Overlap of Convex Polytopes under Rigid Motion*

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Abstract

We present an algorithm to compute an approximate overlap of two convex polytopes P_1 and P_2 in $\mathbb{R}^3$ under rigid motion. Given any $\varepsilon \in (0, 1/2]$, our algorithm runs in $O(\varepsilon^{-3} n \log^{3.5} n)$ time with probability $1 - n^{-O(1)}$ and returns a $(1-\varepsilon)$-approximate maximum overlap, provided that the maximum overlap is at least $\lambda \cdot \max\{|P_1|, |P_2|\}$ for some given constant $\lambda \in (0, 1]$.

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1 Introduction

Shape matching is a common task in many object recognition applications. The particular problem of matching convex shapes has been used in tracking regions in an image sequence [8] and measuring symmetry of a convex body [6]. A translation or rigid motion of one shape is sought to maximize some similarity measure with another shape. The *overlap* of the two convex shapes—the volume of their intersection—is a robust similarity measure [12]. In this paper, we consider the problem of approximating the maximum overlap of two convex polytopes in $\mathbb{R}^3$ under rigid motion.

Efficient algorithms have been developed for two convex polygons of n vertices in the plane. De Berg et al. [5] developed an algorithm to find the maximum overlap of two convex polygons under translation in $O(n \log n)$ time. Ahn et al. [3] presented two algorithms to find a $(1-\varepsilon)$-approximate maximum overlap, one for the translation case and another for the rigid motion case. They assume that the polygon vertices are stored in arrays in clockwise order around the polygon boundaries. Ahn et al.'s algorithms run in $O(\varepsilon^{-1} \log n + \varepsilon^{-1} \log(1/\varepsilon))$ time for the translation case and $O(\varepsilon^{-1} \log n + \varepsilon^{-2} \log(1/\varepsilon))$ time for the rigid motion case. Finding the exact maximum overlap under rigid motion seems difficult. A brute force approach is to subdivide the space of rigid motion $[-\pi, \pi] \times \mathbb{R}^2$ into cells so that the intersecting pairs of polygon edges do not change within a cell. The hope is to obtain a formula for maximum overlap within a cell as the intersection does not change combinatorially. Unfortunately, the subdivision of $[-\pi, \pi] \times \mathbb{R}^2$ has curved edges and facets, which makes it a challenge to obtain formulae for maximum overlap in the cells.

Fewer algorithmic results are known concerning the maximum overlap of two convex polytopes in $\mathbb{R}^d$ for $d \geqslant 3$. Let n be the number of hyperplanes defining the convex polytopes.

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Ahn et al. [4] developed an algorithm to find the maximum overlap of two convex polytopes under translation in $O(n^{(d^2+d-3)/2}\log^{d+1} n)$ expected time. Recently, Ahn, Cheng and Reinbacher [2] have obtained substantially faster algorithms to align two convex polytopes under translation in $\mathbb{R}^3$ and $\mathbb{R}^d$ for $d \geqslant 4$. In both cases, the overlap computed is no less than the optimum minus ε, where ε is an arbitrarily small constant fixed in advanced. The running times are $O(n\log^{3.5} n)$ for $\mathbb{R}^3$ and $O(n^{\lfloor d/2\rfloor+1}\log^d n)$ for $d \geqslant 4$, and these time bounds hold with probability $1-n^{-O(1)}$. There is no specific prior result concerning the maximum overlap of convex polytopes under rigid motion. Vigneron [13] studied the optimization of algebraic functions and one of the applications is the alignment of two possibly non-convex polytopes under rigid motion. For any $\varepsilon \in (0,1)$ and for any two convex polytopes with n defining hyperplanes, Vigneron's method can return in $O\big(\varepsilon^{-\Theta(d^2)}n^{\Theta(d^3)}(\log\frac{n}{\varepsilon})^{\Theta(d^2)}\big)$ time an overlap under rigid motion that is at least $1-\varepsilon$ times the optimum. Finding the exact overlap is even more challenging in $\mathbb{R}^3$.

In this paper, we present a new algorithm to approximate the maximum overlap of two convex polytopes P_1 and P_2 in $\mathbb{R}^3$ under rigid motion. For the purpose of shape matching, it often suffices to know that two input shapes are very dissimilar if this is the case. Therefore, we are only interested in matching P_1 and P_2 when their maximum overlap under rigid motion is at least $\lambda\cdot\max\{|P_1|, |P_2|\}$ for some given constant $\lambda \in (0,1]$, where $|P_i|$ denotes the volume of P_i. Under this assumption, given any $\varepsilon \in (0,1/2]$, our algorithm runs in $O(\varepsilon^{-3}n\log^{3.5} n)$ time with probability $1-n^{-O(1)}$ and returns a rigid motion that achieves a $(1-\varepsilon)$-approximate maximum overlap. The assumption can be verified as follows. Run our algorithm using $\lambda/2$ instead of λ. Check if the overlap output by our algorithm is at least $(1-\varepsilon)\lambda\cdot\max\{|P_1|, |P_2|\}$. If not, we know that the assumption is not satisfied. If yes, the maximum overlap is at least $(\lambda/2)\cdot\max\{|P_1|, |P_2|\}$ and our algorithm's output is a $(1-\varepsilon)$-approximation because we used $\lambda/2$ in running the algorithm.

Our high-level strategy has two steps. First, sample a set of rotations. Second, for each sampled rotation, apply it and then apply the almost optimal translation computed by Ahn et al.'s algorithm [2]. Finally, return the best answer among all rigid motions tried. If one uses a very fine uniform discretization of the rotation space, it is conceptually not difficult to sample rotations so that the resulting approximation is good. The problem is that such a discretization inevitably leads to a running time that depends on some geometric parameters of P_1 and P_2. In order to obtain a running time that depends on n and ε only, we cannot use a uniform discretization of the entire rotation space. Indeed, our contribution lies in establishing some structural properties that allow us to discretize a small subset of the rotation space, and exploiting this discretization in the analysis to prove the desired approximation. This approach is also taken in the 2D case in [3], but our analysis is not an extension of that in [3] as the three-dimensional situation is different and more complicated.

2 Similar Polytopes

In this section, we show that P_1 and P_2 are "similar" under the assumption that their maximum overlap is at least $\lambda\cdot\max\{|P_1|,|P_2|\}$. We use the Löwner-John ellipsoid [11] to identify the three axes of P_1 and P_2. For every convex body P in $\mathbb{R}^d$, it is proven by Löwner that there is a unique ellipsoid E containing P with minimum volume. Then John proved that $\frac{1}{d}E$ is contained in P. There are various algorithms for finding an ellipsoid of this flavor.

▶ **Lemma 1** ([11]). *Let P be a convex body with m vertices in $\mathbb{R}^3$. For every $\eta > 0$, an ellipsoid $\mathcal{E}(P)$ can be computed in $O(m/\eta)$ time such that $\frac{1}{3(1+\eta)}\mathcal{E}(P) \subset P \subset \mathcal{E}(P)$.*

For $i \in \{1,2\}$, we use $\mathcal{E}(P_i)$ to denote the ellipsoid guaranteed by Lemma 1 for P_i. There are three mutually orthogonal directed lines α_i, β_i and γ_i through the center of $\mathcal{E}(P_i)$ such that $|\alpha_i \cap \mathcal{E}(P_i)|$ and $|\gamma_i \cap \mathcal{E}(P_i)|$ are the shortest and longest, respectively, among all possible directed lines through the center of $\mathcal{E}(P_i)$. After fixing α_i and γ_i, there are two choices for β_i and any one will do. We call these lines the α_i-, β_i-, and γ_i-axes of P_i. The lengths $a_i = |\alpha_i \cap \mathcal{E}(P_i)|$, $b_i = |\beta_i \cap \mathcal{E}(P_i)|$, and $c_i = |\gamma_i \cap \mathcal{E}(P_i)|$ are the three *principal diameters* of $\mathcal{E}(P_i)$. Notice that $a_i \leqslant b_i \leqslant c_i$. Define $a_{\min} = \min\{a_1, a_2\}$, $a_{\max} = \max\{a_1, a_2\}$, $b_{\min} = \min\{b_1, b_2\}$, $b_{\max} = \max\{b_1, b_2\}$, $c_{\min} = \min\{c_1, c_2\}$, and $c_{\max} = \max\{c_1, c_2\}$. The following technical result will be needed.

