On Generalized Heawood Inequalities for Manifolds: A Van Kampen–Flores-type Nonembeddability Result*

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Abstract

The fact that the complete graph K_5 does not embed in the plane has been generalized in two independent directions. On the one hand, the solution of the classical *Heawood problem* for graphs on surfaces established that the complete graph K_n embeds in a closed surface M if and only if $(n-3)(n-4) \leq 6b_1(M)$, where $b_1(M)$ is the first $\mathbb{Z}_2$-Betti number of M. On the other hand, Van Kampen and Flores proved that the k-skeleton of the n-dimensional simplex (the higher-dimensional analogue of K_{n+1}) embeds in $\mathbb{R}^{2k}$ if and only if $n \leq 2k+2$.

Two decades ago, Kühnel conjectured that the k-skeleton of the n-simplex embeds in a compact, $(k-1)$-connected $2k$-manifold with kth $\mathbb{Z}_2$-Betti number b_k only if the following *generalized Heawood inequality* holds: $\binom{n-k-1}{k+1} \leq \binom{2k+1}{k+1} b_k$. This is a common generalization of the case of graphs on surfaces as well as the Van Kampen–Flores theorem.

In the spirit of Kühnel's conjecture, we prove that if the k-skeleton of the n-simplex embeds in a $2k$-manifold with kth $\mathbb{Z}_2$-Betti number b_k, then $n \leq 2b_k\binom{2k+2}{k} + 2k + 5$. This bound is weaker than the generalized Heawood inequality, but does not require the assumption that M is $(k-1)$-connected. Our proof uses a result of Volovikov about maps that satisfy a certain homological triviality condition.

1998 ACM Subject Classification G. Mathematics of Computing, I.3.5 Computational Geometry and Object Modeling

Keywords and phrases Heawood Inequality, Embeddings, Van Kampen–Flores, Manifolds

Digital Object Identifier 10.4230/LIPIcs.SOCG.2015.476

1 Introduction

Given a closed surface M, it is a natural question to determine the maximum integer n such that the complete graph K_n can be embedded (drawn without crossings) into M (e.g., $n = 4$ if $M = S^2$ is the 2-sphere, and $n = 7$ if M is a torus). This classical problem was raised in the late 19th century by Heawood [9] and Heffter [10] and completely settled in the

* The work by Z. P. was partially supported by the Charles University Grant SVV-2014-260103. The work by Z. P. and M. T. was partially supported by the project CE-ITI (GACR P202/12/G061) of the Czech Science Foundation and by the ERC Advanced Grant No. 267165. Part of the research work of M. T. was conducted at IST Austria, supported by an *IST Fellowship*. The work by U.W. was partially supported by the Swiss National Science Foundation (grants SNSF-200020-138230 and SNSF-PP00P2-138948).

31st International Symposium on Computational Geometry (SoCG'15).
Editors: Lars Arge and János Pach; pp. 476–490
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

1950–60's through a sequence of works by Gustin, Guy, Mayer, Ringel, Terry, Welch, and Youngs (see [22, Ch. 1] for a discussion of the history of the problem and detailed references). Heawood already observed that if K_n embeds into M then

$$(n-3)(n-4) \le 6b_1(M) = 12 - 6\chi(M), \tag{1}$$

where $\chi(M)$ is the Euler characteristic of M and $b_1(M) = 2 - \chi(M)$ is the first $\mathbb{Z}_2$-*Betti number* of M, i.e., the dimension of the first homology group $H_1(M;\mathbb{Z}_2)$ (here and throughout the paper, we work with homology with $\mathbb{Z}_2$-coefficients).[1] Conversely, for surfaces M other than the Klein bottle, the inequality is tight, i.e., K_n embeds into M if and only if (1) holds; this is a hard result, the bulk of the monograph [22] is devoted to its proof. (The exceptional case, the Klein bottle, has $b_1 = 2$, but does not admit an embedding of K_7, only of K_6.)

