

Provable Advantage for Quantum Strategies in Random Symmetric XOR Games*

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Abstract

Non-local games are widely studied as a model to investigate the properties of quantum mechanics as opposed to classical mechanics. In this paper, we consider a subset of non-local games: symmetric XOR games of n players with 0-1 valued questions. For this class of games, each player receives an input bit and responds with an output bit without communicating to the other players. The winning condition only depends on XOR of output bits and is constant w.r.t. permutation of players.

We prove that for almost any n -player symmetric XOR game the entangled value of the game is $\Theta\left(\frac{\sqrt{\ln n}}{n^{1/4}}\right)$ adapting an old result by Salem and Zygmund on the asymptotics of random trigonometric polynomials. Consequently, we show that the classical-quantum gap is $\Theta(\sqrt{\ln n})$ for almost any symmetric XOR game.

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1 Introduction

Non-local games provide a simple way to test the difference between quantum mechanics and the classical world. A prototypical example of a non-local game is the CHSH game [6] (based on the CHSH inequality of [5]). In the CHSH game, we have two players who cannot communicate between themselves but may share common random bits or a bipartite quantum state (which has been exchanged before the beginning of the game). A referee sends one uniformly random bit $a \in \{0, 1\}$ to the 1st player and an independent uniformly random bit $b \in \{0, 1\}$ to the 2nd player. Players respond by sending one-bit answers $x, y \in \{0, 1\}$. They win in the following 2 cases:

- (a) If at least one of a, b is equal to 0, players win if they produce x, y such that $x = y$;
- (b) If $a = b = 1$, players win if they produce x, y such that $x \neq y$;

Classically, CHSH game can be won with probability at most 0.75. In contrast, if players use an entangled quantum state, they can win the game with probability $\frac{1}{2} + \frac{1}{2\sqrt{2}} = 0.85\dots$

Other non-local games can be defined by changing the number of players, the number of possible questions and answers and the winning condition. Many non-local games have been studied and, in many cases, strategies that use an entangled quantum state outperform any classical strategy.

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Recently [1], it has been shown that, for a large class of non-local games, quantum strategies are better than any classical strategy for almost all games in this class. Namely, [1] considered 2-player games in which the questions a, b are taken from the set $\{1, 2, \dots, n\}$ and the winning condition is either $x = y$ or $x \neq y$, depending on a, b . (Games with a winning condition of such form are called XOR games.) [1] showed that, for $1 - o(1)$ fraction of all such games, the entangled value of the game is at least 1.2... times its classical value.

Then [2], it was discovered that a similar effect might hold for another class of games: n -player symmetric XOR games with binary questions. Namely, [2] showed a gap between entangled and classical values of order $\Omega(\sqrt{\log n})$ - assuming that a non-rigorous argument about the entangled value is correct.

In this paper, we make this gap rigorous, by proving upper and lower bounds on the entangled value of a random game in this class. We show that, with a high probability, the entangled value is equal to $\Theta\left(\frac{\sqrt{\log n}}{n^{1/\mu}}\right)$. The quantum-vs-classical gap of $\Theta(\sqrt{\log n})$ follows by combining this with the fact that the classical value is of the order $\Theta\left(\frac{1}{n^{1/\mu}}\right)$ for almost any random game (shown in [2]).

To prove this result, we use an expression for the entangled value of a symmetric n -player XOR game with entangled answers from [3]. This expression reduces finding the entangled value to maximizing the absolute value of a polynomial in one complex variable. If conditions for the XOR game are chosen at random, this expression reduces to random trigonometric polynomials studied in [7].

Although maxima of random trigonometric polynomials have been studied in [7], they have been studied under different conditions. For this reason, we cannot apply the results from [7] directly. Instead, we adapt the ideas from [7] to prove a bound on maxima of random trigonometric polynomials that would be applicable in our setting.

