

Communication Complexity, Linear Optimization, and lower bounds for the nonnegative rank of matrices

Edited by

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Abstract

This report documents the program and the outcomes of Dagstuhl Seminar 13082 “Communication Complexity, Linear Optimization, and lower bounds for the nonnegative rank of matrices”.

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1 Executive Summary

LeRoy B. Beasley

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The nonnegative rank is a measure of the complexity of a matrix that has applications ranging from Communication Complexity to Combinatorial Optimization. At the time of the proposal of the seminar, known lower bounds for the nonnegative rank were either trivial (rank lower bound) or known not to work in many important cases (bounding the nondeterministic communication complexity of the support of the matrix).

Over the past couple of years in Combinatorial Optimization, there has been a surge of interest in lower bounds on the sizes of Linear Programming formulations. A number of new methods have been developed, for example characterizing nonnegative rank as a variant of randomized communication complexity. The link between communication complexity and nonnegative rank was also instrumental recently in proving exponential lower bounds on the sizes of extended formulations of the Traveling Salesman polytope, answering a longstanding open problem.

This seminar brought together researchers from Matrix Theory, Combinatorial Optimization, and Communication Complexity to promote the transfer of tools and methods between these fields. The focus of the seminar was on discussions, open problems and talks surveying the basic tools and techniques from each area.

In the short time since the seminar, its participants have made progress on a number of open problems.



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Editors: LeRoy B. Beasley, Hartmut Klauck, Troy Lee, and Dirk Oliver Theis



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Program Overview

Background lectures on the connection between matrix factorizations to Communication Complexity and to Combinatorial Optimization were given by the organizers. More importantly, a number of participants contributed their latest research on factorization ranks. In this section, we summarize these talks.

Extended Formulations and Linear Optimization

Hamza Fawzi

Many lower bounds on the nonnegative rank only make use of the zero/nonzero pattern of the matrix. For certain applications, in particular for the extended formulation size lower bounds for approximation problems, nonnegative rank lower bounds need to be shown for matrices that are strictly positive. Hamza discussed an interesting approach to nonnegative rank lower bounds via conic programming that does not only rely on the zero/nonzero structure of the matrix. The bound is in many ways analogous to the trace norm lower bound for rank, but making use of the stronger fact that the factorization is nonnegative leads to a copositive program rather than a semidefinite one. For computing the bound in practice, Hamza discussed ways to approximate the bound by semidefinite programs, and examples of using this in practice.

Sam Fiorini

There is a rich theory on the hardness of approximating NP-optimization problems up to certain factors, given complexity assumptions like $P \neq NP$. Very recently a similar topic has emerged in the study of polytopes. Sam talked about tradeoffs between the approximation ratio and the size of linear formulations. One notable result in Sam's talk was that approximating CLIQUE to within $n^{1/2-\epsilon}$ requires extended formulations of exponential size.

Complexity

Nati Linial

On the first day, Nati Linial treated us to a survey of higher dimensional analogs of familiar combinatorial objects. For example, we are very familiar with permutation matrices, those matrices with entries from $\{0, 1\}$ with exactly one 1 in every row and column, and know that there are $n! = ((1 + o(1))n/e)^n$ many of them. What about 3-dimensional tensors with entries from $\{0, 1\}$ and exactly one 1 along every row, column and shaft? Such 2-dimensional permutations turn out to coincide with latin squares and it is known that there are $((1 + o(1))n/e^2)^{n^2}$ many of them. This relies on some beautiful work on the minimum permanent of doubly stochastic matrices. Nati conjectures that the formula generalizes to count the number of d -dimensional permutations, described by a $d+1$ -tensor with one 1 along every line. That is, that the number of d -dimensional permutations is $((1 + o(1))n/e^d)^{n^d}$. He is able to show such an upper bound, but the lower bound remains open.

Sebastian Pokutta

In order to prove that extended formulations for approximating optimization problems need to be large, communication and information complexity are important tools. In his talk Sebastian described a new approach on how to prove lower bounds on the nonnegative rank of matrices corresponding to the unique disjointness problem when perturbed. He gave tight lower bounds using a new information theoretic fooling set method.