The question naturally generalizes to higher dimension: Let $\Delta_n^{(k)}$ denote the k-skeleton of the n-simplex, the natural higher-dimensional generalization of $K_{n+1} = \Delta_n^{(1)}$ (by definition $\Delta_n^{(k)}$ has $n+1$ vertices and every subset of at most $k+1$ vertices form a face). Given a $2k$-dimensional manifold M, what is the largest n such that $\Delta_n^{(k)}$ embeds (topologically) into M? This line of enquiry started in the 1930's when Van Kampen [23] and Flores [5] showed that $\Delta_{2k+2}^{(k)}$ does not embed into $\mathbb{R}^{2k}$ (the case $k = 1$ corresponding to the non-planarity of K_5). Somewhat surprisingly, little else seems to known, and the following conjecture of Kühnel [12, Conjecture B] regarding a *generalized Heawood inequality* remains unresolved:

▶ **Conjecture 1** (Kühnel). *Let $n, k \ge 1$ be integers. If $\Delta_n^{(k)}$ embeds in a compact, $(k-1)$-connected $2k$-manifold M with kth $\mathbb{Z}_2$-Betti number $b_k(M)$ then*

$$\binom{n-k-1}{k+1} \le \binom{2k+1}{k+1} b_k(M) \tag{2}$$

The classical Heawood inequality (1) and the Van Kampen–Flores Theorem correspond the special cases $k = 1$ and $b_k = 0$, respectively. Kühnel states Conjecture 1 in slightly different form in terms of Euler characteristic of M rather than $b_k(M)$. Our formulation is an equivalent form. The $\mathbb{Z}_2$-coefficients are not important in the statement of the conjecture but they are convenient for our further progress.

New result. Here, we prove an estimate in the spirit of the generalized Heawood inequality (2), with a quantitatively weaker bound. Note that our bound holds (at no extra cost) under weaker hypotheses.

A somewhat technical but useful relaxation is that instead of embeddings, we consider the following slightly more general notion (which also helps with setting up our proof method). Let K be a finite simplicial complex and let $|K|$ be its underlying space (geometric

[1] The inequality (1), which by a direct calculation is equivalent to $n \le c(M) := \lfloor (7 + \sqrt{1 + \beta_1(M)})/2 \rfloor$, is closely related to the *Map Coloring Problem* for surfaces (which is the context in which Heawood originally considered the question). Indeed, it turns out that for surfaces M *other than the Klein bottle*, $c(M)$ is the maximum chromatic number of any graph embeddable into M. For $M = S^2$ the 2-sphere (i.e., $b_1(M) = 0$), this is the *Four-Color Theorem* [1, 2]; for other surfaces (i.e., $b_1(M) > 0$) this was originally stated (with an incomplete proof) by Heawood and is now known as the *Map Color Theorem* or *Ringel–Youngs Theorem* [22]. Interestingly, for surfaces $M \ne S^2$, there is a fairly short proof, based on edge counting and Euler characteristic, that the chromatic number of any graph embeddable into M is at most $c(M)$ (see [22, Thms. 4.2 and 4.8]). The hard part of the proof of the Ringel–Youngs Theorem is to show that for every M (except for the Klein bottle) $K_{c(M)}$ embeds into M.

realization). We define an *almost-embedding* of K into a (Hausdorff) topological space X to be a continuous map $f : |K| \to X$ such that any two disjoint simplices $\sigma, \tau \in K$ have disjoint images, $f(\sigma) \cap f(\tau) = \emptyset$. We stress that the condition for being an almost-embedding depends on the actual simplicial complex (the triangulation), not just the underlying space. That is, if K and L are two different complexes with $|K| = |L|$ then a map $f : |K| = |L| \to X$ may be an almost-embedding of K into X but not an almost-embedding of L into X. Note also that every embedding is an almost-embedding as well. Our main result is as follows:

▶ **Theorem 2.** *Let $n, k \geq 1$ be integers. If $\Delta_n^{(k)}$ almost-embeds into a $2k$-manifold M with kth $\mathbb{Z}_2$-Betti number $b_k(M)$, then*

$$n \leq 2\binom{2k+2}{k} b_k(M) + 2k + 5. \qquad (3)$$

As remarked above, this bound is weaker than the conjectured generalized Heawood inequality (2) and is clearly not optimal (as we already see in the special cases $k = 1$ or $b_k = 0$). On the other hand, apart from applying more generally to almost-embeddings, the hypotheses of Theorem 2 are weaker than those of Conjecture 1 in that we do not assume the manifold M to be $(k-1)$-connected. We conjecture that this connectedness assumption is not necessary for Conjecture 1, i.e., that (2) holds whenever $\Delta_n^{(k)}$ almost-embeds into a $2k$-manifold M. The intuition is that $\Delta_n^{(k)}$ is $(k-1)$-connected and therefore the image of an almost-embedding cannot "use" any parts of M on which nontrivial homotopy classes of dimension less than k are supported.

Previous work. The following special case of Conjecture 1 was proved by Kühnel [12, Thm. 2] (and served as a motivation for the general conjecture): Suppose that P is an n-dimensional simplicial convex polytope, and that there is a subcomplex of the boundary ∂P of P that is *k-Hamiltonian* (i.e., that contains the k-skeleton of P) and that is a triangulation of M, a $2k$-dimensional manifold. Then inequality (2) holds. To see that this is indeed a special case of Conjecture 1, note that ∂P is a *piecewise linear* (*PL*) sphere of dimension $n-1$, i.e., ∂P is combinatorially isomorphic to some subdivision of $\partial \Delta_n$ (and, in particular, $(n-2)$-connected). Therefore, the k-skeleton of P, and hence M, contains a subdivision of $\Delta_n^{(k)}$ and is $(k-1)$-connected.

In this special case and for $n \geq 2k+2$, equality in (2) is attained if and only if P is a simplex. More generally, equality is attained whenever M is a triangulated $2k$-manifold on $n+1$ vertices that is *$k+1$-neighborly* (i.e., any subset of at most $k+1$ vertices form a face, in which case $\Delta_n^{(k)}$ is a subcomplex of M). Some examples of $(k+1)$-neighborly $2k$-manifolds are known, e.g., for $k = 1$ (the so-called *regular cases* of equality for the Heawood inequality [22]), for $k = 2$ [15, 14] (e.g., a 3-neighborly triangulation of the complex projective plane) and for $k = 4$ [3], but in general, a characterization of the higher-dimensional cases of equality for (2) (or even of those values of the parameters for which equality is attained) seems rather hard (which is maybe not surprising, given how difficult the construction of examples of equality is already for $k = 1$).

Proof technique. Our proof of Theorem 2 strongly relies on a different generalization of the Van Kampen–Flores Theorem, due to Volovikov [24], regarding maps into general manifolds but under an additional homological triviality condition:

▶ **Theorem 3** (Volovikov)**.** *Let M be a $2k$-dimensional manifold and let $f : |\Delta_{2k+2}^{(k)}| \to M$ be a continuous map such that the induced homomorphism $f_* \colon H_k(\Delta_{2k+2}^{(k)}; \mathbb{Z}_2) \to H_k(M; \mathbb{Z}_2)$ is trivial. Then f is not an almost-embedding, i.e., there exist two disjoint simplices $\sigma, \tau \in \Delta_{2k+2}^{(k)}$ such that $f(\sigma) \cap f(\tau) \neq \emptyset$.*

Note that the homological triviality condition is automatically satisfied if $H_k(M;\mathbb{Z}_2)=0$, e.g., if $M=\mathbb{R}^{2k}$ or $M=S^{2k}$. On the other hand, without the homological triviality condition, the assertion is in general not true for other manifolds (e.g., K_5 embeds into every closed surface different from the sphere, or $\Delta_8^{(2)}$ embeds into the complex projective plane).

Theorem 3 is only a special of the main result in [24]; it is obtained by setting $j=q=2$, $m=2k$, $s=k+1$ and $N=2k+2$ in item 3 of Volovikov's main result (beware that k from Volovikov's condition "there exists a natural number k" is different from our k).