2 Definitions

A non-local game with n players proceeds as follows:

- 1) Players are separated so that they cannot communicate – hence the name *non-local*,
- 2) The players receive inputs $x_1, x_2, \dots, x_n \in I$ where I is the set of possible inputs. i -th player receives x_i ,
- 3) The players respond with outputs $y_1, y_2, \dots, y_n \in O$ where O is the set of possible outputs.
- 4) The winning condition $P(x_1, \dots, x_n, y_1, \dots, y_n)$ is consulted to determine whether the players win or lose. The condition is known to everyone at the start of the game.

The players are informed of the rules of the game and they can agree upon a strategy and exchange other information. In the classical case players may only use shared randomness. In the quantum case they can use an entangled quantum state which is distributed to the players before the start of the game.

We will restrict ourselves to the case when $I = O = \{0, 1\}$ and the vector of inputs (x_1, \dots, x_n) is chosen uniformly at random. In an XOR game, the winning condition $P(x_1, \dots, x_n, y_1, \dots, y_n)$ depends only on x_1, \dots, x_n and the parity of the output bits $\bigoplus_{j=1}^n y_j$. A game is *symmetric* if the winning condition does not change if x_1, \dots, x_n are permuted.

The winning conditions of a symmetric XOR game can be described by a list of $n+1$ bits: $G = (G_0, G_1, \dots, G_n)$, where the players win if and only if $G_i = \bigoplus_{j=1}^n y_j$ when $\sum_{j=1}^n x_j = i$.

The *entangled value* of the game $Val_Q(G)$ is the probability of winning minus the probability of losing in the conditions that the players can use a shared quantum-physical system.

In this paper, we study the value of symmetric XOR games when the winning condition G is chosen randomly from the uniform distribution of all $(n + 1)$ -bit lists. We use the following lemma (which follows from a more general result by Werner and Wolf for non-symmetric XOR games [9]):

► **Lemma 1** (See [3]). *The entangled value of a symmetric XOR game [3] is*

$$\text{Val}_Q(G) = \max_{|\lambda|=1} \left| \sum_{j=0}^n (-1)^{G_j} p_j \lambda^j \right| \quad (1)$$

where p_j is the probability that players are given an input vector (x_1, \dots, x_n) with j variables $x_i = 1$.

In our case, since (x_1, \dots, x_n) is uniformly random, we have $p_j = \frac{\binom{n}{j}}{2^n}$.

In the following sections we introduce additional notation to keep the proofs more concise as well as to keep in line with the original proofs in [7]:

The *Rademacher system* is a set of functions $\{\varphi_m(t)\}$ for $m = 1, \dots, n$ over $0 \leq t \leq 1$ such that $\varphi_m(t) = (-1)^k$, where k is the m -th digit after the binary point (in the fractional part of t) of the binary expansion of t . Rademacher system will turn out to be a convenient way to state that $\{G_j\}$ are random variables that follow a uniform distribution: if t is chosen randomly from a uniform distribution on $0 \leq t \leq 1$, then $\{\varphi_m(t)\}_{m=1}^{n+1}$ generates a uniformly random element from $\{+1, -1\}^{n+1}$. That in turn corresponds to coefficients $(-1)^{G_j}$ in eq. (1) being picked randomly.

Furthermore, we define

$$r_m = \binom{n}{m} \quad (n \text{ will be clear from context}),$$

$$R_n = \sum_{m=0}^n r_m^2,$$

$$T_n = \sum_{m=0}^n r_m^4,$$

$$P_n(x, t) = \sum_{m=0}^n r_m \varphi_{m+1}(t) \cos mx,$$

$$M_n(t) = \max_{0 \leq x < 2\pi} |P_n(x, t)|.$$

3 Main Result

By adapting the work of Salem and Zygmund [7] on the asymptotics of random trigonometric polynomials, we show

► **Theorem 2.**

$$\lim_{n \rightarrow \infty} \Pr[M_n(t) \geq C_1 \sqrt{R_n \ln n}] = 1.$$

► **Theorem 3.**

$$\lim_{n \rightarrow \infty} \Pr[M_n(t) \leq C_2 \sqrt{R_n \ln n}] = 1.$$

Our proof yields $C_1 = \frac{1}{4\sqrt{3}}$ and $C_2 = 2$.