▶ **Lemma 2.** *For $i \in \{1,2\}$, let R_i be a box with side lengths a_i, b_i, and c_i. The maximum overlap of R_1 and R_2 under rigid motion is at most $\sqrt{3}a_{\min}b_{\min}c_{\min}$.*

Proof. Without loss of generality, we suppose that a_1 is $a_{\min}$, that is, $a_1 \leq a_2$. If $b_{\min} = b_1$ and $c_{\min} = c_1$, then the maximum overlap of R_1 and R_2 are automatically $a_{\min}b_{\min}c_{\min}$ or less. There are three cases left: (i) $b_{\min} = b_2$ and $c_{\min} = c_2$, (ii) $b_{\min} = b_1$ and $c_{\min} = c_2$, and (iii) $b_{\min} = b_2$ and $c_{\min} = c_1$.

Let the *ab*-, *bc*-, and *ca*-planes of R_i be the plane through the center of R_i and parallel to the facets of side length a_i and b_i, b_i and c_i, and c_i and a_i respectively, where $i \in \{1,2\}$. Place R_1 and R_2 such that their overlap is maximum.

Case 1: $b_{\min} = b_2$ and $c_{\min} = c_2$. Let θ be the nonobtuse angle between the normal lines of *bc*-plane of R_1 and the *ab*-plane of R_2.

Suppose that $\theta \leqslant \pi/4$. Refer to Figure 1(a). Consider the two facets of R_1 that are parallel to its *bc*-plane. The supporting planes of these two facets bound an infinite slab with width a_1. Consider the facets of R_2 that are parallel to its *ab*-plane. Sweeping these two facets along the normal line of the *ab*-plane of R_2 produces an infinite rectangular cylinder. The intersection of the slab and the cylinder is a parallelepiped that contains $R_1 \cap R_2$, We can assume that b_2 is parallel to the *bc*-plane of R_1 because the volume of the parallelepiped does not change while we rotate the cylinder around the normal line of *ab*-plane of R_2. Then, the parallelepiped's volume is $a_1a_2b_2/\cos\theta \leqslant a_1b_2c_2/\cos\theta \leqslant \sqrt{2}a_{\min}b_{\min}c_{\min}$.

Suppose that $\theta > \pi/4$. Refer to Figure 1(b). The angle between the normal lines of the *bc*-plane of R_1 and the *bc*-plane of R_2 is $\pi/2 - \theta$. We construct the infinite slab as in the above. We sweep the two facets of R_2 that are parallel to its *bc*-plane to obtain an infinite rectangular cylinder instead. The volume of the parallelepiped at the intersection of the slab and this new cylinder is $a_1b_2c_2/\cos(\pi/2-\theta) \leqslant \sqrt{2}a_{\min}b_{\min}c_{\min}$.

Case 2: $b_{\min} = b_1$ and $c_{\min} = c_2$. Let θ be the nonobtuse angle between the normal lines of the *ab*-planes of R_1 and R_2.

Suppose that $\theta \leqslant \arccos(1/\sqrt{3})$. Refer to Figure 1(c). Consider the two facets of R_2 that are parallel to its *ab*-plane. The supporting planes of these two facets bound an infinite slab with width c_2. Consider the facets of R_1 that are parallel to its *ab*-plane. Sweeping these two facets along the normal line of the *ab*-plane of R_1 produces an infinite rectangular cylinder. As in case 1, we can assume that b_1 is parallel to the *ab*-plane of R_2. The intersection of the slab and the cylinder is a parallelepiped that contains $R_1 \cap R_2$, and the parallelepiped's volume is $a_1b_1c_2/\cos\theta \leqslant \sqrt{3}a_{\min}b_{\min}c_{\min}$.

Suppose that $\theta > \arccos(1/\sqrt{3})$. Refer to Figure 1(d). Let φ be the nonobtuse angle between the normal lines of the *ab*-plane of R_1 and the *bc*-plane of R_2. Let ψ be the nonobtuse angle between the normal lines of the *ab*-plane of R_1 and the *ac*-plane of R_2. Since $\cos^2\theta + \cos^2\varphi + \cos^2\psi = 1$, the sum $\cos^2\varphi + \cos^2\psi$ is at least $2/3$, which implies that $\cos\varphi$

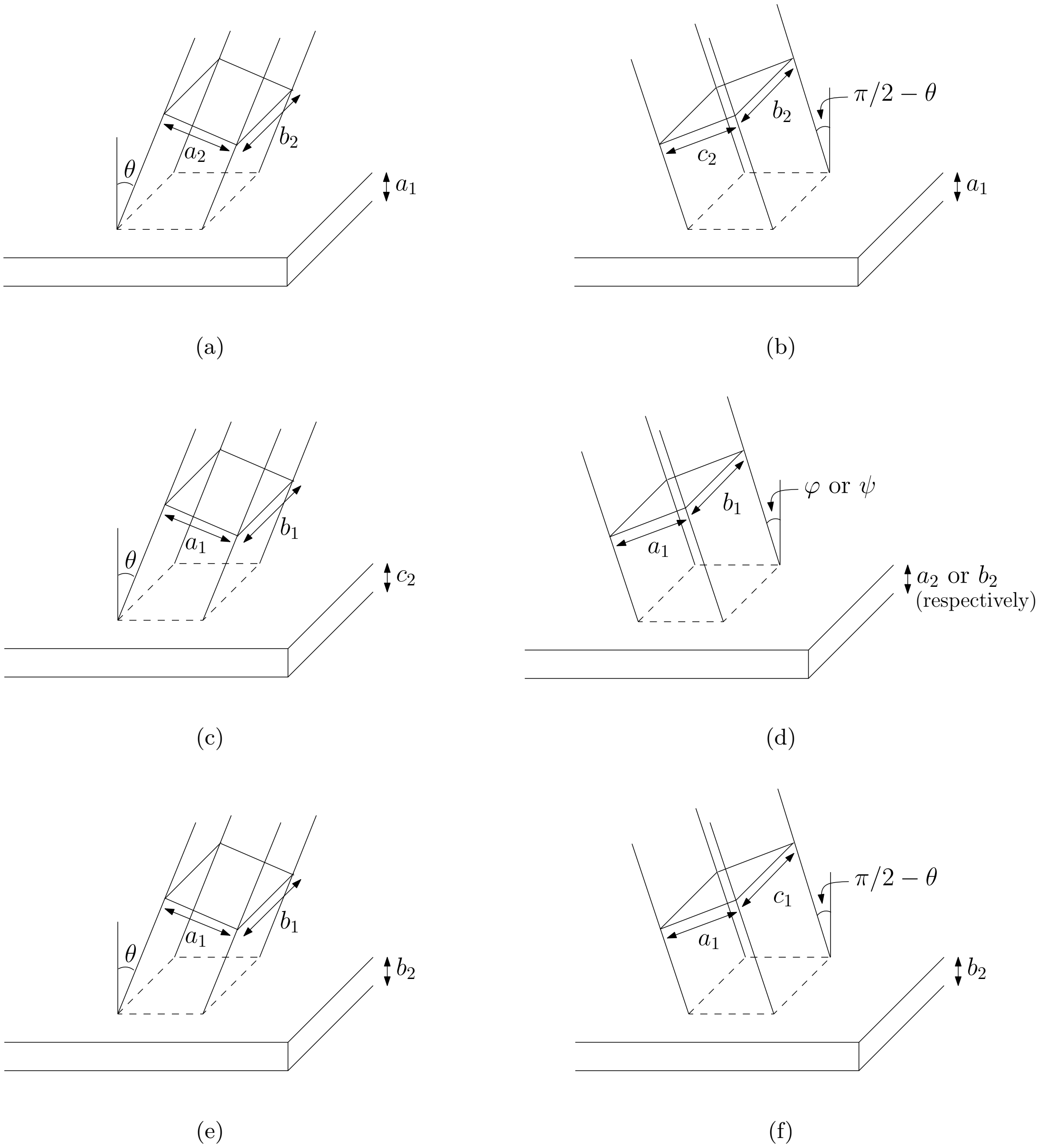

Figure 1 Illustrations for the proof of Lemma 2.

or $\cos\psi$ is at least $1/\sqrt{3}$. We construct the infinite rectangular cylinder as in the previous paragraph. If $\cos\varphi \geqslant 1/\sqrt{3}$, we take the slab bounded by the supporting planes of the two facets of R_2 that are parallel to its bc-plane. If $\cos\psi \geqslant 1/\sqrt{3}$, we take the slab bounded by the supporting planes of the two facets of R_2 that are parallel to its ac-plane. $R_1 \cap R_2$ is contained in the parallelepiped at the intersection of the slab and the cylinder, whose volume is at most $a_1a_2b_1/\cos\varphi$ if $\cos\varphi \geqslant 1/\sqrt{3}$, or $a_1b_1b_2/\cos\varphi$ if $\cos\varphi \geqslant 1/\sqrt{3}$. In either case, the volume is at most $\sqrt{3}a_1b_1c_2 = \sqrt{3}a_{\min}b_{\min}c_{\min}$.