In addition, Volovikov [24] formulates the triviality condition in terms of cohomology, i.e., he requires that $f^*: H^k(M;\mathbb{Z}_2)\to H^k(\Delta_{2k+2}^{(k)};\mathbb{Z}_2)$ is trivial. However, since we are working with field coefficients and the (co)homology groups in question are finitely generated, the homological triviality condition (which is more convenient for us to work with) and the cohomological one are equivalent.[2]

The key idea of our approach is to show that if n is large enough and f is a mapping from $\Delta_n^{(k)}$ to M, then there is an almost-embedding g from $\Delta_s^{(k)}$ to $|\Delta_n^{(k)}|$ for some prescribed value of s such that the composed map $f\circ g\colon \Delta_s\to M$ satisfies Volovikov's condition. More specifically, the following is our main technical lemma:

▶ **Lemma 4.** *Let $k,s\geq 1$ and $b\geq 0$ be integers. There exists a value $n_0:=n_0(k,b,s)$ with the following property. Let $n\geq n_0$ and let f be a mapping of $|\Delta_n^{(k)}|$ into a manifold M with kth $\mathbb{Z}_2$-Betti number at most b. Then there exists a subdivision D of $\Delta_s^{(k)}$ and a simplicial map $g_{\text{simp}}\colon D\to\Delta_n^{(k)}$ with the following properties.*

1. *The induced map on the geometric realizations $g\colon |D|\to|\Delta_n^{(k)}|$ is an almost-embedding from $\Delta_s^{(k)}$ to $|\Delta_n^{(k)}|$ (note that $|D|=|\Delta_s^{(k)}|$).*
2. *The homomorphism $(f\circ g)_*: H_k(\Delta_s^{(k)})\to H_k(M)$ is trivial (see Section 2 below for the precise interpretation of $(f\circ g)_*$).*

The value n_0 can be taken as $\binom{s}{k}b(s-2k)+2s-2k+1$.

Therefore, if $s\geq 2k+2$, then $f\circ g$ cannot be an almost-embedding by Volovikov's theorem. We deduce that f is not an almost-embedding either, and Theorem 2 immediately follows. This deduction requires the following lemma as in general, a composition of two almost-embeddings needs not be an almost-embedding.

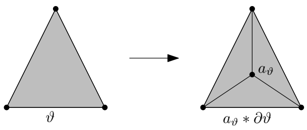
▶ **Lemma 5.** *Let K and L be simplicial complexes and X a topological space. Suppose g is an almost-embedding of K into $|L|$ and f is an almost-embedding of L into X. Then $f\circ g$ is an almost-embedding of K into X, provided that g is the realization of a simplicial map g_{simp} from some subdivision K' of K to L.*

We prove Lemma 4 in Section 4 thus completing the proof of Theorem 2. Before that, in Section 3 we first present a simpler version of that proof that introduces the main ideas in a simpler setting, and yields a weaker bound for n_0 (see Equation(4)). Further related questions and problems will be discussed in Section 5.

[2] More specifically, by the Universal Coefficient Theorem [21, 53.5], $H_k(\,\cdot\,;\mathbb{Z}_2)$ and $H^k(\,\cdot\,;\mathbb{Z}_2)$ are dual vector spaces, and f^* is the adjoint of f_*, hence triviality of f_* implies that of f^*. Moreover, if the homology group $H_k(X;\mathbb{Z}_2)$ of a space X is finitely generated (as is the case for both $\Delta_n^{(k)}$ and M, by assumption) then it is (non-canonically) isomorphic to its dual vector space $H^k(X;\mathbb{Z}_2)$. Therefore, f_* is trivial if and only if f^* is.

2 Preliminaries

We begin by fixing some terminology and notation. We will use $\mathrm{card}(U)$ to denote the cardinality of a set U.



