denote by $\mathcal{P}_{st}$ the set of all (simple) directed *st-passenger paths* from origin s to destination t in G and by $\mathcal{P} = \bigcup_{(s,t) \in D} \mathcal{P}_{st}$ the set of all *passenger paths*.

A *direct connection st-passenger path for line* ℓ or an *st-dcpath* is an *st*-passenger path p of the form $p = (s, a_0, v_1, \ldots, v_r, a_r, t)$ where $s, t, v_i \in V$, $a_0, a_i \in A$, $e(a_i) \in \ell$, $i = 1, \ldots, r$, $r \in \mathbb{N}_0$, i.e., passengers can travel along p from origin s directly to destination t via line

ℓ without transfers. Let $\mathcal{P}_{st}^{0,\ell}$ be the set of st-dcpaths for line ℓ, $\mathcal{P}^{0,\ell} = \bigcup_{(s,t)\in D} \mathcal{P}_{st}^{0,\ell}$, and $\mathcal{P}^{0,\mathcal{L}} = \bigcup_{\ell\in\mathcal{L}} \mathcal{P}^{0,\ell}$; note that $|\mathcal{P}_{st}^{0,\ell}| = 1$ if ℓ is simple and $|\delta^+(s)| = |\delta^-(t)| = 1$. Let further $\mathcal{P}_{st}^{0,\ell}(a) = \{p \in \mathcal{P}_{st}^{0,\ell} : a \in p\}$ be the set of st-dcpaths for line ℓ that pass over arc a, and let $\mathcal{L}_{st}^{0}(a) = \{\ell \in \mathcal{L} : \mathcal{P}_{st}^{0,\ell}(a) \neq \emptyset\}$ be the set of all lines that support an st-dcpath via arc a. A path p is a *direct connection st-passenger path* (st-dcpath), if it is an st-dcpath for some line ℓ. Let $\mathcal{P}_{st}^{0}$ be the set of st-dcpaths, and $\mathcal{P}^{0} = \bigcup_{(s,t)\in D} \mathcal{P}_{st}^{0}$ their union. For a dcpath $p \in \mathcal{P}^0$, we set the travel time to the sum of the arc travel times $\tau_{p,0} = \sum_{a\in p} \tau_a$. For an st-passenger path $p \in \mathcal{P}$, we set the travel time to the sum of the arc travel times plus a summand $\sigma(p)$ to arrive at a travel time of $\tau_{p,1} = \sigma(p) + \sum_{a\in p} \tau_a$, where $\sigma(p) = \sigma$ if p does not contain a transfer arc, and 0 otherwise, since we already incorporated a penalty on transfer arcs.

2.1 Direct Line Connection Model

We will first introduce a model that computes a line plan and a passenger routing minimizing a weighted sum of line operation costs and passenger traveling times. This model accounts exactly for the number of travelers on direct connections according to the model assumptions. Introducing path flow variables $z_{p,0}^{\ell}$ and $y_{p,1}$ for the number of passengers that travel on dcpath p on line ℓ and on path p with at least one transfer, respectively, and $x_{\ell,f} \in \{0,1\}$ for the operation of line ℓ at frequency f, we state a *direct line connection model* for integrated line planning and passenger routing as follows:

$$\text{(DLC)} \quad \min \lambda \sum_{\ell\in\mathcal{L}} \sum_{f\in\mathcal{F}} c_{\ell,f}\, x_{\ell,f} + (1-\lambda) \left(\sum_{p\in\mathcal{P}^0} \sum_{\ell\in\mathcal{L}: p\in\mathcal{P}^{0,\ell}} \tau_{p,0}\, z_{p,0}^{\ell} + \sum_{p\in\mathcal{P}} \tau_{p,1}\, y_{p,1} \right)$$

$$\sum_{\ell\in\mathcal{L}} \sum_{p\in\mathcal{P}_{st}^{0,\ell}} z_{p,0}^{\ell} + \sum_{p\in\mathcal{P}_{st}} y_{p,1} = d_{st} \qquad \forall\,(s,t)\in D \tag{1}$$