Since the seminar, Sebastian and his co-author Gabór Braun have made available a preprint containing these results [3].

Hans Raj Tiwary

There are entire books of NP-complete problems and explicit reductions between them. For the extension complexity of the associated polytopes, however, this book is still slowly being written—usually by arguing that P is a projection of Q or finding P as a face of Q . Hans discussed the intriguing possibility of automatically turning an NP gadget reduction into a polytope reduction. While still not a general theory, Hans can currently do this for many NP-hard problems and their associated polytopes.

Nicolas Gillis

Nicolas spoke about the problem of actually computing a non-negative factorization of a nonnegative matrix. This talk was important to seminar participants on small matrices, allowing them to test the quality of their lower bounds against upper bounds. On small matrices, these upper bounds can be found computationally. The problem also has applications to compression of images, to identifying topics in documents, even to identifying the mineral composition of rocks from spectral data (hyper-spectral imaging). Nicolas discussed specifically the case of separable matrices. An n -by- n matrix M is r -separable if it has a factorization $M = WH$ where W is n -by- r , H is r -by- N and moreover W is a subset of the columns of M . Such types of factorization can be more useful in practice. Nicolas talked about a linear programming approach to this problem that is polynomial time and moreover outperforms previous approaches in practice.

Matrix Theory

Alexander Guterman

Alexander Guterman gave a survey talk on various matrix ranks over semirings. A big focus was on tropical algebra over the real number with operations $a \oplus b = \max a, b$ and $a \otimes b = a + b$. Tropical algebra provides a way of formulating many hard combinatorial optimization problems (like scheduling problems) in terms of a very elegant linear algebraic type language. In tropical linear algebra there are varying notions of linear independence, for example Gondran-Minoux independence, weak linear independence, and strong linear independence. Each of these gives rise to a different notion of rank of a matrix and a hierarchy of these ranks is known.

Yaroslav Shitov

Yaroslav continued talking about tropical matrix rank, in particular the tropical factorization rank. This is defined as the minimum k such that $A = B \otimes C$ for a n -by- k matrix B and k -by-

n matrix C . Note that in tropical matrix multiplication $(B \otimes C)(i, j) = \min_t B(i, t) + C(t, j)$. Yaroslav mentioned a very interesting application of the tropical factorization rank. Say that we are given an instance of the traveling salesman problem, with distances specified by a matrix A , and moreover we are given a tropical factorization $A = B \otimes C$ that witnesses that A has constant factorization rank. Then the resulting traveling salesman instance can be solved in polynomial time! This is a result of Barvinok, Johnson, Woeginger, and Woodroofe. Yaroslav also showed that the problem of detecting if the tropical factorization rank of a matrix is at most 8 is NP-hard.

Richard Robinson

In his talk, Richard Robinson gave a characterization, among all nonnegative matrices, of the extreme-ray / facet slack matrices of polyhedral cones, and vertex/facet slack matrices of polytopes. This characterization leads to an algorithm for deciding whether a given matrix is a vertex/facet slack matrix. The underlying decision problem is equivalent to the polyhedral verification problem whose complexity is unknown.

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3 Overview of Talks

3.1 New lower bounds on nonnegative rank using conic programming

Hamza Fawzi (MIT – Cambridge, US)

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Joint work of Fawzi, Hamza; Parrilo, Pablo

Main reference H. Fawzi, P.A. Parrilo, “New lower bounds on nonnegative rank using conic programming,” arXiv:1210.6970v1 [math.OC], 2012.

URL <http://arxiv.org/abs/1210.6970v1>

We propose a new lower bound on the nonnegative rank which, unlike most existing lower bounds, does not explicitly rely on the matrix sparsity pattern and applies to nonnegative matrices with arbitrary support. Our lower bound is expressed as the solution of a copositive programming problem and can be relaxed to obtain polynomial-time computable lower bounds using semidefinite programming. The idea involves computing a certain nuclear norm with nonnegativity constraints which allows to lower bound the nonnegative rank, in the same way the standard nuclear norm gives lower bounds on the standard rank. We compare our lower bound with existing ones, and we show examples of matrices where our lower bound performs better than currently known ones.