We also recall that the *stellar subdivision* of a maximal face ϑ in a simplicial complex K is obtained by removing ϑ from K and adding a cone $a_\vartheta * (\partial\vartheta)$, where a_ϑ is a newly added vertex, the apex of the cone (see the figure on the left).

Throughout this paper we only work with homology groups and Betti numbers over $\mathbb{Z}_2$, and for simplicity, we will for the most part drop the coefficient group $\mathbb{Z}_2$ from the notation. Moreover, we will need to switch back and forth between singular and simplicial homology. More precisely, if K is a simplicial complex then $H_*(K)$ will mean the simplicial homology of K, whereas $H_*(X)$ will mean the singular homology of a topological space X. In particular, $H_*(|K|)$ denotes the singular homology of the underlying space $|K|$ of a complex K. We use analogous conventions for $C_*(K), C_*(X)$ and $C_*(|K|)$ on the level of chains, and likewise for the subgroups of cycles and boundaries, respectively.[3] Given a cycle c, we denote by $[c]$ the homology class it represents.

A mapping $h\colon |K| \to X$ induces a chain map $h_\sharp^{\mathrm{sing}}\colon C_*(|K|) \to C_*(X)$ on the level of singular chains; see [8, Chapter 2.1]. There is also a canonical chain map $\iota_K\colon C_*(K) \to C_*(|K|)$ inducing the isomorphism of $H_*(K)$ and $H_*(|K|)$, see again [8, Chapter 2.1]. We define $h_\sharp\colon C_*(K) \to C_*(X)$ as $h_\sharp := h_\sharp^{\mathrm{sing}} \circ \iota_K$. The three chain maps mentioned above also induce maps h_*^{sing}, $(\iota_K)_*$, and h_* on the level of homology satisfying $h_* = h_*^{\mathrm{sing}} \circ (\iota_K)_*$.

We also need a technical lemma saying that our maps compose, in a right way, on the level of homology.

▶ **Lemma 6.** *Let K and L be simplicial complexes and X a topological space. Let j_{simp} be a simplicial map for K to L, $j\colon |K| \to |L|$ be the continuous map induced by j_{simp} and $h\colon |L| \to X$ be another continuous map. Then $h_* \circ (j_{\mathrm{simp}})_* = (h \circ j)_*$ where $(j_{\mathrm{simp}})_*\colon H_*(K) \to H_*(L)$ is the map induced by j_{simp} on the level of simplicial homology and h_* and $(h \circ j)_*$, as explained above.*

3 Proof of Lemma 4 with a weaker bound on n_0

Let k, b, s be fixed integers. We consider a $2k$-manifold M with kth Betti number b, a map $f : |\Delta_n^{(k)}| \to M$. The strategy of our proof of Lemma 4 is to start by designing an auxiliary chain map

$$\varphi\colon C_*\left(\Delta_s^{(k)}\right) \to C_*\left(\Delta_n^{(k)}\right).$$

that behaves as an almost-embedding, in the sense that whenever σ and σ' are disjoint k-faces of Δ_s, $\varphi(\sigma)$ and $\varphi(\tau)$ have disjoint supports, and such that for every $(k+1)$-face τ of Δ_s the homology class $[(f_\sharp \circ \varphi)(\partial\tau)]$ is trivial. We then use φ to design a subdivision D of $\Delta_s^{(k)}$ and a simplicial map $g_{\mathrm{simp}} : D \to \Delta_n^{(k)}$ that induces a map $g : |D| \to |\Delta_n^{(k)}|$ with the

[3] We remark that throughout this paper, we will only work with spaces that are either (underlying spaces of) simplicial complexes or topological manifolds. Such spaces are homotopy equivalent to CW complexes [20, Corollary 1], and so on the matter of homology, it does not really matter which (ordinary, i.e., satisfying the dimension axiom) homology theory we use as they are all naturally equivalent for CW complexes [8, Thm. 4.59]. However the distinction between the simplicial and the singular setting will be relevant on the level of chains.

desired properties: g is an almost-embedding and $(f \circ g)_*([\partial\tau])$ is trivial for all $(k+1)$-faces τ of Δ_s. Since the cycles $\partial\tau$, for $(k+1)$-faces τ of Δ_s, generate all k-cycles of $\Delta_s^{(k)}$, this implies that $(f \circ g)_*$ is trivial.