Case 3: $b_{\min} = b_2$ and $c_{\min} = c_1$. Let θ be the nonobtuse angle between the normal lines of the ab-plane of R_1 and the ac-plane of R_2.

Suppose that $\theta \leqslant \pi/4$. Refer to Figure 1(e). Consider the two facets of R_2 that are parallel to its ac-plane. The supporting planes of these two facets bound an infinite slab with width b_2. Consider the facets of R_1 that are parallel to its ab-plane. Sweeping these two facets along the normal line of the ab-plane of R_1 produces an infinite rectangular cylinder. As in case 1, we can assume that a_1 is parallel to the ac-plane of R_2. The intersection of the slab and the cylinder is a parallelepiped that contains $R_1 \cap R_2$, and the parallelepiped's volume is $a_1b_1b_2/\cos\theta \leqslant a_1c_1b_2/\cos\theta \leqslant \sqrt{2}a_{\min}b_{\min}c_{\min}$.

Suppose that $\theta > \pi/4$. Refer to Figure 1(f). We keep the same slab in the previous paragraph. Sweep the two facets of R_1 that are parallel to its ac-plane to obtain an infinite rectangular cylinder. $R_1 \cap R_2$ is contained in the parallelepiped at the intersection of the slab and the cylinder. This parallelepiped has volume $a_1c_1b_2/\cos(\pi/2-\theta) \leqslant \sqrt{2}a_{\min}b_{\min}c_{\min}$. ◀

We are ready to show that P_1 and P_2 are similar in the sense that the respective principal diameters are within a constant factor of each other.

▶ **Lemma 3.** *If the maximum overlap of P_1 and P_2 under rigid motion is $\lambda \cdot \max\{|P_1|, |P_2|\}$ or more, then the ratios a_1/a_2, b_1/b_2, and c_1/c_2 are between $\lambda/(2^7\sqrt{3})$ and $2^7\sqrt{3}/\lambda$.*

Proof. It follows from Lemma 1 that $|P_i| \geq \frac{4}{3}\pi \cdot 2^{-3}3^{-3}(1+\eta)^{-3} \cdot a_ib_ic_i$ for $i \in \{1,2\}$. By setting η such that $3(1+\eta) < 4$, we obtain: for $i \in \{1,2\}$, $|P_i| \geq \frac{4}{3}\pi \cdot 2^{-3}4^{-3} \cdot a_ib_ic_i$. The maximum overlap of P_1 and P_2 under rigid motion is at most $\sqrt{3}a_{\min}b_{\min}c_{\min}$ by Lemma 2. Thus, for $i \in \{1,2\}$, $\sqrt{3}a_{\min}b_{\min}c_{\min} \geq \lambda|P_i| \geq (3/\pi)\cdot\lambda|P_i| \geqslant \lambda a_ib_ic_i/2^7$. It follows that

$$a_1/a_2 \;\leqslant\; (2^7\sqrt{3}/\lambda)\cdot(b_{\min}/b_1)\cdot(c_{\min}/c_1) \;\leqslant\; 2^7\sqrt{3}/\lambda \qquad (1)$$

$$a_1/a_2 \;\geqslant\; (\lambda/(2^7\sqrt{3}))\cdot(b_1/b_{\min})\cdot(c_1/c_{\min}) \;\geqslant\; \lambda/(2^7\sqrt{3}) \qquad (2)$$

We can similarly show that b_1/b_2 and c_1/c_2 are between $\lambda/(2^7\sqrt{3})$ and $2^7\sqrt{3}/\lambda$. ◀

3 Sampling Rigid Motions

A rigid motion can be viewed as a rotation of P_1 and P_2 followed by a translation of P_2. A rotation is a relative motion between P_1 and P_2, and indeed, it is more convenient to rotate both P_1 and P_2 for our purposes. We set the initial positions of P_1 and P_2 so that the centers of $\mathcal{E}(P_1)$ and $\mathcal{E}(P_2)$ coincide and the α_1- and α_2-axes, β_1- and β_2-axes, and the γ_1- and γ_2-axes are aligned, respectively. Without loss of generality, assume that the $\alpha_1\beta_1$-plane is horizontal initially. (So is the $\alpha_2\beta_2$-plane as it coincides with the $\alpha_1\beta_1$-plane initially.)

A rotation R_* acts on the pair (P_1, P_2) and produces a new pair $R_*(P_1, P_2)$. R_* is decomposed into three simpler rotations R_β, R_α, and R_γ parametrized by three angles $\theta_\beta, \theta_\alpha, \theta_\gamma \in [-\pi, \pi]$, respectively, such that $R_\gamma(R_\beta(P_1))$ is the rotated P_1 by R_* and $R_\alpha(P_2)$ is the rotated P_2 by R_*. Let $\angle(u, v)$ be the angle between two oriented axes u and v. So

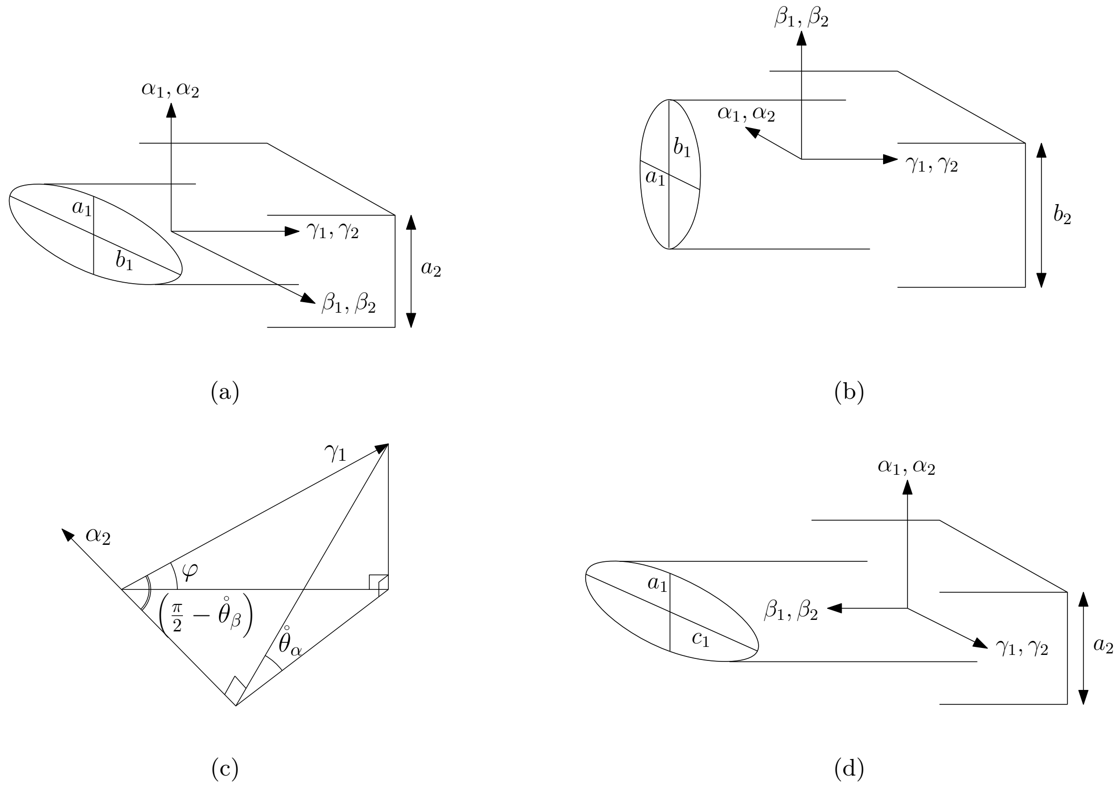

Figure 2 Illustrations for the proof of Lemma 4.