3.2 Approximation Limits of Linear Programs (Beyond Hierarchies)

Samuel Fiorini (University of Brussels, BE)

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Joint work of Braun, Gabor; Fiorini, Samuel; Pokutta, Sebastian; Steurer, David

Main reference G. Braun, S. Fiorini, S. Pokutta, D. Steurer, “Approximation Limits of Linear Programs (Beyond Hierarchies),” arXiv:1204.0957v2 [cs.CC], 2013.

URL <http://arxiv.org/abs/1204.0957v2>

We develop a framework for proving approximation limits of polynomial-size linear programs from lower bounds on the nonnegative ranks of suitably defined matrices. This framework yields unconditional impossibility results that are applicable to any linear program as opposed to only programs generated by hierarchies. Using our framework, we prove that $O(n^{1/2-\epsilon})$ -approximations for CLIQUE require linear programs of size $2^{n^{\Omega(\epsilon)}}$. (This lower bound applies to linear programs using a certain encoding of CLIQUE as a linear optimization problem.) Moreover, we establish a similar result for approximations of semidefinite programs by linear programs.

Our main technical ingredient is a quantitative improvement of Razborov’s rectangle corruption lemma (1992) for the high error regime, which gives strong lower bounds on the nonnegative rank of certain perturbations of the unique disjointness matrix.

3.3 Robust Near-Separable Nonnegative Matrix Factorization Using Linear Optimization

Nicolas Gillis (UC Louvain-la-Neuve, BE)

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Main reference N. Gillis, R. Luce, “Robust Near-Separable Nonnegative Matrix Factorization Using Linear Optimization,” arXiv:1302.4385v1 [stat.ML], 2013.

URL <http://arxiv.org/abs/1302.4385v1>

Nonnegative matrix factorization (NMF) has been shown recently to be tractable under the separability assumption [1], which amounts for the columns of the input data matrix to belong to the convex cone generated by a small number of columns. Since then, several algorithms have been proposed to handle this subclass of NMF problems under any small perturbation of the input matrix, see for example [2] and the references therein. In particular, [3] proposed a linear programming (LP) model, referred to as HottTopixx; see also [4]. However, HottTopixx has two important drawbacks: (i) the input matrix has to be normalized, and (ii) the factorization rank has to be known in advance. In [5], we generalize HottTopixx in order to resolve these two drawbacks, that is, we propose a new LP model which does not require normalization and detects the factorization rank automatically. Moreover, the new LP model is more flexible, significantly more tolerant to noise, and can easily be adapted to handle outliers and other noise models. Finally, we show on several synthetic datasets that it outperforms HottTopixx while competing favorably with two state-of-the-art methods.

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- 4 N. Gillis, ‘Robustness Analysis of HottTopixx, a Linear Programming Model for Factoring Nonnegative Matrices’, arXiv:1211.6687.
- 5 N. Gillis and R. Luce, ‘Robust Near-Separable Nonnegative Matrix Factorization Using Linear Optimization’, <http://arxiv.org/abs/1302.4385>.

3.4 On the Geometric Interpretation of the Nonnegative Rank

Francois Glineur (UC Louvain, BE)

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Joint work of Gillis, Nicolas; Glineur, Francois

Main reference On the Geometric Interpretation of the Nonnegative Rank, Nicolas Gillis, François Glineur, Linear Algebra and its Applications, Volume 437, Issue 11 (1 December 2012), Elsevier.

A geometric bound for the nonnegative rank Nicolas Gillis and François Glineur, UCLouvain (Belgium)

We start with a brief introduction to the nonnegative rank and recall what is currently known about its computational complexity. We then present a geometric view of the nonnegative rank computation, which leads us to introduce the concept of restricted nonnegative rank. This allows us to derive a new lower bound for the nonnegative rank. In particular, it

is easily computable for slack matrices, and provides a lower bound on the size of extended formulations for any d -polytope with a given number of facets and vertices. To conclude, we report the results of recent computational experiments attempting to factorize numerically slack matrices of low-dimensional polytopes.