$$\sum_{\ell\in\mathcal{L}} \sum_{p\in\mathcal{P}^{0,\ell}: a\in p} z_{p,0}^{\ell} + \sum_{p\in\mathcal{P}: a\in p} y_{p,1} \leq \sum_{\ell\in\mathcal{L}: e(a)\in\ell} \sum_{f\in\mathcal{F}} \kappa_{\ell,f}\, x_{\ell,f} \qquad \forall\, a\in A \tag{2}$$

$$\sum_{p\in\mathcal{P}^{0,\ell}: a\in p} z_{p,0}^{\ell} \leq \sum_{f\in\mathcal{F}} \kappa_{\ell,f}\, x_{\ell,f} \qquad \forall\,\ell\in\mathcal{L},\ e(a)\in\ell \tag{3}$$

$$\sum_{f\in\mathcal{F}} x_{\ell,f} \leq 1 \qquad \forall\,\ell\in\mathcal{L} \tag{4}$$

$$x_{\ell,f} \in \{0,1\} \qquad \forall\,\ell\in\mathcal{L},\ f\in\mathcal{F} \tag{5}$$

$$z_{p,0}^{\ell} \geq 0 \qquad \forall\,\ell\in\mathcal{L},\ p\in\mathcal{P}^{0,\ell} \tag{6}$$

$$y_{p,1} \geq 0 \qquad \forall\, p\in\mathcal{P}. \tag{7}$$

The model (DLC) minimizes a weighted sum of line operation costs and passenger travel times; $0 \leq \lambda \leq 1$ is a *weight parameter*. Note that the st-passenger path variables $y_{p,1}$ incur a penalty for each transfer arc and exactly one transfer penalty otherwise, i.e., the number of transfers may be underestimated. Equations (1) enforce the passenger flow. Inequalities (2) guarantee sufficient total transportation capacity on each arc. Constraints (3) ensure sufficient transportation capacity for direct connection passenger paths on each arc of each line. Inequalities (4) ensure that a line is operated at one frequency at most.

Model (DLC) includes a variable $z_{p,0}^{\ell}$ for the assignment of each direct connection passenger path p to a direct connection line ℓ. A line of length k is usually a direct connection line for $O(k^2)$ OD-pairs, such that the number of variables is much larger than the number of lines; moreover, choices between several possible direct connection lines for each dcpath produce lots of degeneracy. To overcome these problems, we will now compress the model by relaxing the explicit assignment of dcpaths to direct connection lines. We describe in the following Subsection 2.2 an approximative construction, that we will use for computation, and argue in Subsection 2.3 how it can be made exact.