$\angle(u, v) \in [0, \pi]$. The idea is to first rotate P_1 and P_2 to fix the angle between γ_1 and the $\alpha_2\beta_2$-plane, and then rotate P_1 around γ_1. The detailed specification of R_* is as follows.

1. Rotate P_1 around the β_1-axis in the clockwise direction as viewed from infinity in β_1's direction by the angle θ_β. This is the rotation R_β which fixes the angle $\angle(\gamma_1, \alpha_2)$.
2. Rotate P_2 around the α_2-axis in the clockwise direction as viewed from infinity in α_2's direction by the angle θ_α. This is the rotation R_α which fixes the angle $\angle(\gamma_1, \beta_2)$. Notice that the angle $\angle(\gamma_1, \alpha_2)$ is not affected by R_α.
3. Rotate P_1 around the γ_1-axis in the clockwise direction as viewed from infinity in γ_1's direction by the angle θ_γ. This is the rotation R_γ. Notice that the angles $\angle(\gamma_1, \alpha_2)$ and $\angle(\gamma_1, \beta_2)$ are unaffected by R_γ.

The order of the applications of R_β, R_α and R_γ matters—the result of applying R_β, R_α and R_γ in this order can differ from the result of applying the same three rotations in another order. Every rotation in $\mathbb{R}^3$ is specified by a triple $(\theta_\beta, \theta_\alpha, \theta_\gamma) \in [-\pi, \pi] \times [-\pi, \pi] \times [-\pi, \pi]$.

▶ **Lemma 4.** *Let P_1 and P_2 be convex polytopes in $\mathbb{R}^3$. Let $\mathring{R}_*$ be the rotation part of an optimal rigid motion $\mathring{M}$ that maximizes the intersection volume of P_1 and P_2. Let $\mathring{\theta}_\beta$, $\mathring{\theta}_\alpha$ and $\mathring{\theta}_\gamma$ be the three angles in the representation of $\mathring{R}_*$. If $2^{13}3^5 a_{\min} \leqslant \lambda^2 c_{\min}/\sqrt{2}$, then*

$$|\sin \mathring{\theta}_\beta| \leq \frac{2^{13}3^5 a_{\min}}{\lambda^2 c_{\min}}, \quad |\sin \mathring{\theta}_\alpha| \leq \frac{2^{14}3^5 b_{\min}}{\sqrt{2}\lambda^2 c_{\min}}, \quad |\sin \mathring{\theta}_\gamma| \leq \frac{2^{13}3^5 a_{\min}}{\lambda^2 b_{\min}}.$$

Proof. By Lemma 1, if we position P_1 and P_2 such that the centers of $\mathcal{E}(P_1)$ and $\mathcal{E}(P_2)$ coincide and the respective axes of P_1 and P_2 are aligned, the overlap of P_1 and P_2 contains an ellipsoid with principal diameters $a_{\min}/3$, $b_{\min}/3$, and $c_{\min}/3$. Let $\mathring{T}$ be the translation part of the optimal rigid motion. Thus, $|\mathring{R}_\gamma(\mathring{R}_\beta(P_1)) \cap \mathring{T}(\mathring{R}_\alpha(P_2))| \geq \frac{4}{3}\pi a_{\min} b_{\min} c_{\min}/(2^3 3^3)$.

Enclose $\mathcal{E}(P_1)$ in an elliptic cylinder C such that the base of C has principal diameters a_1 and b_1, and the axis of C is aligned with the γ_1-axis of P_1. Enclose $\mathcal{E}(P_2)$ with a infinite slab S that has thickness a_2 and is parallel to the $\beta_2\gamma_2$-plane. Refer to Figure 2(a). When we apply $\mathring{R}_*$, we get $|\mathring{R}_\gamma(\mathring{R}_\beta(P_1)) \cap \mathring{R}_\alpha(P_2)| \leqslant |\mathring{R}_\gamma(\mathring{R}_\beta(C)) \cap \mathring{R}_\alpha(S)|$. Only $\mathring{R}_\beta$ has an effect on the volume of $|\mathring{R}_\gamma(\mathring{R}_\beta(C)) \cap \mathring{R}_\alpha(S)|$ because $\mathring{R}_\alpha$ does not change the shape of the intersection, and $\mathring{R}_\gamma$ does not change the volume of the intersection as long as $\mathring{\theta}_\beta \notin \{0, \pi, -\pi\}$. $\mathring{R}_\gamma(\mathring{R}_\beta(C)) \cap \mathring{R}_\alpha(S)$ has base area $\pi a_1 b_1/(2^2|\sin\mathring{\theta}_\beta|)$ and height a_2, so $|\mathring{R}_\gamma(\mathring{R}_\beta(C)) \cap \mathring{R}_\alpha(S)| = \pi a_1 a_2 b_1/(2^2|\sin\mathring{\theta}_\beta|)$. Obviously, applying the translation part $\mathring{T}$ of the optimal rigid motion to $\mathring{R}_\alpha(S)$ has no impact on the intersection volume. Therefore, $|\mathring{R}_\gamma(\mathring{R}_\beta(C)) \cap \mathring{R}_\alpha(S)| \geqslant |\mathring{R}_\gamma(\mathring{R}_\beta(P_1)) \cap \mathring{T}(\mathring{R}_\alpha(P_2))| \geqslant \frac{4}{3}\pi a_{\min} b_{\min} c_{\min}/(2^3 3^3)$. We conclude that

$$\frac{4\pi}{2^3 3^4} a_{\min} b_{\min} c_{\min} \leqslant \frac{\pi}{2^2|\sin\mathring{\theta}_\beta|} a_1 a_2 b_1 \quad\Rightarrow\quad |\sin\mathring{\theta}_\beta| \leqslant \frac{3^4 a_1 a_2 b_1}{2 a_{\min} b_{\min} c_{\min}} \leqslant \frac{2^{13} 3^5 a_{\min}}{\lambda^2 c_{\min}}.$$

The last inequality follows from Lemma 3.

The assumption of $2^{13}3^5 a_{\min} \leqslant \lambda^2 c_{\min}/\sqrt{2}$ is needed in bounding $|\sin\mathring{\theta}_\alpha|$ and $|\sin\mathring{\theta}_\gamma|$. By this assumption, $|\sin\mathring{\theta}_\beta| \leqslant 1/\sqrt{2}$ and $\mathring{\theta}_\beta \in [0, \pi/4] \cup (-\pi, -3\pi/4]$. To bound $\sin\mathring{\theta}_\alpha$, we similarly enclose $\mathcal{E}(P_1)$ with an elliptic cylinder C and $\mathcal{E}(P_2)$ with a slab S, except that we swap the positions of α_i and β_i. The height of the slab S enclosing $\mathcal{E}(P_2)$ is thus b_2. Refer to Figure 2(b). R_α makes C tilt at an acute angle φ to S, while R_γ has no effect on the volume of the intersection of C and S as long as $\mathring{\theta}_\alpha \notin \{0, \pi, -\pi\}$. The maximum value of $\sin\varphi$ is $|\sin\mathring{\theta}_\alpha|$ when $\mathring{\theta}_\beta = 0$ or π or $-\pi$; and $\sin\varphi$ is minimized when $\mathring{\theta}_\beta = \pi/4$ or $-3\pi/4$. Refer to Figure 2(c); by elementary trigonometry, the minimum value of $\sin\varphi$ is $|\sin\mathring{\theta}_\alpha|/\sqrt{2}$. As in the last paragraph, $|\mathring{R}_\gamma(\mathring{R}_\beta(C)) \cap \mathring{R}_\alpha(S)| = \pi a_1 b_1 b_2/(2^2\sin\varphi) \leqslant \pi a_1 b_1 b_2/(2\sqrt{2}|\sin\mathring{\theta}_\alpha|)$. Therefore,

$$\frac{4\pi}{2^3 3^4} a_{\min} b_{\min} c_{\min} \leqslant \frac{\pi}{2\sqrt{2}|\sin\mathring{\theta}_\alpha|} a_1 b_1 b_2 \quad\Rightarrow\quad |\sin\mathring{\theta}_\alpha| \leqslant \frac{3^4 a_1 b_1 b_2}{\sqrt{2} a_{\min} b_{\min} c_{\min}} \leqslant \frac{2^{14} 3^5 b_{\min}}{\sqrt{2}\lambda^2 c_{\min}}.$$