Paper about the first half of the talk: On the Geometric Interpretation of the Nonnegative Rank, Nicolas Gillis, François Glineur, *Linear Algebra and its Applications*, Volume 437, Issue 11 (1 December 2012), Elsevier. 10.1016/j.laa.2012.06.038

Code to attempt factorisation of slack or other nonnegative matrices
<http://sites.google.com/site/nicolasgillis/code> (direct link <http://bit.ly/13hvCA2>)

3.5 Matrix ranks over semirings

Alexander Guterman (Moscow State University, RU)

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Tropical algebra (sometimes called max algebra) is a set of real numbers equipped with the maximum operation instead of usual addition and addition instead of usual multiplication. Under these operations this is an algebraic structure called a semiring. The other typical examples of such structures are non-negative integers, non-negative reals, boolean algebras. Semirings naturally appear in different problems of communication complexity, scheduling theory, optimization, dynamical systems, etc. Semiring arithmetics allows to reduce non-linear problems to the linear problems but over semirings. To investigate these problems it is necessary to develop linear algebra over semirings. This subject is very actual nowadays. Different rank functions over various classes of semirings are intensively investigated during the last decades. We plan to introduce and investigate some of them, in particular, factor rank, tropical rank, nonnegative rank, determinantal rank, Gondran-Minoux rank. We plan to compare these functions and discuss their interrelations. Among the other topics we shall discuss our recent joint research results with Marianne Akian, LeRoy Beasley, Stéphane Gaubert, and Yaroslav Shitov.

3.6 Constructing Extended Formulations for Stable Set Polytopes via Decomposition Rules

Kanstantsin Pashkovich (University of Padova, IT)

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 Joint work of Conforti, Michele; Gerards, Bert

We develop decomposition/composition tools for describing stable set polytopes as polynomially sized linear programs. Some of these tools are well-known but need some extra work to yield polynomial “decomposition schemes”. We apply the tools to graphs that contain no even hole and no cap.

3.7 On lower bounds for extended formulations

Sebastian Pokutta (Universität Erlangen-Nürnberg, DE)

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Joint work of Braun, Gabor; Pokutta, Sebastian

Main reference G. Braun, S. Pokutta, “Common information and unique disjointness,” ECCC TR13-056, 2013.

URL <http://eccc.hpi-web.de/report/2013/056/>

Communication complexity and information theoretic approaches have been at the core of many recent lower bound proofs for the size of extended formulations of certain polytopes. One of the most important problems in this context is the unique disjointness problem which is closely related to the extension complexity of the correlation polytope and (a natural encoding of) the clique problem. We will provide an overview of recent results by Braun, Fiorini, Pokutta and Steurer as well as those of Braverman and Moitra. Based on the BM approach we then present a generalized information theoretic framework which decouples polyhedral/geometric aspects from the underlying combinatorics and opens up several routes for establishing more general lower bounds.

3.8 Which nonnegative matrices are slack matrices?

Richard Robinson (University of Washington, US)

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Joint work of Gouveia, João; Grappe, Roland; Kaibel, Volker; Pashkovich, Kanstantsin; Robinson, Richard Z.; Thomas, Rekha R.

Main reference J. Gouveia, R. Grappe, V. Kaibel, K. Pashkovich, R.Z. Robinson, R.R. Thomas, “Which Nonnegative Matrices Are Slack Matrices?,” arXiv:1303.5670v1 [math.OC], 2013.

URL <http://arxiv.org/abs/1303.5670v1>

In this note we characterize the slack matrices of cones and polytopes among all nonnegative matrices. This leads to an algorithm for deciding whether a given matrix is a slack matrix. The underlying decision problem is equivalent to the polyhedral verification problem whose complexity is unknown.