We will now show how these two theorems lead to an asymptotic bound for the entangled value of a random game.

► **Corollary 4.** *For almost all n -player symmetric XOR games the entangled value of the game is $\Theta\left(\frac{\sqrt{\ln n}}{n^{1/4}}\right)$.*

Proof. From Lemma 1,

$$\text{Val}_Q(G) \geq \max_{|\lambda|=1} \left| \Re \left(\sum_{j=0}^n \frac{(-1)^{G_j} \binom{n}{j} \lambda^j}{2^n} \right) \right| = \max_{\alpha \in [0; 2\pi]} \left| \sum_{j=0}^n \frac{(-1)^{G_j} \binom{n}{j} \cos j\alpha}{2^n} \right|,$$

and

$$\begin{aligned} \text{Val}_Q(G) &\leq \max_{|\lambda|=1} \left| \Re \left(\sum_{j=0}^n \frac{(-1)^{G_j} \binom{n}{j} \lambda^j}{2^n} \right) \right| + \max_{|\lambda|=1} \left| \Im \left(\sum_{j=0}^n \frac{(-1)^{G_j} \binom{n}{j} \lambda^j}{2^n} \right) \right| = \\ &= \max_{\alpha \in [0; 2\pi]} \left| \sum_{j=0}^n \frac{(-1)^{G_j} \binom{n}{j} \cos j\alpha}{2^n} \right| + \max_{\alpha \in [0; 2\pi]} \left| \sum_{j=0}^n \frac{(-1)^{G_j} \binom{n}{j} \sin j\alpha}{2^n} \right|. \end{aligned}$$

For a random game $\{(-1)^{G_j}\}$ follow the same distribution as $\{\varphi_{j+1}(t)\}$ for t uniformly distributed from interval $[0; 1]$. Therefore Theorem 2 and Theorem 3 apply. Note that Theorem 3 is true for cosines as well as sines since we only use that $\cos^2 x \leq 1$, and so

$$\lim_{n \rightarrow \infty} \Pr \left[C_1 \frac{\sqrt{R_n \ln n}}{2^n} \leq \text{Val}_Q(G) \leq 2C_2 \frac{\sqrt{R_n \ln n}}{2^n} \right] = 1. \quad (2)$$

Finally,

$$\frac{\sqrt{R_n \ln n}}{2^n} = \frac{\sqrt{\binom{2n}{n} \ln n}}{2^n} \sim \frac{\sqrt{\frac{4^n}{\sqrt{\pi n}} \ln n}}{2^n} = \sqrt{\frac{\ln n}{\sqrt{\pi n}}}.$$

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4 Proof of Upper and Lower Bounds

We now proceed to prove theorems 2 and 3. Our proof is based on an old result by Salem and Zygmund [7], in which they prove bounds on the asymptotics of random trigonometric polynomials in a different setting (in which the coefficients r_m are not allowed to depend on n).

Due to the difference in the two settings, we cannot immediately apply the results from [7]. Instead, we prove corresponding theorems for our setting, re-using the parts of proof from [7] which also work in our case and replacing other parts with different arguments.

► **Lemma 5** (From [7]). *Let $f_n(t) = \sum_{m=0}^n c_m \varphi_{m+1}(t)$, where $\{\varphi_{m+1}(t)\}$ is the Rademacher system and c_m are real constants. Let $C_n = \sum_{m=0}^n c_m^2$, $D_n = \sum_{m=0}^n c_m^4$ and let λ be any real number. Then*

$$e^{\frac{1}{2}\lambda^2 C_n - \lambda^4 D_n} \leq \int_0^1 e^{\lambda f_n(t)} dt \leq e^{\frac{1}{2}\lambda^2 C_n}.$$

► **Lemma 6** (From [7]). Let $g(x, y)$, $a \leq x \leq b$, $c \leq y \leq d$, be a bounded real function. Suppose that

$$|g(x, y)| \leq A, \quad \int_c^d \int_a^b g^2(x, y) \, dx \, dy = B.$$

Then, for any positive number μ ,

$$\frac{\int_c^d \int_a^b e^{\mu g(x, y)} \, dx \, dy}{(b-a)(d-c)} \leq 1 + \mu\sqrt{B} + \frac{B}{A^2} e^{\mu A}.$$