2.2 Direct Connection Model

To construct a compact approximation of (DLC), we eliminate the assignment of passenger paths to particular lines by aggregating the dcpath variables as $y_{p,0} = \sum_{\ell \in \mathcal{L}} z^{\ell}_{p,0}$. To this purpose, consider for each OD-pair $(s,t) \in D$ the set $\mathcal{P}^0_{st}$ of all st-dcpaths and unite them to construct what we call a *direct connection st-passenger routing graph* $G^0_{st} = (V^0_{st}, A^0_{st}) = \bigcup_{p \in \mathcal{P}^0_{st}} (V(p), A(p))$, where $V(p)$ and $A(p)$ denote the nodes and arcs of dcpath p, respectively. Note that G^0_{st} can be constructed in polynomial time. We proceed by considering *all* st-paths in G^0_{st} as *relaxed st-dcpaths* (st-rdcpaths); let $\mathcal{P}^{0+}_{st}$ be the set of all such rdcpaths-paths, $\mathcal{P}^{0+}_{st}(a) = \{p \in \mathcal{P}^{0+}_{st} : a \in p\}$ the set of all st-rdcpaths via arc a, and $\mathcal{P}^{0+} = \bigcup_{(s,t)\in D} \mathcal{P}^{0+}_{st}$. Obviously, $\mathcal{P}^{0+}_{st} \supseteq \mathcal{P}^0_{st}$, i.e., $\mathcal{P}^{0+}_{st}$ overestimates the number of direct connections between origin s and destination t. We say that OD-pairs (s,t) and (u,v) are *dc-equivalent with respect to arc a*, if $\mathcal{L}^0_{uv}(a) = \mathcal{L}^0_{st}(a)$, i.e., if the st- and the uv-rdcpaths are supported by the same set of lines. We further say that OD-pair (u,v) is *dc-dominated with respect to arc a* by OD-pair (s,t) if $\mathcal{L}^0_{uv}(a) \subseteq \mathcal{L}^0_{st}(a)$. Denote by $[s,t]_a$ and $[s,t]^{\leq}_a$ the corresponding equivalence class and domination set, respectively, i.e., $(u,v) \in [s,t]_a$ if $\mathcal{L}^0_{uv}(a) = \mathcal{L}^0_{st}(a)$ and $(u,v) \in [s,t]^{\leq}_a$ if $\mathcal{L}^0_{uv}(a) \subseteq \mathcal{L}^0_{st}(a)$. Let finally $D(a) = \{[s,t]_a\}$ be the set of equivalence classes for dc-equivalent OD-pairs w.r.t. a. Introducing line-independent rdcpath variables $y_{p,0}$ for the number of direct travelers on path p, this flow must satisfy the following *direct connection constraints* for each arc a and each class $[s,t]_a$ of equivalent OD-pairs:

$$\sum_{(u,v)\in[s,t]^{\leq}_a} \sum_{p \in \mathcal{P}^{0+}_{uv}(a)} y_{p,0} \leq \sum_{\ell \in \mathcal{L}^0_{st}(a)} \sum_{f \in \mathcal{F}} \kappa_{\ell,f}\, x_{\ell,f} \qquad \forall\, a \in A, [s,t]_a \in D(a). \tag{8}$$

These constraints enforce sufficient transportation capacity to route all uv-rdcpaths, $(u,v) \in [s,t]^{\leq}_a$, via arc a. Using variables $y_{p,0}$ instead of $z^{\ell}_{p,0}$, and substituting constraints (3) by the direct connection constraints (8), we obtain the following *direct connection model*:

$$\text{(DC)} \quad \min\ \lambda \sum_{\ell \in \mathcal{L}} \sum_{f \in \mathcal{F}} c_{\ell,f}\, x_{\ell,f} + (1-\lambda) \left(\sum_{p \in \mathcal{P}^{0+}} \tau_{p,0}\, y_{p,0} + \sum_{p \in \mathcal{P}} \tau_{p,1}\, y_{p,1} \right)$$

$$\sum_{p \in \mathcal{P}^{0+}_{st}} y_{p,0} + \sum_{p \in \mathcal{P}_{st}} y_{p,1} = d_{st} \qquad \forall\, (s,t) \in D \tag{9}$$

$$\sum_{p \in \mathcal{P}^{0+}: a \in p} y_{p,0} + \sum_{p \in \mathcal{P}: a \in p} y_{p,1} \leq \sum_{\ell \in \mathcal{L}: e(a) \in \ell} \sum_{f \in \mathcal{F}} \kappa_{\ell,f}\, x_{\ell,f} \qquad \forall\, a \in A \tag{10}$$

$$\sum_{(u,v)\in[s,t]^{\leq}_a} \sum_{p \in \mathcal{P}^{0+}_{uv}(a)} y_{p,0} \leq \sum_{\ell \in \mathcal{L}^0_{st}(a)} \sum_{f \in \mathcal{F}} \kappa_{\ell,f}\, x_{\ell,f} \qquad \forall\, a \in A, [s,t]_a \in D(a) \tag{8}$$

$$\sum_{f \in \mathcal{F}} x_{\ell,f} \leq 1 \qquad \forall\, \ell \in \mathcal{L} \tag{11}$$