3.9 Matrix factorization over semirings

Yaroslav Shitov (Moscow State University, RU)

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Joint work of Guterman, Alexander; Shitov, Yaroslav

My recent work in matrix theory has been devoted to studying matrix factorizations over nonnegative numbers, and also over tropical and other semirings. The problem of factoring tropical matrices is useful in tropical geometry as well as finds applications in optimization and phylogenetics. Thus a question of describing the computational complexity of tropical factorization arises. That question has been answered recently, and it has been shown that tropical factorization is hard. I also studied different problems on nonnegative matrix factorizations, and a number of techniques from semiring linear algebra allowed to make some progress on those problems. The behavior and computational complexity of rank functions on tropical matrices have been discussed also in our recent joint paper with Alexander Guterman.

3.10 On the extension complexity of combinatorial polytopes

Hans Raj Tiwary (University of Brussels, BE)

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Joint work of Avis, David; Tiwary, Hans Raj

Main reference D. Avis, H.R. Tiwary, “On the extension complexity of combinatorial polytopes,”
arXiv:1302.2340v2 [math.CO], 2013.

URL <http://arxiv.org/abs/1302.2340v2>

In this talk I will describe a lifting argument to show exponential extension complexity for a number of NP-complete problems including subset-sum and three dimensional matching. We obtain a relationship between the extension complexity of the cut polytope of a graph and that of its graph minors. Using this we are able to show exponential extension complexity for the cut polytope of a large number of graphs, including those used in quantum information and suspensions of cubic planar graphs.

4 Problem discussion sessions, and subsequent developments

Here we report on the status of questions which were presented during the Problem Sessions. Near the end of this section, we discuss developments on problems, which were not presented during the Problem Sessions, but discussed during the seminar.

4.1 Real vs. rational nonnegative rank

Presented by Dirk Oliver Theis; problem based on a problem by Cohen and Rothblum from 1991. Give a non-trivial bound for $\text{rk}_{\mathbb{Q}_+}(A) - \text{rk}_{\mathbb{R}_+}(A)$! For example, is it true that for every rational nonnegative matrix A we have $\text{rk}_{\mathbb{Q}_+}(A) \leq \text{rk}_{\mathbb{R}_+}(A) + 1$?

The original question asks for equality between the two ranks, but currently no non-trivial bounds for $\text{rk}_{\mathbb{Q}_+}(A) - \text{rk}_{\mathbb{R}_+}(A)$ or even $\text{rk}_{\mathbb{Q}_+}(A) / \text{rk}_{\mathbb{R}_+}(A)$ are known.

In the discussion, Nati Linial pointed to Micha Perles discovery of non-rational polytopes, and the studies by Richter-Gebert and others of the realization spaces of polytopes.

A related problem is the following.

Complex vs. real positive semidefinite rank. One can ask a similar question for the positive semidefinite rank. The positive semidefinite rank over \mathbb{R} of a matrix $A \in \mathbb{R}^{m \times n}$ is the minimal r such that there are $B_i \in \mathbb{R}^{r \times r}$ for $i = 1, \dots, m$ and $C_j \in \mathbb{R}^{r \times r}$ for $j = 1, \dots, n$ such that $A(i, j) = \text{Tr}(B_i^* C_j)$. The positive semidefinite rank over \mathbb{C} is defined analogously with $B_i, C_j \in \mathbb{C}^{r \times r}$. Is the positive semidefinite rank over \mathbb{R} equal to positive semidefinite rank over \mathbb{C} ? This is the simplest in a family of questions: The same needs to be asked for rational vs. real positive semidefinite rank. In the discussion, Nicolas Gillis pointed out that this question also must be settled for the copositive ranks. In this case one looks for a factorization $A(i, j) = \text{Tr}(B_i^* C_j)$ where each B_i is copositive and each C_j is completely positive.