The analysis for $\sin\mathring{\theta}_\gamma$ is similar. We enclose $\mathcal{E}(P_1)$ with an elliptic cylinder C and $\mathcal{E}(P_2)$ with a slab S as shown in Figure 2(d). Notice that the height of S is a_2. R_β has no effect on the volume of the intersection of C and S as long as $\mathring{\theta}_\gamma \notin \{0, \pi, -\pi\}$, and R_α does not change the shape of intersection of C and S. R_γ makes C tilt at an acute angle $\mathring{\theta}_\gamma$ to S. Therefore, $|\mathring{R}_\gamma(\mathring{R}_\beta(C)) \cap \mathring{R}_\alpha(S)| = \pi a_1 a_2 c_1/(2^2|\sin\mathring{\theta}_\gamma|)$. Therefore,

$$\frac{4\pi}{2^3 3^4} a_{\min} b_{\min} c_{\min} \leqslant \frac{\pi}{2^2|\sin\mathring{\theta}_\gamma|} a_1 a_2 c_1 \quad\Rightarrow\quad |\sin\mathring{\theta}_\gamma| \leqslant \frac{3^4 a_1 a_2 c_1}{2 a_{\min} b_{\min} c_{\min}} \leqslant \frac{2^{13} 3^5 a_{\min}}{\lambda^2 b_{\min}}.$$

◀

Lemma 4 tells us that if $2^{13}3^5 a_{\min} \leqslant \lambda^2 c_{\min}/\sqrt{2}$, the angles θ_β, θ_α and θ_γ can only vary in some appropriate subsets of $[-\pi, \pi]$. This allows us to discretize only a small subrange of $[-\pi, \pi]$ in designing our approximation algorithm, which helps to reduce the running time. If $2^{13}3^5 a_{\min} > \lambda^2 c_{\min}/\sqrt{2}$, the lengths a_i, b_i and c_i are within constant factors of each other, and it suffices to discretize the range $[-\pi, \pi]$ uniformly in this case. In the following, we first define the angular ranges I_β, I_α and I_γ for θ_β, θ_α and θ_γ respectively, and then discuss the discretization of these three ranges.

- If $2^{13}3^5 a_{\min} > \lambda^2 c_{\min}/\sqrt{2}$, then $I_\beta = I_\alpha = I_\gamma = [-\pi, \pi]$.

- If $2^{13}3^5 a_{\min} \leqslant \lambda^2 c_{\min}/\sqrt{2}$, then for $\xi \in \{\beta, \alpha, \gamma\}$,

$$I_\xi = [0, f_\xi] \cup [\pi - f_\xi, \pi] \cup [-f_\xi, 0] \cup [-\pi, -\pi + f_\xi],$$

where:
- $f_\beta = \arcsin(2^{13}3^5 a_{\min}/(\lambda^2 c_{\min}))$.
- $f_\alpha = \arcsin(2^{14}3^5 b_{\min}/(\sqrt{2}\lambda^2 c_{\min}))$ if $2^{14}3^5 b_{\min} \leqslant \sqrt{2}\lambda^2 c_{\min}$; otherwise, $f_\alpha = \pi$.
- $f_\gamma = \arcsin(2^{13}3^5 a_{\min}/(\lambda^2 b_{\min}))$ if $2^{13}3^5 a_{\min} \leqslant \lambda^2 b_{\min}$; otherwise, $f_\gamma = \pi$.

The rotation part $\mathring{R}_*$ of the optimal rigid motion belongs to $I_\beta \times I_\alpha \times I_\gamma$ according to Lemma 4. We sample angle triples from $I_\beta \times I_\alpha \times I_\gamma$ at intervals of $\Delta_\beta\varepsilon, \Delta_\alpha\varepsilon, \Delta_\gamma\varepsilon$ respectively:

$$\Delta_\beta = \frac{a_{\min} b_{\min} c_{\min}}{2^4 3^5 b_1 c_1^2}, \quad \Delta_\alpha = \frac{1}{2} \cdot \frac{a_{\min} b_{\min} c_{\min}}{3^5 a_2 c_2^2}, \quad \Delta_\gamma = \frac{1}{2} \cdot \frac{a_{\min} b_{\min} c_{\min}}{3^5 b_1^2 c_1}.$$

Let S_ξ denote the set of angles sampled from I_ξ for $\xi \in \{\beta, \alpha, \gamma\}$. Our strategy is to try all rotations in $S_\beta \times S_\alpha \times S_\gamma$ and for each such rotation, find the best translation to maximize the overlap. It remains to show that the best rigid motion obtained by this strategy gives a $(1 - \varepsilon)$-approximation.

In the lemma below, the rotation center p can be outside of C unlike in Lemma 4 from [1]. The proof is similar to that of Lemma 4 in [1].

▶ **Lemma 5.** *Let C be a convex set in $\mathbb{R}^2$, and let C' be a copy of C, rotated by an angle δ around a point p that is at distance l or less from any point in C. Then $|C \backslash C'| \leq (\pi\delta l/2) \cdot \mathrm{diam}(C) + \pi\delta^2 l^2/8$.*

Proof. We denote by D the symmetric difference between C and C'. Note that $|D| = |C| - |C \cap C'| + |C'| - |C \cap C'| = 2(|C| - |C \cap C'|) \geq 2|C \backslash C'|$. Let C_r be the rotated copy of C by an angle $\delta/2$ around p. Let T_r be the set of points that are at distance at most $\delta l/2$ from the boundary of C_r. Note that any point q in D is obtained from a point on the boundary of C_r by a rotation around p by an angle at most $\delta/2$. Since the distance $d(p, q)$ is at most l, q is an element of T_r. Thus $D \subset T_r$. Because the Minkowski sum of the boundary of C and a disk of radius r has area less than or equal to $2r\mathrm{peri}(C) + \pi r^2$, the area of T_r is at most $\delta l \mathrm{peri}(C) + \pi\delta^2 l^2/4$. Since $\mathrm{peri}(C_r) = \mathrm{peri}(C)$ and $\mathrm{peri}(C) \leq \pi\mathrm{diam}(C)$, we obtain that $|T_r| \leq \pi\delta l\mathrm{diam}(C) + \pi\delta^2 l^2/4$. Since $D \subset T_r$, it implies that $|D| \leq \pi\delta l\mathrm{diam}(C) + \pi\delta^2 l^2/4$. Therefore $|C \backslash C'| \leq \frac{1}{2}|D| \leq (\pi\delta l/2)\mathrm{diam}(C) + \pi\delta^2 l^2/8$. ◀

The following lemma is another extension of Lemma 4 in [1]. It shows that two copies of a convex polyhedron have small symmetric difference if there is a bound on the Hausdorff distance between them.