$$x_{\ell,f} \in \{0,1\} \qquad \forall\, \ell \in \mathcal{L},\ f \in \mathcal{F} \tag{12}$$

$$y_{p,0} \geq 0 \qquad \forall\, p \in \mathcal{P}^{0+} \tag{13}$$

$$y_{p,1} \geq 0 \qquad \forall\, p \in \mathcal{P}. \tag{14}$$

2.3 Model Discussion

To relate the models (DLC) and (DC), we show now that (DC) is a relaxation of the projection of model (DLC) onto the space of the dcpath variables. This can be seen as follows. For each st-dcpath $p \in \mathcal{P}^0_{st}$, link the flow variables $y_{p,0}$ and $z^\ell_{p,0}$ via equations

$$y_{p,0} = \sum_{\ell \in \mathcal{L} : p \in \mathcal{P}^{0,\ell}_{st}} z^\ell_{p,0}. \tag{15}$$

Consider the polytopes

$$\begin{aligned}
P &:= \{(x, y_1, z) \in \mathbb{R}^{(\mathcal{L}\times\mathcal{F})\times\mathcal{P}\times\mathcal{P}^{0,\mathcal{L}}}_{\geq 0} \mid \text{(DLC)}(1)-(4), (6)-(7)\} \\
PQ &:= \{(x, y_0, y_1, z) \in \mathbb{R}^{(\mathcal{L}\times\mathcal{F})\times\mathcal{P}^0\times\mathcal{P}\times\mathcal{P}^{0,\mathcal{L}}}_{\geq 0} \mid (15), \text{(DLC)}(1)-(4), (6)-(7)\} \\
Q &:= \{(x, y_0, y_1) \in \mathbb{R}^{(\mathcal{L}\times\mathcal{F})\times\mathcal{P}^0\times\mathcal{P}}_{\geq 0} \mid \exists z \in \mathbb{R}^{\mathcal{P}^{0,\mathcal{L}}}_{\geq 0} \text{ s.t. } (x, y_0, y_1, z) \in PQ\}.
\end{aligned}$$

P is the solution set of the LP relaxation of (DLC). PQ extends this set into a higher-dimensional space by adding the aggregate flow variables $(y_{p,0})$; hence, P is the projection of PQ onto the space of $(x_{\ell,f}, y_{p,1}, z^\ell_{p,0})$ variables. Q is the projection of PQ onto the space of $(x_{\ell,f}, y_{p,1}, y_{p,0})$ variables, i.e., Q describes exactly the feasible combinations of line plans and aggregate direct connection passenger flows.

Let $Q = \{Ax + By \leq b\}$; then adding constraints $Ax + By \leq b$ to model (DC) and using dcpaths instead of rdcpaths produces a strengthening of the direct connection model (DC) that is equivalent to the direct line connection model (DLC), i.e., that handles all direct connections correctly. Note that the cuts in the system $Ax + By \leq b$ can be separated using Benders decomposition, i.e., this construction is algorithmic. Model (DC) is a relaxation that considers a larger set of paths $\mathcal{P}^{0+}_{st} \supseteq \mathcal{P}^0_{st}$ and replaces the Benders cut system $Ax + By \leq b$ by the smaller, explicit, and purely combinatorial set of direct connection constraints (8). This makes model (DC) algorithmically tractable. One can show that the pricing problem for passenger path variables is a shortest path problem in G^0_{st} for direct connection passenger paths, and a constrained shortest path problem in G for paths with at least one transfer.