4.2 Square-root rank

Proposed by Richard Robinson. Given nonnegative matrix $M \in \mathbb{R}_+^{p \times q}$, we say that $A \in \mathbb{R}^{p \times q}$ is a Hadamard square-root if $(A)_{ij}^2 = M_{ij}$. Define $\text{rk}_{\sqrt{\cdot}}(M)$ as the minimum rank of A such that A is Hadamard square-root of M . This is equivalent to a version of the positive semidefinite rank where the matrices in the factorization are constrained to have rank 1. Hence, $\text{rk}_{PSD}(M) \leq \text{rk}_{\sqrt{\cdot}}(M)$ holds. For vertex-facet slack matrices of polytopes, $\text{rk}(S) \leq \text{rk}_{\sqrt{\cdot}}(S)$. A large number of observations have led Richard to conjecture the following.

If $\text{rk}(S) = \text{rk}_{\sqrt{\cdot}}(S)$ holds for the slack matrix S of a polytope, then the entries in the hadamard square-roots in $\text{rk}_{\sqrt{\cdot}}(S)$ can be taken to be nonnegative.

In the discussion, Samuel Fiorini suggested to look specifically at the matrix $M_{ab} = (1 - a^T b)^2$.

4.3 Positive semidefinite rank of matrices defined by polynomials

Proposed by Troy Lee. What is the positive semidefinite rank of the matrix

$$M(x, y) = (x^t y - 1)(x^t y - 2),$$

where x, y range over all $\{0, 1\}^n$?. The motivation is that such a matrix M is a submatrix of the slack matrix of the correlation polytope. One can define a whole family of submatrices of the slack matrix of the correlation polytope by taking a quadratic polynomial p which is nonnegative on nonnegative integers, and letting $M(x, y) = p(|x \cap y|)$. In the discussion, Sam Fiorini pointed out that to show a strong lower bound on M one would have to focus on more than just the entries of the matrix which take values in some small set. He also mentioned that this matrix can be approximated by one that does have low positive semidefinite rank, namely the matrix $N(x, y) = (x^t y - 3/2)^2$.

A toy version of this problem asks about the positive semidefinite rank of the n -by- n matrix $M_n(i, j) = (i - j - 1)(i - j - 2)$. Seminar participant João Gouveia [8] answered a question of Lee and Theis [11] by showing that the psd rank of M_n goes to infinity with n .

4.4 A query complexity problem

Proposed by Raghav Kulkarni. For $f: \{0, 1\}^n \rightarrow \{0, 1\}$ and $z \in \{0, 1\}^n$, we say that i th bit of z is *sensitive* if $f(z_1, \dots, \bar{z}_i, \dots, z_n) \neq f(z_1, \dots, z_i, \dots, z_n)$. Let $s(f, z)$ be the number of sensitive bits of z and $s(f) = \max_z \{s(f, z)\}$ the maximum number of sensitive bits of any argument. These concepts arise in the context of decision tree complexity. For $x, y \in \{0, 1\}^n$ let $f(x, y)$ be a 2-parameter function, and let $M_f(x, y) = f(x, y)$ be the corresponding matrix. Raghav conjectures that $\log \text{rk}_{\mathbb{R}}(M_f) \leq \text{poly}(s(f))$.

In the following discussion, Hartmut Klauck asked about block sensitivity and Raghav said the conjecture is true with sensitivity replaced by block sensitivity. Hartmut also suggested easier versions of the conjecture where, for example, the rank is replaced by sign rank which is the minimum rank of a matrix that entrywise agrees with the target matrix in sign. Nati Linial then asked if assuming the log-rank conjecture is true implies anything for this conjecture.

4.5 A rigidity-type question

Proposed by Adi Shraibman. The famous matrix rigidity problem of Valiant asks to explicitly construct matrices with high rank and such that if a constant fraction of the entries are arbitrarily changed, the rank remains high. While probabilistic constructions exist, finding explicit constructions remains a hard open problem.

Consider the following variant of the problem known as discrepancy games. Here you start with an empty $n \times n$ matrix. Two players, Balancer (+1) and Unbalancer (−1), take turns assigning entries of the matrix to their associated value. Balancer wants to make all combinatorial rectangles balanced, while Unbalancer wants to make them unbalanced. In this game it is known that Balancer can get ensure an upper bound of $s^{3/4}$ on discrepancy after s rounds.