▶ **Lemma 6.** *Let C be a convex polyhedron in $\mathbb{R}^3$, and let C' be a copy of C such that the Hausdorff distance between C and C' is at most l. Let c and b be the first and second largest principal diameters of $\mathcal{E}(\mathcal{C})$. Then $|C \backslash C'| \leq \frac{4}{3}\pi l^3 + 2\pi bcl + 2\pi^2 cl^2$.*

Proof. Note that any point q in $C \backslash C'$ is in distance at most l from a point on the boundary of C'. Therefore $|C \backslash C'|$ has volume less than or equal to the volume of the Minkowski sum of the boundary of C' and a ball of radius l. Let V be the set of vertices, E be the set of edges, and F be the set of facets of C'. For every $j \in [0, 3]$ and every j-face f of C', define the interior angle $\varphi(f)$ to be the fraction of an arbitrarily small sphere centered at an interior point of f, that lies inside C'. For example, $\varphi(f) = 1$ if $f = C'$, $\varphi(f) = 1/2$ if f is a facet,

$\varphi(f)$ is the ratio of the internal directed angle at f to 2π if f is an edge, and $\varphi(f)$ is the solid angle at f if f is a vertex. The exterior angle $\theta(f)$ is defined to be $1/2 - \varphi(f)$. The Minkowski sum of the boundary of C' and a ball of radius l has volume less than

$$\frac{4}{3}\pi l^3 + 2l \cdot \sum_{f\in F} \mathrm{area}(f) + \sum_{e\in E} \pi l^2 \cdot \mathrm{length}(e) \cdot 2\pi\theta(e)$$

where area(f) is the area of a facet f and length(e) is the length of an edge e. Since C' is a convex polyhedron, the total surface area of C' does not exceed the surface area of the ellipsoid $E(C')$ containing C' which surface area is less than πbc [7] because $E(C')$ and $E(C)$ are identical. So we obtain the volume bound as below.

$$\frac{4}{3}\pi l^3 + 2\pi bcl + \sum_{e\in E} \pi l^2 \cdot \mathrm{length}(e) \cdot 2\pi\theta(e)$$

The last term can be bounded as follow. By the Gram-Euler theorem [9, 10], we know that

$$\sum_{j\in[0,3]} \sum_{j\text{-face } f \text{ of } C'} (-1)^j \varphi(f) = 0.$$

Therefore,

$$\sum_{v\in V} \varphi(v) - \sum_{e\in E} \varphi(e) + \sum_{f\in F} 1/2 = 1.$$

Since $\sum_{v\in V} \theta(v) = 1$, $\sum_{v\in V} \varphi(v) = |V|/2 - \sum_{v\in V} \theta(v) = |V|/2-1$, and $\sum_{e\in E} \theta(e) = |E|/2 - \sum_{e\in E} \varphi(E) = 2-(|V|-|E|+|F|)/2$. By the Euler's formula, $|V|-|E|+|F|$ is 2. It follows that $\sum_{e\in E} \theta(e) = 1$. Therefore, $|C\backslash C'| \le \frac{4}{3}\pi l^3 + 2\pi bcl + \sum_{e\in E} \pi l^2 \cdot \mathrm{length}(e) \cdot 2\pi\theta(e) \le \frac{4}{3}\pi l^3 + 2\pi bcl + \sum_{e\in E} \pi cl^2 \cdot 2\pi\theta(e) = \frac{4}{3}\pi l^3 + 2\pi bcl + 2\pi^2 cl^2 \sum_{e\in E} \theta(e) = \frac{4}{3}\pi l^3 + 2\pi bcl + 2\pi^2 cl^2$. ◀

The next result proves the correctness of our strategy to find a $(1-\varepsilon)$-approximately maximum overlap of P_1 and P_2 under rigid motion.

▶ **Lemma 7.** *Let P_1 and P_2 be convex polytopes in $\mathbb{R}^3$. Let ε be a constant from the range $(0, 1/2)$. Suppose that the maximum overlap of P_1 and P_2 under rigid motion is at least $\lambda \cdot \max\{|P_1|, |P_2|\}$ for some constant $\lambda \in (0, 1]$. Then, there exists a rotation $\tilde{R}_* \in S_\beta \times S_\alpha \times S_\gamma$ and a translation $\tilde{T}$ such that $|\tilde{R}_\gamma(\tilde{R}_\beta(P_1)) \cap \tilde{T}(\tilde{R}_\alpha(P_2))|$ is at least $1-\varepsilon$ times the maximum overlap of P_1 and P_2 under rigid motion.*

Proof. The rotation part $\mathring{R}_*$ of the optimal rigid motion is represented by a triple of angles $(\mathring{\theta}_\beta, \mathring{\theta}_\alpha, \mathring{\theta}_\gamma) \in I_\beta \times I_\alpha \times I_\gamma$. For $\xi \in \{\beta, \alpha, \gamma\}$, let $\tilde{\theta}_\xi$ be the closest interval endpoint in S_ξ to $\mathring{\theta}_\xi$. Then, $(\tilde{\theta}_\beta, \tilde{\theta}_\alpha, \tilde{\theta}_\gamma)$ defines a rotation $\tilde{R}_*$. Let $\tilde{R}_\alpha$, $\tilde{R}_\beta$, and $\tilde{R}_\gamma$ denote the three simple rotations that comprise $\tilde{R}_*$.

Let $\tilde{T}$ denote the translation that maximizes the overlap of $\tilde{R}_\gamma(\tilde{R}_\beta(P_1))$ and $\tilde{R}_\alpha(P_2)$. Let $\mathring{T}$ denote the translation that maximizes the overlap of $\mathring{R}_\gamma(\mathring{R}_\beta(P_1))$ and $\mathring{R}_\alpha(P_2)$. Therefore, $|\tilde{R}_\gamma(\tilde{R}_\beta(P_1)) \cap \tilde{T}(\tilde{R}_\alpha(P_2))| \ge |\tilde{R}_\gamma(\tilde{R}_\beta(P_1)) \cap \mathring{T}(\tilde{R}_\alpha(P_2))|$. We analyze the difference between the maximum overlap and the approximate overlap as follows.

$$\begin{aligned}
& |\mathring{R}_\gamma(\mathring{R}_\beta(P_1)) \cap \mathring{T}(\mathring{R}_\alpha(P_2))| - |\tilde{R}_\gamma(\tilde{R}_\beta(P_1)) \cap \tilde{T}(\tilde{R}_\alpha(P_2))| \\
\leqslant\ & |\mathring{R}_\gamma(\mathring{R}_\beta(P_1)) \cap \mathring{T}(\mathring{R}_\alpha(P_2))| - |\tilde{R}_\gamma(\tilde{R}_\beta(P_1)) \cap \mathring{T}(\tilde{R}_\alpha(P_2))| \\
=\ & |\mathring{R}_\gamma(\mathring{R}_\beta(P_1)) \cap \mathring{R}_\alpha(\mathring{T}(P_2))| - |\tilde{R}_\gamma(\tilde{R}_\beta(P_1)) \cap \tilde{R}_\alpha(\mathring{T}(P_2))| \\
=\ & |\mathring{R}_\gamma(\mathring{R}_\beta(P_1)) \cap \mathring{R}_\alpha(\mathring{T}(P_2))| - |\tilde{R}_\gamma(\mathring{R}_\beta(P_1)) \cap \mathring{R}_\alpha(\mathring{T}(P_2))| + \qquad (3) \\
& |\tilde{R}_\gamma(\mathring{R}_\beta(P_1)) \cap \mathring{R}_\alpha(\mathring{T}(P_2))| - |\tilde{R}_\gamma(\mathring{R}_\beta(P_1)) \cap \tilde{R}_\alpha(\mathring{T}(P_2))| + \qquad (4) \\
& |\tilde{R}_\gamma(\mathring{R}_\beta(P_1)) \cap \tilde{R}_\alpha(\mathring{T}(P_2))| - |\tilde{R}_\gamma(\tilde{R}_\beta(P_1)) \cap \tilde{R}_\alpha(\mathring{T}(P_2))|. \qquad (5)
\end{aligned}$$