The purpose of this section is to give a first implementation of the above strategy that proves Lemma 4 with a bound of

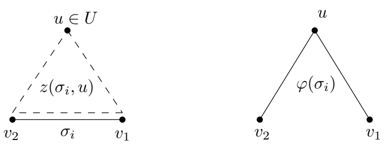
$$n_0 \geq \left(\binom{s+1}{k+1} - 1\right) 2^{b\binom{s+1}{k+1}} + s + 1. \tag{4}$$

In Section 4 we then improve this bound to $\binom{s}{k}b(s-2k) + 2s - 2k + 1$ at the cost of some technical complications.

Throughout the rest of this paper we use the following notations. We let $\{v_1, v_2, \ldots, v_{n+1}\}$ denote the set of vertices of Δ_n and we assume that Δ_s is the induced subcomplex of Δ_n on $\{v_1, v_2, \ldots, v_{s+1}\}$. We let $U = \{v_{s+2}, v_{s+3}, \ldots, v_{n+1}\}$ denote the set of vertices of Δ_n *unused* by Δ_s. We let $m = \binom{s+1}{k+1}$ and denote by $\sigma_1, \sigma_2, \ldots, \sigma_m$ the k-faces of Δ_s.

3.1 Construction of φ

For every face ϑ of Δ_s of dimension at most $k-1$ we set $\varphi(\vartheta) = \vartheta$. We then "route" each σ_i by mapping it to its stellar subdivision with an apex $u \in U$, *i.e.* by setting $\varphi(\sigma_i)$ to $\sigma_i + z(\sigma_i, u)$ where $z(\sigma_i, u)$ denotes the cycle $\partial(\sigma_i \cup \{u\})$. The picture on the left shows the case $k = 1$, the support of $z(\sigma_i, u)$ is dashed on the left, and the support of the resulting $\varphi(\sigma_i)$ is on the right.



We ensure that φ behave as an almost-embedding by using a different apex $u \in U$ for each σ_i. The difficulty is to choose these m apices in a way that $[f_\sharp(\varphi(\partial\tau))]$ is trivial for every $(k+1)$-face τ of Δ_s. To that end we associate to each $u \in U$ the sequence

$$\mathbf{v}(u) := ([f_\sharp(z(\sigma_1, u))], [f_\sharp(z(\sigma_2, u))], \ldots, [f_\sharp(z(\sigma_m, u))]) \in H_k(M)^m,$$

and we denote by $\mathbf{v}_i(u)$ the ith element of $\mathbf{v}(u)$. We work with $\mathbb{Z}_2$-homology, so $H_k(M)^m$ is finite; more precisely, its cardinality equals 2^{bm}. From $n \geq n_0 = (m-1)2^{bm} + s + 1$ we get that $\operatorname{card}(U) \geq (m-1)\operatorname{card}(H_k(M)^m) + 1$. The pigeonhole principle then guarantees that there exist m distinct vertices $u_1, u_2, \ldots, u_m$ of U such that $\mathbf{v}(u_1) = \mathbf{v}(u_2) = \cdots = \mathbf{v}(u_m)$. We use u_i to "route" σ_i and put

$$\varphi(\sigma_i) := \sigma_i + z(\sigma_i, u_i). \tag{5}$$

We finally extend φ linearly to $C_*\left(\Delta_s^{(k)}\right)$.

▶ **Lemma 7.** *φ is a chain map and $\left[f_\sharp\big(\varphi(\partial\tau)\big)\right] = 0$ for every $(k+1)$-face $\tau \in \Delta_s$.*

Before proving the lemma, we establish a simple claim that will be also useful later on.