Here is another variant that is open. In this case we begin with a $\{-1, +1\}$ valued matrix with discrepancy $n^{3/2}$. Say a Hadamard matrix. Balancer picks certain +1's. Unbalancer picks certain −1's. Over the course of the game, can Balancer maintain discrepancy to be less than $s^{3/4}$ after s moves? In this variation you cannot rely on strategy stealing.

4.6 Extension complexity of stable set polytopes of split-graph-free perfect graphs

Proposed by Samuel Fiorini. A split graph is a graph in which the vertices can be partitioned into a clique and an independent set. Let H be a split graph, and consider the class of all graphs G not containing H as an induced subgraph. What is the extension complexity of the stable set polytopes of this class of graphs? This problem is motivated by a recent result of Bousquet, Lagoutte, and Thomassé [2]. They provide a certificate of size $O(\log(n))$ proving that a clique and an independent set do not intersect for H -free graphs.

4.7 Matrices with low non-negative rank are a low-dimensional subset in the manifold of rank-bounded matrices

It is an easy fact that the nonnegative rank is semicontinuous: If A_j tends to A , then $\text{rk}_+(A) \leq \liminf \text{rk}_+(A_j)$. Let n, k be nonnegative integers such that $3 \leq k \ll n$, and consider the set of $n \times n$ matrices

$$\{A \in \mathbb{R}_+^{n \times n} \mid \text{rk } A = k, \text{rk}_+ A \leq n - 1\}.$$

Does this set contain interior points within the manifold $\{A \in \mathbb{R}_+^{n \times n} \mid \text{rk } A = k\}$ of nonnegative rank- k matrices? (The “ $n - 1$ ” is somewhat arbitrary, and should be replaced by an appropriate function of n .) This question asks for the dimension of the set of matrices of “small” nonnegative rank as a semialgebraic subset of the variety of rank- k $n \times n$ matrices. In the discussions, people expressed that the intuitively obvious answer to the question is yes. But a recent result of Yaroslav Shitov proves that, for $k = 3$, *every* such matrix A has nonnegative rank at most $6n/7$ [14]. However, the question with “ $n - 1$ ” may still be true for large k .

4.8 Fooling-sets and rank

Let A be an $n \times n$ matrix over a field \mathbb{K} satisfying $A_{kk} = 1$ for all k and $A_{k\ell}A_{\ell k} = 0$ whenever $k \neq \ell$. Dietzfelbinger et al. (1996) proved that $n \leq \text{rk}(A)^2$. The question raised by Dietzfelbinger et al. is whether this bound is asymptotically ($n \rightarrow \infty$) tight.

This problem was fully settled by Friesen and Theis in the case of nonzero characteristic shortly before the seminar [7]. In summary, the following is known.

	characteristic of \mathbb{K}		
	0	2	≥ 3
A 0/1 entries	open	tight	open
A arbitrary entries	open	tight	tight

The major open question is when the characteristic is 0 and the entries are arbitrary. At the time of the seminar, the best separation was by Klauck and de Wolf [10] who gave an example where $\text{rk}(A) \leq n^{0.613\dots}$ with integral entries of small modulus. After the seminar, in the case of characteristic zero, Troy Lee was able to improve the best known bound to $\text{rk}(A) \leq n^{0.594\dots}$ with a method that warrants future investigation.

4.9 Polygons — or, more generally, rank-3 matrices

A problem which was discussed intensely during the seminar was the extension complexity of polygons, where, at the time of the seminar, a lower bound of $\Omega(\sqrt{n})$ was known, and the trivial upper bound n .