Furthermore, when $\int_c^d \int_a^b g(x, y) \, dx \, dy = 0$,

$$\frac{\int_c^d \int_a^b e^{\mu g(x, y)} \, dx \, dy}{(b-a)(d-c)} \leq 1 + \frac{B}{A^2} e^{\mu A}. \quad (3)$$

► **Lemma 7** (From [7]). Let x be real and $P(x) = \sum_{m=0}^n \alpha_m \cos mx + \beta_m \sin mx$ be a trigonometric polynomial of order n , with real or imaginary coefficients. Let M denote the maximum of $|P(x)|$ and let θ be a positive number less than 1. Then there exists an interval of length not less than $\frac{1-\theta}{n}$ in which $|P(x)| \geq \theta M$.

► **Lemma 8** (From [7]). Let $\varphi(x) \geq 0$, and suppose that

$$\int_0^1 \varphi(x) \, dx \geq A > 0, \quad \int_0^1 \varphi^2(x) \, dx \leq B$$

(clearly, $A^2 \leq B$). Let $0 < \delta < 1$. Then

$$\Pr[\varphi(x) \geq \delta A \mid 0 \leq x \leq 1] \geq (1 - \delta)^2 \frac{A^2}{B}.$$

► **Lemma 9.**

$$\frac{\sum_{i=0}^n \binom{n}{i}^4}{\left(\sum_{i=0}^n \binom{n}{i}^2\right)^2} \leq \frac{4}{3} n^{-\frac{1}{2}}$$

Proof. If n is even:

$$\begin{aligned} \frac{\sum_{i=0}^n \binom{n}{i}^4}{\left(\sum_{i=0}^n \binom{n}{i}^2\right)^2} &\leq \frac{\sum_{i=0}^n \binom{n}{i}^2 \binom{n}{n/2}^2}{\left(\sum_{i=0}^n \binom{n}{i}^2\right)^2} = \frac{\binom{n}{n/2}^2}{\binom{2n}{n}} \leq \\ &\leq \frac{\left(\frac{2^n}{\sqrt{3^{\frac{n}{2}}+1}}\right)^2}{\frac{4^n}{\sqrt{4n}}} \leq \frac{\sqrt{4n}}{3^{\frac{n}{2}}+1} \leq \frac{4}{3} n^{-\frac{1}{2}} \end{aligned}$$

If n is odd:

$$\begin{aligned} \frac{\sum_{i=0}^n \binom{n}{i}^4}{\left(\sum_{i=0}^n \binom{n}{i}^2\right)^2} &\leq \frac{\sum_{i=0}^n \binom{n}{i}^2 \binom{n}{\lfloor n/2 \rfloor}^2}{\left(\sum_{i=0}^n \binom{n}{i}^2\right)^2} = \frac{\left(\frac{\binom{n+1}{\frac{n+1}{2}}}{2}\right)^2}{\binom{2n}{n}} \leq \\ &\leq \frac{\left(\frac{2^{n+1}}{2\sqrt{3^{\frac{n+1}{2}}+1}}\right)^2}{\frac{4^n}{\sqrt{4n}}} \leq \frac{\sqrt{4n}}{3^{\frac{n+1}{2}}+1} \leq \frac{4}{3} n^{-\frac{1}{2}} \end{aligned}$$

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Proof of Theorem 2. Set $I_n(t) = \frac{1}{2\pi} \int_0^{2\pi} e^{\lambda P_n(x,t)} dx$. We proceed to give an upper bound for $\int_0^1 I_n(t) dt$ and lower bound for $\int_0^1 I_n^2(t) dt$ using Lemma 5. Then we will plug in these bounds in Lemma 8 for $\varphi = I_n$.