If a point p lies in $\mathring{R}_\gamma(\mathring{R}_\beta(P_1)) \cap \mathring{R}_\alpha(\mathring{T}(P_2))$ but not in $\tilde{R}_\gamma(\mathring{R}_\beta(P_1)) \cap \mathring{R}_\alpha(\mathring{T}(P_2))$, then $p \in \mathring{R}_\gamma(\mathring{R}_\beta(P_1))$ but $p \notin \tilde{R}_\gamma(\mathring{R}_\beta(P_1))$. The common rotation $\mathring{R}_\beta$ can be ignored. Thus,

$$|\mathring{R}_\gamma(\mathring{R}_\beta(P_1)) \cap \mathring{R}_\alpha(\mathring{T}(P_2))| \;-\; |\tilde{R}_\gamma(\mathring{R}_\beta(P_1)) \cap \mathring{R}_\alpha(\mathring{T}(P_2))| \leqslant |\mathring{R}_\gamma(P_1) \setminus \tilde{R}_\gamma(P_1)|.$$

Similar reasoning shows that

$$\begin{aligned}
|\tilde{R}_\gamma(\mathring{R}_\beta(P_1)) \cap \mathring{R}_\alpha(\mathring{T}(P_2))| - |\tilde{R}_\gamma(\mathring{R}_\beta(P_1)) \cap \tilde{R}_\alpha(\mathring{T}(P_2))| &\leqslant |\mathring{R}_\alpha(P_2) \setminus \tilde{R}_\alpha(P_2)|, \\
|\tilde{R}_\gamma(\mathring{R}_\beta(P_1)) \cap \tilde{R}_\alpha(\mathring{T}(P_2))| - |\tilde{R}_\gamma(\tilde{R}_\beta(P_1)) \cap \tilde{R}_\alpha(\mathring{T}(P_2))| &\leqslant |\tilde{R}_\gamma(\mathring{R}_\beta(P_1)) \setminus \tilde{R}_\gamma(\tilde{R}_\beta(P_1))|.
\end{aligned}$$

Let H be a plane perpendicular to the γ_1-axis of P_1 that intersects $\mathring{R}_\gamma(P_1)$ and $\tilde{R}_\gamma(P_1)$. The convex polygon $H \cap \mathring{R}_\gamma(P_1)$ is rotated from the convex polygon $H \cap \tilde{R}_\gamma(P_1)$ by an angle at most $\varepsilon\Delta_\gamma$ around a point in $H \cap \tilde{R}_\gamma(P_1)$. The diameter of $H \cap \tilde{R}_\gamma(P_1)$ is at most b_1. Since the rotation center in $H \cap \tilde{R}_\gamma(P_1)$ is at distance at most $b_1/2$ from any point in $H \cap \mathring{R}_\gamma(P_1)$, Lemma 5 can be applied. Thus $|(H \cap \mathring{R}_\gamma(P_1)) \setminus (H \cap \tilde{R}_\gamma(P_1))| \leqslant (\pi/2)\varepsilon b_1^2 \Delta_\gamma$, which implies that

$$|\mathring{R}_\gamma(P_1) \setminus \tilde{R}_\gamma(P_1)| \leqslant \int_{-c_1}^{c_1} \left(\frac{\pi}{2}\right) \cdot \varepsilon b_1^2 \Delta_\gamma \;\mathrm{d}x = \pi\varepsilon b_1^2 c_1 \Delta_\gamma \leqslant \left(\frac{1}{3}\right) \cdot \frac{\pi}{2 \cdot 3^4} \varepsilon a_{\min} b_{\min} c_{\min}.$$

Similar reasoning shows that

$$|\mathring{R}_\alpha(P_2) \setminus \tilde{R}_\alpha(P_2)| \leqslant \int_{-a_2}^{a_2} \left(\frac{\pi}{2}\right) \cdot \varepsilon c_2^2 \Delta_\alpha \;\mathrm{d}x = \pi\varepsilon a_2 c_2^2 \Delta_\alpha \leqslant \left(\frac{1}{3}\right) \cdot \frac{\pi}{2 \cdot 3^4} \varepsilon a_{\min} b_{\min} c_{\min}.$$

Substitute these results into (3)–(5) gives

$$\begin{aligned}
& |\mathring{R}_\gamma(\mathring{R}_\beta(P_1)) \cap \mathring{T}(\mathring{R}_\alpha(P_2))| \;-\; |\tilde{R}_\gamma(\tilde{R}_\beta(P_1)) \cap \tilde{T}(\tilde{R}_\alpha(P_2))| \\
\leqslant\;\; & |\tilde{R}_\gamma(\mathring{R}_\beta(P_1)) \setminus \tilde{R}_\gamma(\tilde{R}_\beta(P_1))| + \left(\frac{2}{3}\right) \cdot \frac{\pi}{2 \cdot 3^4} \varepsilon a_{\min} b_{\min} c_{\min}.
\end{aligned}$$

We bound $|\tilde{R}_\gamma(\mathring{R}_\beta(P_1)) \setminus \tilde{R}_\gamma(\tilde{R}_\beta(P_1))|$ as follows. This set difference may be non-empty because the γ_1-axis of $\mathring{R}_\beta(P_1)$ can make an angle up to $\varepsilon\Delta_\beta$ with the γ_1-axis of $\tilde{R}_\beta(P_1)$. This slight misalignment causes the results to be different after rotating $\mathring{R}_\beta(P_1)$ and $\tilde{R}_\beta(P_1)$ around their respective γ_1-axes by the same angle. Let x be a point of P_1. To apply the Lemma 6, we want to bound the distance between $\tilde{R}_\gamma(\mathring{R}_\beta(x))$ and $\tilde{R}_\gamma(\tilde{R}_\beta(x))$.

Let $\mathring{x}$ be $\mathring{R}_\beta(x)$ and let $\tilde{x}$ be $\tilde{R}_\beta(x)$. Then $\|\mathring{x} - \tilde{x}\| \leqslant \varepsilon c_1 \Delta_\beta/2$. When $\tilde{R}_\gamma$ is applied, the point $\mathring{x}$ and $\tilde{x}$ are rotated in the planes orthogonal to the γ-axes of $\mathring{R}_\beta(P_1)$ and $\tilde{R}_\beta(P_1)$ respectively. We denote these planes $\mathring{H}$ and $\tilde{H}$, which contain $\mathring{x}$ and $\tilde{x}$ respectively and are orthogonal to the γ-axes of $\mathring{R}_\beta(P_1)$ and $\tilde{R}_\beta(P_1)$ respectively. Let $\mathring{c}$ be the intersection between $\mathring{H}$ and the γ-axis of $\mathring{R}_\beta(P_1)$. Let $\tilde{c}$ be the intersection of $\tilde{H}$ and the γ-axis of $\tilde{R}_\beta(P_1)$. Note that $\|\mathring{c} - \tilde{c}\| \leqslant \varepsilon c_1 \Delta_\beta/2$, $\|\mathring{c} - \mathring{x}\| \leqslant b_1/2$, and $\|\tilde{c} - \tilde{x}\| \leqslant b_1/2$. Another fact is that $\|\mathring{c} - \mathring{x}\| = \|\tilde{c} - \tilde{x}\|$. Therefore, we can imagine that $\tilde{R}_\gamma$ rotates $\mathring{x}$ on the boundary of a disk $\mathring{D}$ on $\mathring{H}$ with center $\mathring{c}$ and radius $r \leqslant b_1/2$. Similarly, $\tilde{R}_\gamma$ rotates $\tilde{x}$ on the boundary of a disk $\tilde{D}$ on $\tilde{H}$ with center $\tilde{c}$ and radius r. The distance $\|\mathring{x} - \tilde{x}\|$ is at most $\varepsilon c_1 \Delta_\beta/2$. Move $\mathring{D}$ to align the points $\mathring{c}$ and $\tilde{c}$ and also the points $\mathring{x}$ and $\tilde{x}$. Let D denote the moved $\mathring{D}$. Let y be the point on the boundary of D that correspoinds to $\tilde{R}_\gamma(\mathring{x})$. Let z be the point $\tilde{R}_\gamma(\tilde{x})$. By the triangle inequality, the distance between $\tilde{R}_\gamma(\mathring{R}_\beta(x))$ and $\tilde{R}_\gamma(\tilde{R}_\beta(x))$ is at most $\|\mathring{c} - \tilde{c}\| + \|\mathring{x} - \tilde{x}\| + \|y - z\| \leqslant \varepsilon c_1 \Delta_\beta + \|y - z\|$. Using the spherical sine law, one can show that $\|y - z\| \leqslant (\pi b_1/2)\varepsilon\Delta_\beta$ because $\tilde{R}_\gamma$ rotates by an angle of magnitude π or less. So the distance been $\tilde{R}_\gamma(\mathring{R}_\beta(x))$ and $\tilde{R}_\gamma(\tilde{R}_\beta(x))$ is at most $((\pi/2) \cdot b_1 + c_1)\varepsilon\Delta_\beta < (2b_1 + c_1)\varepsilon\Delta_\beta$. The above relation holds for every point $x \in P_1$, which means that the Hausdorff distance between $\tilde{R}_\gamma(\mathring{R}_\beta(P_1))$ and $\tilde{R}_\gamma(\tilde{R}_\beta(P_1))$ is at most