The following problem by Beasley & Laffey [1] is both more general and more specific: Given any sub-semiring S of \mathbb{R}_+ and $n \geq 6$, is there a matrix $A \in \mathbb{M}_{n \times n}(S)$ such that $\text{rk} A = 3$ and $\text{rk}_S(A) = n$? At the time of the seminar, this was open even for $S = \mathbb{R}_+$. Following the seminar, participant Yaroslav Shitov [14] has settled this problem for the semiring \mathbb{R}_+ : every rank-3 nonnegative $n \times n$ matrix with $n \geq 7$ has nonnegative rank at most $6n/7$.

4.10 Euclidean distance matrices

Do “generic”/“random” Euclidean distance matrices¹ have full nonnegative rank?²

For $d = 1$, Shitov’s above mentioned result gives a negative answer to the question.

4.11 How does nonnegative rank behave under tensor products?

This question is interesting on its own, and also has application to communication complexity. A very strong conjecture, discussed at the workshop, would be that the nonnegative rank is multiplicative, as the rank is. That is, that $\text{rk}_+(A \otimes B) = \text{rk}_+(A) \text{rk}_+(B)$.

Collaboration between two seminar participants, Nicolas Gillis and Hamza Fawzi showed that this strong conjecture is false. Specifically, Nicolas wrote software for computing the

¹ A d -dimensional, Euclidean distance matrix of size n is defined by points x_1, \dots, x_n in d -dimensional Euclidean space. Its entries are $(\|x_k - x_\ell\|)_{k,\ell}$.

² An affirmative proof of this for $d = 1$ by Lin and Chu (2011) is fatally flawed.

nonnegative rank of small-size matrices, and using this software Hamza was able to disprove the conjecture.

4.12 Vertex/facet slack matrices vs. general nonnegative matrices

Some lower bounds can be proved for matrices which arise from vertex/facet slack matrices of polytopes. It is an open question whether some of the bounds behave fundamentally different in the case of vertex/facet slack matrices (cf. e.g., 4.2). As a first step towards resolving this type of questions, a linear-algebraic characterization of these matrices was obtained by João Gouveia, Richard Robinson, and Rekha Thomas, and presented during the seminar. It turned out that Volker Kaibel, Roland Grappe, and Kanstantsin Pashkovich had a similar approach. Their results were combined in the recent paper [9].

4.13 Extensions/factorizations over the positive semidefinite cone

In the year preceding the seminar, matrix factorizations over other cones than $(\mathbb{R}_+)^r$ have gained importance, specifically the cone of positive semidefinite matrices. From the combinatorial optimization point of view, a very basic question there is, whether there exist polytopes whose extension complexity is exponential in the dimension — the same question for $(\mathbb{R}_+)^r$ was settled by Rothvoß [13]. In a very recent paper, seminar participant Sebastian Pokutta, together with two coauthors, Jop Briët and Daniel Dadush, have answered that question in the affirmative [4].

In the context of factorizations over other cones, recently, the conference participants Samuel Fiorini and Hans Raj Tiwary [6] have observed that a 2012 theorem by Alexander Maksimenko [12] together with a result by Samuel Burer [5] implies that every 0/1 polytope whose vertex set can be described by a polynomial predicate has a polynomial sized copositive extension.

5 Conclusion

A natural approach to solving hard combinatorial optimization problem is to give a formulation as a linear program and solve it using standard techniques. An important topic initiated by Yannakakis is to investigate the size of extended formulations of optimization problems.

Recently the theory of representing hard optimization problems via extended formulations has seen much progress. Techniques from communication complexity and matrix theory have been essential to investigate how large extended formulations need to be, finally improving on Yannakakis' seminal results.

The seminar brought together researchers from the areas of optimization theory, complexity theory, and matrix theory, to further collaboration on these and newly emerging topics. Exciting progress was reported on proving lower bounds for the nonnegative rank, on the hardness of approximation using extended formulations, and on new notions of matrix ranks.

For the future we hope that similar progress will soon be made on the topic of using semidefinite programming to solve hard optimization problems. Intriguingly this problem is connected to quantum communication complexity.

Dagstuhl provided a wonderful environment for many informal discussions as well as talks, plus an exciting open problems session. The opportunity to have this seminar was well appreciated by the participants, many of them who were new to the center.

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