First, the lower bound clause of Lemma 5 applied to $I_n(t)$ gives for any real λ (we will assign its value later, at our convenience),

$$\begin{aligned}
\int_0^1 I_n(t) dt &= \int_0^1 \left(\frac{1}{2\pi} \int_0^{2\pi} e^{\lambda P_n(x,t)} dx \right) dt = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 e^{\lambda P_n(x,t)} dt dx \geq \\
&\geq \frac{1}{2\pi} \int_0^{2\pi} e^{\frac{1}{2}\lambda^2 \sum_{m=0}^n (r_m \cos mx)^2 - \lambda^4 \sum_{m=0}^n (r_m \cos mx)^4} dx \geq \\
&\geq \frac{1}{2\pi} \int_0^{2\pi} e^{\frac{1}{2}\lambda^2 \sum_{m=0}^n (r_m \cos mx)^2 - \lambda^4 T_n} dx = \\
&= \left(e^{\frac{1}{4}\lambda^2 R_n - \lambda^4 T_n} \right) \cdot \frac{1}{2\pi} \int_0^{2\pi} e^{\frac{1}{2}\lambda^2 \sum_{m=0}^n (r_m \cos mx)^2 - \frac{r_m^2}{2}} dx = \\
&= \left(e^{\frac{1}{4}\lambda^2 R_n - \lambda^4 T_n} \right) \cdot \frac{1}{2\pi} \int_0^{2\pi} e^{\frac{1}{4}\lambda^2 \sum_{m=0}^n (r_m^2 \cos 2mx)} dx > \\
&> \left(e^{\frac{1}{4}\lambda^2 R_n - \lambda^4 T_n} \right) \cdot \frac{1}{2\pi} \int_0^{2\pi} \left(1 + \frac{1}{4}\lambda^2 \sum_{m=0}^n (r_m^2 \cos 2mx) \right) dx \geq \\
&\geq \left(e^{\frac{1}{4}\lambda^2 R_n - \lambda^4 T_n} \right)
\end{aligned}$$

The second step is to establish an upper bound for $\int_0^1 I_n^2(t) dt$. We start out in a similar fashion, by applying Lemma 5:

$$\begin{aligned}
\int_0^1 I_n^2(t) dt &= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \int_0^1 e^{\lambda(P_n(x,t)+P_n(y,t))} dt dx dy \leq \\
&\leq \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} e^{\frac{1}{2}\lambda^2 \sum_{m=0}^n r_m^2 (\cos mx + \cos my)^2} dx dy = \\
&= e^{\frac{1}{2}\lambda^2 (R_n + r_0^2)} \cdot \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} e^{\frac{1}{2}\lambda^2 S_n(x,y)} dx dy
\end{aligned}$$

where

$$S_n(x,y) = \sum_{m=1}^n \left(\frac{1}{2} r_m^2 \cos 2mx + \frac{1}{2} r_m^2 \cos 2my + 2r_m^2 \cos mx \cos my \right).$$

One can verify that

a)

$$\int_0^{2\pi} \int_0^{2\pi} S_n(x,y) dx dy = 0,$$

b)

$$\frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} S_n(x,y)^2 dx dy =$$

$$\begin{aligned}
&= \frac{1}{2\pi} \sum_{m=1}^n \int_0^{2\pi} \left(\frac{1}{2} r_m^2 \cos 2mx \right)^2 dx + \\
&+ \frac{1}{2\pi} \sum_{m=1}^n \int_0^{2\pi} \left(\frac{1}{2} r_m^2 \cos 2my \right)^2 dy + \\
&+ \frac{1}{(2\pi)^2} \sum_{m=1}^n \int_0^{2\pi} \int_0^{2\pi} (2r_m^2 \cos mx \cos my)^2 dx dy = \\
&= \frac{5}{4} T_n,
\end{aligned}$$

c)

$$|S_n(x, y)| \leq 3R_n.$$

We apply eq. 3 from Lemma 6 with function $g = S_n$, $\mu = \frac{1}{2}\lambda^2$, $A = 3R_n$ and $B = \frac{5}{4}T_n$. We get

$$\begin{aligned}
\frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} e^{\frac{1}{2}\lambda^2 S_n(x,y)} dx dy &\leq 1 + \frac{\frac{5}{4}T_n}{9R_n^2} e^{\frac{3}{2}\lambda^2 R_n} \leq \\
&\leq 1 + \frac{T_n}{R_n^2} e^{\frac{3}{2}\lambda^2 R_n}.
\end{aligned}$$