Indeed, consider the solution of the LP relaxation of model (DC) by column generation, i.e., consider the pricing problems for the variables. Associate dual variables π (unbounded), $\mu \geq 0$, $\nu \geq 0$, and $\psi \geq 0$ with constraints (9), (10), (8), and (11) of program (DC). The dual of the LP relaxation of (DC) is

$$\begin{aligned}
\max \sum_{(s,t)\in D} d_{st}\pi_{st} - \sum_{\ell\in\mathcal{L}} \psi_\ell & & \\
\pi_{st} - \sum_{a\in p} \mu_a - \sum_{a\in p} \nu_{a,[s,t]_a} &\leq (1-\lambda)\tau_{p,0} && \forall\, p \in \mathcal{P}^{0+}_{st},\ (s,t) \in D, \\
\pi_{st} - \sum_{a\in p} \mu_a &\leq (1-\lambda)\tau_{p,1} && \forall\, p \in \mathcal{P}_{st},\ (s,t) \in D, \\
\sum_{a:e(a)\in\ell} \kappa_{\ell,f}\mu_a + \sum_{a:e(a)\in\ell} \sum_{[s,t]_a\in D(a)} \kappa_{\ell,f}\nu_{a,[s,t]_a} - \psi_\ell &\leq \lambda c_{\ell,f} && \forall\, \ell \in \mathcal{L},\ f \in \mathcal{F} \\
\mu_a &\geq 0 && \forall\, a \in A \\
\nu_{a,[s,t]_a} &\geq 0 && \forall\, a \in A,\ [s,t]_a \in D(a) \\
\psi_\ell &\geq 0 && \forall\, \ell \in \mathcal{L}.
\end{aligned}$$

The pricing problem for the passenger variables is twofold: Find an st-rdcpath with negative reduced cost or find a path from s to t with at least one transfer and negative reduced cost.

The reduced cost can be computed as follows

$$\bar{\tau}_{p,0} = -\pi_{st} + \sum_{a\in p} \left(\mu_a + \nu_{a,[s,t]_a} + (1-\lambda)\tau_a\right) \tag{16}$$

$$\bar{\tau}_{p,1} = -\pi_{st} + \sum_{a\in p} (\mu_a + (1-\lambda)\tau_a) + (1-\lambda)\sigma(p). \tag{17}$$

In the first case we have to find an (s,t)-rdcpath in G_{st} with weight smaller than π_{st}. The arc weights are set to $\omega_a = \mu_a + \nu_{a,[s,t]_a} + (1-\lambda)\tau_a \geq 0$ for $a \in A_{st}$. This problem can be solved by Dijkstra's algorithm.

In the second case we have to find an st-path in G with weight smaller than π_{st}. The arc weights are set to $\omega_a = \mu_a + (1-\lambda)\tau_a \geq 0$ for $a \in A$. However, the reduced cost depends on whether the path p contains a transfer arc or not; if not we have to add $(1-\lambda)\sigma$ to the weight of the path. This problem can be solved by a constrained shortest path algorithm.

Model (DC) can be seen as a "first order approximation" to the change-and-go approach of Schöbel and Scholl, because (DC) does not consider transfer penalties for the second, third, etc. transfer in a passenger path that can not be attributed to a transfer arc. It further relaxes the assignment of direct connection paths to particular lines. Model (DC) can also be seen a "transfer improvement" of the model of Borndörfer, Grötschel, and Pfetsch [3]. Namely, dropping the direct connection constraints results in a variant of the model of Borndörfer, Grötschel, and Pfetsch, in which each passenger path is handled as a direct connection path unless it contains a transfer arc; we will denote this model by (B). The only difference between (B) and the original model of Borndörfer, Grötschel, and Pfetsch is that line frequencies are handled explicitly in terms of a finite set of possible integral frequencies instead of allowing a continuum of values.

3 Computational Results

In this section, we will show that the direct connection model can be used to solve large-scale line planning problems and that the direct connection constraints strongly improve the number of direct travelers in comparison to models that ignore transfers, in particular, model (B), see Section 2.3.

We consider four transportation networks that we denote as China, Dutch, SiouxFalls, and Potsdam. The instance SiouxFalls uses the graph of the street network with the same name from the Transportation Network Test Problems Library of Bar-Gera [20]. Instances China, Dutch, and Potsdam correspond to public transportation networks. The Dutch network was introduced by Bussieck in the context of line planning [11]. The China instance is artificial; we constructed it as a showcase example, connecting the twenty biggest cities in China by the 2009 high speed train network. The Potsdam instances are real multi-modal public transportation networks for 1998 and 2009.