▶ **Claim 8.** *Let τ be a $(k+1)$-face of Δ_s and let $u \in U$. Let $\sigma_{i_1}, \ldots, \sigma_{i_{k+2}}$ be all the k-faces of τ. Then*

$$\partial\tau + z(\sigma_{i_1}, u) + z(\sigma_{i_2}, u) + \cdots + z(\sigma_{i_{k+2}}, u) = 0. \tag{6}$$

Proof. This follows from expanding the equation $0 = \partial^2(\tau \cup \{u\})$. ◀

Proof of Lemma 7. The map φ is the identity on ℓ-chains with $\ell \leq k-1$ and Equation (5) immediately implies that $\partial\varphi(\sigma) = \partial\sigma$ for every k-simplex σ. It follows that φ is a chain map.

Now let τ be a $(k+1)$-simplex of Δ_s and let $\sigma_{i_1}, \dots, \sigma_{i_{k+2}}$ be its k-faces. We have

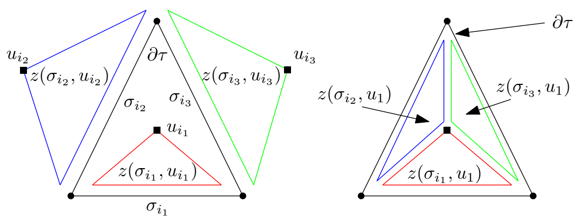
$$f_\sharp \circ \varphi(\partial\tau) = f_\sharp\left(\sum_{j=1}^{k+2} \sigma_{i_j} + z(\sigma_{i_j}, u_{i_j})\right) = f_\sharp(\partial\tau) + \sum_{j=1}^{k+2} f_\sharp\big(z(\sigma_{i_j}, u_{i_j})\big).$$

The u_i's are chosen in such a way that the homology class $\big[f_\sharp\big(z(\sigma_{i_j}, u_\ell)\big)\big] = \mathbf{v}_{i_j}(u_\ell)$ is independent of the value ℓ. When passing to the homology classes in the above identity, we can therefore replace each u_{i_j} with u_1, and obtain,

$$[f_\sharp \circ \varphi(\partial\tau)] = [f_\sharp(\partial\tau)] + \sum_{j=1}^{k+2}\Big[f_\sharp\big(z(\sigma_{i_j}, u_1)\big)\Big] = \Big[f_\sharp\Big(\partial\tau + \sum_{j=1}^{k+2} z(\sigma_{i_j}, u_1)\Big)\Big].$$

This class is trivial by Claim 8.

Here is the idea behind the proof with $k=1$ and $u_{i_1} = u_1$ (same colors represent same homology classes; the class on the right is trivial, because each edge appears twice):



◀

3.2 Construction of D and g

The definition of φ, an in particular Equation (5), suggests to construct our subdivision D of $\Delta_s^{(k)}$ by simply replacing every k-face of $\Delta_s^{(k)}$ by its stellar subdivision. Let a_i denote the new vertex introduced when subdividing σ_i.

We define a simplicial map $g_{\text{simp}}\colon D \to \Delta_n^{(k)}$ by putting $g_{\text{simp}}(v) = v$ for every original vertex v of $\Delta_s^{(k)}$, and $g_{\text{simp}}(a_i) = u_i$ for $i \in [m]$. This g_{simp} induces a map $g\colon |\Delta_s^{(k)}| \to |\Delta_n^{(k)}|$ on the geometric realizations. Since the u_i's are pairwise distinct, g is an embedding[4], so Condition 1 of Lemma 4 holds.

On principle, we would like to derive Condition 2 of Lemma 4 by observing that g 'induces' a chain map from $C_*(\Delta_s^{(k)})$ to $C_*(\Delta_n^{(k)})$ that coincides with φ. Making this a formal statement is thorny because g, as a continuous map, naturally induces a chain map $g_\sharp$ on singular rather than simplicial chains. We can't use directly g_{simp} either, since we are interested in a map from $C_*(\Delta_s^{(k)})$ and not from $C_*(D)$.