And by Lemma 9,

$$1 + \frac{T_n}{R_n^2} e^{\frac{3}{2}\lambda^2 R_n} \leq 1 + \frac{4}{3} n^{-\frac{1}{2}} e^{\frac{3}{2}\lambda^2 R_n}.$$

So far we have established the two prerequisites for Lemma 8:

1)

$$\int_0^1 I_n(t) dt > e^{\frac{1}{4}\lambda^2 R_n - \lambda^4 T_n},$$

2)

$$\int_0^1 I_n^2(t) dt \leq e^{\frac{1}{2}\lambda^2 (R_n + r_0^2)} \left(1 + \frac{4}{3} n^{-\frac{1}{2}} e^{\frac{3}{2}\lambda^2 R_n} \right).$$

The third step is to apply Lemma 8 with $\varphi = I_n$, $A = e^{\frac{1}{4}\lambda^2 R_n - \lambda^4 T_n}$, $B = e^{\frac{1}{2}\lambda^2 (R_n + r_0^2)} \times \left(1 + \frac{4}{3} n^{-\frac{1}{2}} e^{\frac{3}{2}\lambda^2 R_n} \right)$ and $\delta = n^{-\eta}$. This results in

$$\begin{aligned}
\Pr[I_n(t) \geq n^{-\eta} e^{\frac{1}{4}\lambda^2 R_n - \lambda^4 T_n}] &\geq (1 - n^{-\eta})^2 \frac{e^{\frac{1}{2}\lambda^2 R_n - 2\lambda^4 T_n}}{e^{\frac{1}{2}\lambda^2 (R_n + r_0^2)} \left(1 + \frac{4}{3} n^{-\frac{1}{2}} e^{\frac{3}{2}\lambda^2 R_n} \right)} \geq \\
&\geq (1 - n^{-\eta})^2 e^{-2\lambda^4 T_n - \frac{1}{2}\lambda^2 r_0^2} \left(1 - \frac{4}{3} n^{-\frac{1}{2}} e^{\frac{3}{2}\lambda^2 R_n} \right).
\end{aligned}$$

Finally we show that for suitably chosen λ , θ and η the claim follows. Set $\lambda = \theta \sqrt{\frac{\ln n}{R_n}}$ having θ such that $2\sqrt{\eta} < \theta < \sqrt{\frac{1}{3}}$. We deal with the two claims separately:

► **Lemma 10.**

$$I_n(t) \geq n^{-\eta} e^{\frac{1}{4}\lambda^2 R_n - \lambda^4 T_n} \implies M_n(t) \geq C_1 \sqrt{R_n \ln n}.$$

Proof. Note that

$$e^{\lambda M_n(t)} \geq I_n(t) \geq e^{\frac{1}{4}\lambda^2 R_n - \lambda^4 T_n - \eta \ln n}.$$

Thus

$$\begin{aligned} M_n(t) &\geq \frac{1}{4}\lambda R_n - \lambda^3 T_n - \frac{\eta}{\lambda} \ln n = \\ &= \frac{\theta}{4}\sqrt{R_n \ln n} - \theta^3 \sqrt{R_n \ln n} \ln n \frac{T_n}{R_n^2} - \frac{\eta}{\theta} \sqrt{R_n \ln n} = \\ &= \sqrt{R_n \ln n} \left(\frac{\theta}{4} - \theta^3 \frac{4 \ln n}{3\sqrt{n}} - \frac{\eta}{\theta} \right) \rightarrow \sqrt{R_n \ln n} \left(\frac{\theta}{4} - \frac{\eta}{\theta} \right). \end{aligned}$$