For China, Dutch, and SiouxFalls all nodes are considered as terminals, i.e., nodes where lines can start or end. We constructed a line pool by generating for each pair of terminals all lines that satisfy a certain length restriction. To be more precise, the number of edges of a line between two terminals s and t must be less than or equal to k times the number of edges of the shortest path between s and t. For each network, we increased k in three steps to produce three instances with different line pool sizes. For Dutch and China instance number 3 contains all lines, i.e., all paths that are possible in the network. The line pools for the Potsdam network of 1998 are generated for different restrictions on the length of the lines considering the given turning restrictions on crossings. We defined all nodes as terminals that

Table 1 Statistics on the line planning instances. The columns list the instances, the number of non-zero OD pairs, number of OD nodes, number of nodes and edges of the preprocessed passenger routing graph, the number of considered lines, the number of direct connection constraints, and the number of all other constraints.

problem	$\lvert D\rvert$	$\lvert V_O\rvert$	$\lvert V\rvert$	$\lvert A\rvert$	$\lvert\mathcal{L}\rvert$	vars	dc-cons	cons
Dutch1	420	23	23	106	402	1 608	1 832	1 080
Dutch2	420	23	23	106	2 679	10 716	7 544	3 341
Dutch3	420	23	23	106	7 302	29 208	9 736	7 945
China1	379	20	20	98	474	1 896	2 754	1 178
China2	379	20	20	98	4 871	19 484	8 162	5 457
China3	379	20	20	98	19 355	77 420	12 443	19 931
SiouxFalls1	528	24	24	124	866	3 464	4 400	1 779
SiouxFalls2	528	24	24	124	9 397	37 588	16 844	10 197
SiouxFalls3	528	24	24	124	15 365	61 460	21 220	16 145
Potsdam98a	7 734	107	344	2 746	207	776	3 538	9 970
Potsdam98b	7 734	107	344	2 746	1 907	7 572	60 902	11 991
Potsdam98c	7 734	107	344	2 746	4 342	17 313	76 640	14 366
Potsdam2009	4 443	236	851	5 542	3 433	14 140	30 780	12 006

are terminals of operating lines in the year 1998. The Potsdam 2009 instance arose within a project with the Verkehr in Potsdam GmbH (ViP) [19] to optimize the 2010 line plan [2, 5]. The line pool contains all possible lines that fulfill the ViP requirements. The line pools of the Potsdam instances contain also lines for regional and commuter trains. These lines are not operated by ViP and we therefore fix them to given frequencies in our computations.

The other lines can be operated at frequencies 3, 6, 9, and 18; this corresponds to a cycle time of 60, 30, 20, and 10 minutes in a time horizon of 3 hours. Line costs are proportional to line lengths plus a fixed cost term that is used to reduce the number of lines. The objective weighing parameter was set to $\lambda = 0.8$ and the transfer penalty was set to $\sigma = 15$ minutes.

Table 1 gives some statistics about the test instances. The columns labeled $|D|$, $|V_O|$, $|V|$, $|A|$, and $|\mathcal{L}|$ list the number of OD pairs with non-zero demand, OD nodes, nodes, arcs, and lines after some preprocessing. The last three columns give the number of variables and constraints associated with the integer program (DC). Here, “dc-cons” gives the number of direct connection constraints while “cons” gives the number of all other constraints.