$l \leqslant (2b_1+c_1)\varepsilon\Delta_\beta$. We apply the Lemma 6 with $C = \tilde{R}_\gamma(\mathring{R}_\beta(P_1))$ and $C' = \tilde{R}_\gamma(\tilde{R}_\beta(P_1))$. Note that $(2b_1 + c)\varepsilon\Delta_\beta \leqslant 3\varepsilon a_{min}b_{min}c_{min}/(2^4 3^5 b_1 c_1) \leqslant 3\varepsilon a_{min}c_{min}/(2^4 3^5 c_1) \leqslant 3\varepsilon a_{min}/(2^4 3^5)$. Lemma 6 gives

$$\begin{aligned}
|\tilde{R}_\gamma(\mathring{R}_\beta(P_1)) \setminus \tilde{R}_\gamma(\tilde{R}_\beta(P_1))| &\leqslant \frac{4}{3}\pi l^3 + 2\pi b_1 c_1 l + 2\pi^2 c_1 l^2 \\
&< \varepsilon\pi(0.1\varepsilon^2 a_{min}^3 + 3a_{min}b_{min}c_{min} + 0.1\varepsilon a_{min}^2 c_{min})/(2^3 3^5) \\
&< \varepsilon\pi a_{\min}b_{\min}c_{\min}/(2\cdot 3^5).
\end{aligned}$$

Hence, $|\mathring{R}_\gamma(\mathring{R}_\beta(P_1)) \cap \mathring{T}(\mathring{R}_\alpha(P_2))| - |\tilde{R}_\gamma(\tilde{R}_\beta(P_1)) \cap \tilde{T}(\tilde{R}_\alpha(P_2))| < \varepsilon\pi a_{\min}b_{\min}c_{\min}/(2\cdot 3^4)$. Notice that $\frac{1}{3}\mathcal{E}(P_1) \cap \frac{1}{3}\mathcal{E}(P_2)$ lies inside $P_1 \cap P_2$ and has volume $\pi a_{\min}b_{\min}c_{\min}/(2\cdot 3^4)$. It follows that $|\mathring{R}_\gamma(\mathring{R}_\beta(P_1)) \cap \mathring{T}(\mathring{R}_\alpha(P_2))| - |\tilde{R}_\gamma(\tilde{R}_\beta(P_1)) \cap \tilde{T}(\tilde{R}_\alpha(P_2))|$ is at most ε times the maximum overlap of P_1 and P_2 under rigid motion. ◀

4 Main Algorithm

We use last section's result to sample a set of rotations $S_\beta \times S_\alpha \times S_\gamma$ from $I_\beta \times I_\alpha \times I_\gamma$. For each rotation $R_* \in S_\beta \times S_\alpha \times S_\gamma$, we want to compute the best translation to align $R_\gamma(R_\beta(P_1))$ and $R_\alpha(P_2)$, and then keep track of the rigid motion $M = (T, R_*)$ encountered so far that gives the largest overlap. For efficiency purpose, we compute the "almost best" translation using Theorem 8 below. Algorithm 1 shows the pseudocode of our algorithm.

▶ **Theorem 8** ([2]). *Let P_1 and P_2 be two convex polytopes in $\mathbb{R}^3$ specified by n bounding planes. For any $\mu > 0$, we can compute an overlap of P_1 and P_2 under translation that is at most μ less than the optimum. The running time is $O(n\log^{3.5} n)$ with probability $1 - n^{-O(1)}$.*

Algorithm 1 Maximum overlap approximation algorithm

```
procedure MaxOverlap(P_1, P_2, ε)            ▷ return (1 − ε)-optimal rigid motion
    Compute E(P_1) and E(P_2) and align their centers and the respective axes.
    Compute three sets of sampled angles S_β, S_α, and S_γ.
    ans := 0
    M := null
    for all rotation R_* ∈ S_β × S_α × S_γ do
        Compute the translation T to align R_γ(R_β(P_1)) and R_α(P_2) using Theorem 8
        if |R_γ(R_β(P_1)) ∩ T(R_α(P_2))| > ans then
            ans := |R_γ(R_β(P_1)) ∩ T(R_α(P_2))|
            M := (R_*, T)
        end if
    end for
    return M
end procedure
```

▶ **Theorem 9.** *Let P_1 and P_2 be convex polytopes in $\mathbb{R}^3$. Suppose that the maximum overlap of P_1 and P_2 under rigid motion is at least $\lambda \cdot \max\{|P_1|, |P_2|\}$ for some given constant $\lambda \in (0, 1]$. Given any $\varepsilon \in (0, 1/2)$, Algorithm 1 runs in $O(\varepsilon^{-3}\lambda^{-6} n\log^{3.5} n)$ time with probability at least $1 - n^{-O(1)}$ and returns a $(1-\varepsilon)$-approximate maximum overlap of P_1 and P_2 under rigid motion.*

Proof. The solution quality of algorithm 1 is guaranteed by Lemma 7. We analyze its running time as follows. First, it takes $O(n/\varepsilon)$ time to compute the ellipsoids $\mathcal{E}(P_1)$ and $\mathcal{E}(P_2)$. The remaining time spent by Algorithm 1 is $|S_\beta| \cdot |S_\alpha| \cdot |S_\gamma| \cdot n \log^{3.5} n$ with high probability. Thus, it suffices to bound $|S_\beta| \cdot |S_\alpha| \cdot |S_\gamma|$, which is $O\left(\varepsilon^{-3} \cdot |I_\beta||I_\alpha||I_\gamma| \cdot (\Delta_\beta\Delta_\alpha\Delta_\gamma)^{-1}\right)$.

Suppose that $2^{13}3^5 a_{\min} > \lambda^2 c_{\min}/\sqrt{2}$. Then $I_\xi = [-\pi, \pi]$ for $\xi \in \{\beta, \alpha, \gamma\}$. The assumption of $2^{13}3^5 a_{\min} > \lambda^2 c_{\min}/\sqrt{2}$ implies that $a_{\min}$, $b_{\min}$, and $c_{\min}$ are within constant factors of each other. Therefore, $\Delta_\beta\Delta_\alpha\Delta_\gamma = \Theta(1)$, which implies that $|S_\beta|\cdot|S_\alpha|\cdot|S_\gamma| = O(\varepsilon^{-3})$. Thus, the remaining time spent by Algorithm 1 is $O(\varepsilon^{-3} n \log^{3.5} n)$.

Suppose that $2^{13}3^5 a_{\min} \leqslant \lambda^2 c_{\min}/\sqrt{2}$. Then $|I_\beta| = O(a_{\min}/(\lambda^2 c_{\min}))$ and $\Delta_\beta = \Theta(a_{\min}/c_{\min})$, so $|I_\beta|/\Delta_\beta = O(\lambda^{-2})$. By definition, $|I_\alpha| = O(b_{\min}/(\lambda^2 c_{\min}))$ and $\Delta_\alpha = \Theta(b_{\min}/c_{\min})$, so $|I_\alpha|/\Delta_\alpha = O(\lambda^{-2})$. Similarly, $|I_\gamma| = O(a_{\min}/(\lambda^2 b_{\min}))$ and $\Delta_\gamma = \Theta(a_{\min}/b_{\min})$ by definition, which implies that $|I_\gamma|/\Delta_\gamma = O(\lambda^{-2})$. ◀

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