But $\frac{\theta}{4} - \frac{\eta}{\theta} = \text{constant} > 0$. We can choose θ arbitrarily close to $\sqrt{\frac{1}{3}}$ and η arbitrarily close to 0 to obtain $C_1 = \frac{1}{4\sqrt{3}}$. ◀

► **Lemma 11.**

$$\lim_{n \rightarrow \infty} (1 - n^{-\eta})^2 e^{-2\lambda^4 T_n - \frac{1}{2}\lambda^2 r_0^2} \left(1 - \frac{4}{3} n^{-\frac{1}{2}} e^{\frac{3}{2}\lambda^2 R_n} \right) = 1$$

Proof. Since η is positive, $n^{-\eta} \rightarrow 0$. Similarly,

$$e^{-2\lambda^4 T_n - \frac{1}{2}\lambda^2 r_0^2} = e^{-2\theta^4 (\ln n)^2 \frac{T_n}{R_n^2} - \frac{1}{2}\theta^2 r_0^2 \frac{\ln n}{R_n}} \geq e^{-\frac{8}{3\sqrt{n}}\theta^4 (\ln n)^2 - \frac{1}{2}\theta^2 r_0^2 \frac{\ln n}{R_n}} \rightarrow e^0 = 1,$$

and

$$\frac{4}{3} n^{-\frac{1}{2}} e^{\frac{3}{2}\lambda^2 R_n} = \frac{4}{3} n^{-\frac{1}{2}} e^{\frac{3}{2}\theta^2 \ln n} = \frac{4}{3} n^{\frac{3\theta^2 - 1}{2}} \rightarrow 0.$$

Proof of Theorem 3. We will examine $\int_0^1 \int_0^{2\pi} e^{\lambda|P_n(x,t)|} dx dt$. By Lemma 7 there exists $0 < \theta < 1$ such that:

$$\begin{aligned} &\int_0^1 \int_0^{2\pi} e^{\lambda|P_n(x,t)|} dx dt \geq \\ &\geq \int_0^1 \frac{1 - \theta}{n} e^{\theta \lambda M_n(t)} dt. \end{aligned}$$

On the other hand, by Lemma 5 we obtain:

$$\begin{aligned} &\int_0^1 \int_0^{2\pi} e^{\lambda|P_n(x,t)|} dx dt = \\ &= \int_0^{2\pi} \int_0^1 e^{\lambda|P_n(x,t)|} dt dx \leq \\ &\leq \int_0^{2\pi} \int_0^1 e^{\lambda P_n(x,t)} + e^{-\lambda P_n(x,t)} dt dx \leq \\ &\leq \int_0^{2\pi} \int_0^1 2e^{\frac{1}{2}\lambda^2 \sum_{m=0}^n r_m^2 \cos^2 mx} dt dx \leq \\ &\leq \int_0^{2\pi} \int_0^1 2e^{\frac{1}{2}\lambda^2 R_n} dt dx = \\ &= 4\pi e^{\frac{1}{2}\lambda^2 R_n}. \end{aligned}$$

Therefore,

$$\int_0^1 e^{\theta \lambda M_n(t)} dt \leq \frac{4\pi}{1-\theta} e^{\frac{1}{2}\lambda^2 R_n + \ln n}.$$

Have $\lambda = 2\sqrt{\frac{\ln n}{R_n}}$ and multiply both sides by $n^{-4-\eta}$, where $\eta > 0$. Then

$$\int_0^1 e^{\theta \lambda M_n(t) - (4+\eta) \ln n} dt \leq \frac{4\pi}{1-\theta} n^{-(1+\eta)}.$$

The sum over all n converges:

$$\sum_{n=1}^{\infty} \int_0^1 e^{\theta \lambda M_n(t) - (4+\eta) \ln n} dt \leq \sum_{n=1}^{\infty} \frac{4\pi}{1-\theta} n^{-(1+\eta)} < \infty.$$

Since the exponent function is non-negative and the whole sum converges, it is safe to interchange sum and integral:

$$\int_0^1 \sum_{n=1}^{\infty} e^{\theta \lambda M_n(t) - (4+\eta) \ln n} dt < \infty.$$