The instances were solved with a column generation algorithm implemented on the basis of the CIP framework SCIP, version 2.1.0, see [1, 18], using CPLEX 12.4 as LP-solver (in single core mode). Line/frequency variables were enumerated, passenger path variables were priced with Dijkstra’s shortest path algorithm and a labeling algorithm for the constrained shortest path case. We mainly used the default settings of SCIP. We further implemented three special rounding heuristics and preprocessing cuts that account for the demand on single arcs that disconnect at least two OD-nodes as well as the out-going and in-coming demand of an OD-node. Namely, we scale the capacity constraints associated with these demand sets by $\kappa^i f$, for each $f \in \mathcal{F}$, and apply mixed integer rounding. We also added violated cuts of the form

$$\sum_{p \in \mathcal{P}^{0+}_{st}(a)} \frac{y_{p,0}}{d_{st}} \leq \sum_{\ell \in \mathcal{L}^0_{st}(a)} \sum_{f \in \mathcal{F}} x_{\ell,f} \qquad (s,t) \in D,\ a \in A_{st} \tag{18}$$

in each branching node. These cuts can be derived from the direct connection constraints (8). The preprocessing constraints and the cuts (18) improve the root LP value by around 0.1% to

Table 2 Statistics on the computations for model (DC) and (B). The columns list the instances, computation time, number of branching nodes, and the integrality gap.

	(DC)			(B)		
problem	time	nodes	gap	time	nodes	gap
Dutch1	15s	329	opt.	10h	5 940 327	0.03%
Dutch2	<1h	11 532	opt.	10h	815 966	0.04%
Dutch3	10h	57 273	0.05%	10h	151 053	0.08%
China1	10h	814 964	0.32%	10h	3 754 582	0.11%
China2	10h	5 366	0.53%	10h	129 217	0.15%
China3	10h	997	0.47%	10h	37 519	0.18%
SiouxFalls1	10h	458 379	0.10%	<1h	347 999	opt.
SiouxFalls2	10h	13 868	0.09%	10h	110 836	0.01%
SiouxFalls3	10h	3 230	0.10%	10h	44 713	0.00%
Potsdam98a	10h	7 357	0.09%	10h	6 266	0.12%
Potsdam98b	10h	62	0.28%	10h	2 491	0.26%
Potsdam98c	10h	10	0.27%	10h	661	0.25%
Potsdam2010	10h	2	0.81%	10h	2123	0.41%

0.5% for the Dutch and Potsdam instances (which is much). The improvement for the China and SiouxFalls instances is in the order of per mill. Finally, we included additional auxiliary *branching variables* $h_{a,i} \in \mathbb{Z}_{\geq 0}$, $a \in A$, $i \in \mathcal{F}$, that account for the number of lines on arc a with frequency greater than or equal to i, and the corresponding *branching constraints* $\sum_{\ell \in \mathcal{L}: e(a) \in \ell} \sum_{f \in \mathcal{F}: f \geq i} x_{\ell,f} = h_{a,i} \,\forall\, a \in A,\, i \in \mathcal{F}$.

Including these branching variables and constraints combines the possibility to branch on those constraints with the sophisticated branching rules implemented in the SCIP framework. This works well, e.g., it needs nearly half a million branching nodes to solve instance N1 without the branching variables in comparison to less than 500 nodes with the branching variables. Instance N2 could not be solved within 10 hours without branching variables.

We also included the branching variables in the computations for model (B) as well as the preprocessing cuts, and constraints similar to (18) that can be derived from the capacity constraints for each arc. We set a time limit of 10 hours for all instances. All computations were done on computers with an Intel(R) Xeon(R) CPU X5672 with 3.20 GHz, 12 MB cache, and 48 GB of RAM.

Table 2 shows statistics on the number of branching nodes, computation time, and the integrality gap for model (DC) and model (B). Albeit model (DC) seems to be harder to solve (the number of solved branching nodes is usually smaller for (DC) than for (B)), the integrality gaps are similar for (DC) and (B). The Dutch instances 1 and 2 can even be solved to optimality for model (DC); for those instances the direct connection constraints improve the optimization process.