Therefore, for almost all t

$$\sum_{n=1}^{\infty} e^{\theta \lambda M_n(t) - (4+\eta) \ln n} < \infty.$$

Hence, for almost all t there exists n_0 such that for all $n \geq n_0$

$$\theta \lambda M_n(t) - (4 + \eta) \ln n < 0.$$

It follows that

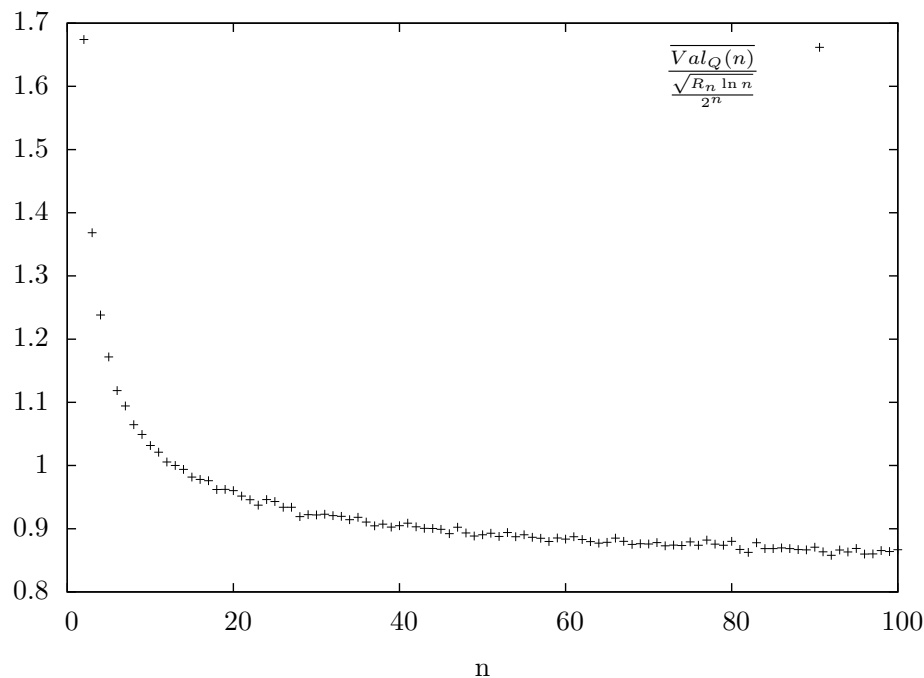
$$\lim_{n \rightarrow \infty} \Pr \left[M_n(t) < \frac{(4 + \eta)}{2\theta} \sqrt{R_n \ln n} \right] = 1.$$

◀

5 Conclusion

We have proven that the entangled value of almost any n -player symmetric XOR game is $\Theta\left(\frac{\sqrt{\ln n}}{n^{1/4}}\right)$ and therefore is by a factor of $\sqrt{\ln n}$ greater than its classical value.

In Fig. 1 we have plotted the sample mean value of the coefficient $\frac{\text{Val}_Q(n)}{\sqrt{R_n \ln n} / 2^n}$ over 10^5 random games for each n up to 100. We point out that the mean value of the coefficient is approaching ≈ 0.85 . It would be interesting to determine if C_1 and $2C_2$ (see, eq. 2) can be further improved and whether the coefficients in fact tend to a common limit near 0.85.



■ **Figure 1** $\frac{\overline{Val_Q(n)}}{\frac{\sqrt{R_n \ln n}}{2^n}}$ for a random sample of n player games.

In this paper we have dealt with a small portion of non-local games. In particular, the case of random non-symmetric games is still open and there has been little progress in multiplayer XOR games with m -ary input. The primary hurdle in the n -player m -ary input setting is that at the moment it lacks a description in terms of algebraic and analytic expressions.

Recently Briët and Vidick have shown large quantum-classical gaps for some 3-player m -ary input XOR games [4]. Despite being able to establish quantum-classical gaps for specific games, for a general 3-player m -ary input XOR game calculating its the entangled value is difficult [8].

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