We evaluate the quality of the solutions of model (DC) and (B) by computing an optimal passenger routing, including penalties for all transfers, in a change-and-go graph similar to that of Schöbel and Scholl [16]. Namely, we construct nodes and arcs for each line individually, i.e., the change-and-go graph contains a copy of each node and arc for every line that contains this node and arc. Further transfer arcs are added between two nodes of different lines whenever a transfer between these two lines is possible on this node. The travel time of all arcs is set to the travel time of the associated arc in G, transfer arcs are additionally penalized by σ. We then fix the frequencies of the lines according to the computed line plan and route the passenger to minimize the total travel and transfer times, i.e., we compute the

Table 3 Evaluation of the solutions of (DC) and (B) of Table 2 in the change-and-go graph. The columns list travel time (in minutes), cost, objective value, number of direct travelers predicted in the considered model, and number of direct travelers in the change-and-go graph.

problem	travel time	cost	obj.	dir. trav. of model	exact dir. trav.
Dutch1 (DC)	$1.279 \cdot 10^7$	68 900	2613305	179 496	179 496
Dutch1 (B)	$1.333 \cdot 10^7$	57 800	2711770	183 582	148 924
Dutch2 (DC)	$1.279 \cdot 10^7$	66 900	2612122	180 484	179 384
Dutch2 (B)	$1.319 \cdot 10^7$	57 500	2683071	183 582	156 251
Dutch3 (DC)	$1.279 \cdot 10^7$	66 900	2612122	180 484	179 384
Dutch3 (B)	$1.319 \cdot 10^7$	57 500	2683071	183 582	156 251
China1 (DC)	$1.259 \cdot 10^7$	267 937	2732445	749 736	716 040
China1 (B)	$1.559 \cdot 10^7$	233 268	3304432	759 950	509 526
China2 (DC)	$1.258 \cdot 10^7$	247 241	2714438	759 936	709 145
China2 (B)	$1.559 \cdot 10^7$	233 268	3304432	759 950	509 526
China3 (DC)	$1.245 \cdot 10^7$	244 361	2684860	759 950	714 728
China3 (B)	$1.559 \cdot 10^7$	233 268	3304432	759 950	509 526
SiouxFalls1 (DC)	$3.267 \cdot 10^6$	9 205	660675	360 600	358 888
SiouxFalls1 (B)	$3.633 \cdot 10^6$	8 295	733288	360 600	335 355
SiouxFalls2 (DC)	$3.392 \cdot 10^6$	5 787	682996	360 600	360 178
SiouxFalls2 (B)	$3.776 \cdot 10^6$	5 178	759365	360 600	326 625
SiouxFalls3 (DC)	$3.431 \cdot 10^6$	4 899	690200	360 600	355 068
SiouxFalls3 (B)	$3.695 \cdot 10^6$	4 283	742397	360 600	334 052
Potsdam98a (DC)	$5.076 \cdot 10^6$	27 044	1036865	70 513	71 075
Potsdam98a (B)	$5.102 \cdot 10^6$	29 018	1043617	83 702	68 900
Potsdam98b (DC)	$4.836 \cdot 10^6$	33 484	993938	78 745	79 511
Potsdam98b (B)	$4.970 \cdot 10^6$	28 302	1016610	84 879	73 983
Potsdam98c (DC)	$4.829 \cdot 10^6$	32 544	991772	79 694	79 576
Potsdam98c (B)	$4.952 \cdot 10^6$	29 320	1013779	84 979	74 356
Potsdam2010 (DC)	$1.032 \cdot 10^6$	9 314	213769	38 152	38 001
Potsdam2010 (B)	$1.073 \cdot 10^6$	8 734	221549	41 052	35 285

correct number of transfers for all passengers in a system optimum routing. Table 3 shows the result of this evaluation for the best solutions computed with model (DC) and model (B), respectively,

It can be seen that the exact number of direct travelers is very close to the number of direct travelers predicted by model (DC), which is exactly what we wanted to achieve. The only bigger differences (of around 7%) are in the China instances. However, the China instances also display the largest prediction improvement in comparison to model (B), namely, around 40%. Over all instances, model (DC) significantly improves the number of direct travelers in comparison to (B); the improvement is around 7% for the Potsdam and SiouxFalls instances and around 15% to 20% for the Dutch instances.

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