We handle this technicality as follows. Let $\rho\colon C_*(\Delta_s^{(k)}) \to C_*(D)$ be the chain map that sends each simplex ϑ of $\Delta_s^{(k)}$ to the sum of simplices of D of the same dimension that subdivide it. This map induces an isomorphism ρ_* in homology, and $\varphi = (g_{\text{simp}})_\sharp \circ \rho$ where $(g_{\text{simp}})_\sharp\colon C_*(D) \to C_*(\Delta_n^{(k)})$ denotes the (simplicial) chain map induced by g_{simp}. We thus have in homology

$$f_* \circ \varphi_* = f_* \circ (g_{\text{simp}})_* \circ \rho_*$$

[4] We use the full strength of almost-embeddings when proving Lemma 4 with the better bound on n_0.

and since ρ_* is an isomorphism and $f_* \circ \varphi_*$ is trivial, Lemma 7 yields that $f_* \circ (g_{\text{simp}})_*$ is also trivial. Since $f_* \circ (g_{\text{simp}})_* = (f \circ g)_*$ by Lemma 6, $(f \circ g)_*$ is trivial as well. This concludes the proof of Lemma 4 with the weaker bound.

4 Proof of Lemma 4

We now prove Lemma 4 with the bound claimed in the statement, namely

$$n_0 = \binom{s}{k} b(s - 2k) + 2s - 2k + 1.$$

Let k, b, s be fixed integers. We consider a $2k$-manifold M with kth Betti number b, a map $f : |\Delta_n^{(k)}| \to M$, and we assume that $n \geq n_0$.

The proof follows the same strategy as in Section 3 : we construct a chain map $\varphi\colon C_*(\Delta_s^{(k)}) \to C_*(\Delta_n^{(k)})$ such that the homology class $[(f_\sharp \circ \varphi)(\partial\tau)]$ is trivial for all $(k+1)$-faces τ of Δ_s, then upgrade φ to a continuous map $g\colon |\Delta_s^{(k)}| \to |\Delta_n^{(k)}|$ with the desired properties.

When constructing φ, we refine the arguments of Section 3 to "route" each k-face using not only one, but several vertices from U; this makes finding "collisions" easier, as we can use linear algebra arguments instead of the pigeonhole principle. This comes at the cost that when upgrading g, we must content ourselves with proving that it is an almost-embedding. This is sufficient for our purpose and has an additional benefit: the same group of vertices from U may serve to route several k-faces provided they pairwise intersect in $\Delta_k^{(s)}$.

4.1 Construction of φ

We use the same notation regarding $v_1, \ldots, v_{n+1}$, Δ_n, Δ_s, U, $m = \binom{s+1}{k+1}$ and $\sigma_1, \sigma_2, \ldots, \sigma_m$ as in Section 3.

Definition of multipoints and the map v. As we said we plan to route k-faces of Δ_s through collections of vertices from U, we will call these collections multipoints. It turns out that this is useful for our needs only if these multipoints have an odd cardinality. In order to easily proceed with later computations, we define multipoints as vectors rather than subsets of U as below.

Let $C_0(U)$ denote the $\mathbb{Z}_2$-vector space of formal linear combinations of vertices from U. A *multipoint* is an element of $C_0(U)$ with an odd number of non-zero coefficients. The multipoints form an affine subspace of $C_0(U)$ which we denote by $\mathcal{M}$. The *support*, $\sup(\mu)$, of a multipoint $\mu \in \mathcal{M}$ is the set of vertices $v \in U$ with non-zero coefficient in μ. We say that two multipoints are *disjoint* if their supports are disjoint.

For any k-face σ_i and any multipoint μ we define:

$$z(\sigma_i, \mu) := \sum_{u \in \sup(\mu)} z(\sigma_i, u) = \sum_{u \in \sup(\mu)} \partial(\sigma_i \cup \{u\}).$$

Now, we proceed as in Section 3 but replace the unused points by the multipoints of $\mathcal{M}$ and the cycles $z(\sigma_i, u)$ with the cycles $z(\sigma_i, \mu)$. Since $\mathbb{Z}_2$ is a field, $H_k(M)^m$ is a vector space and we can replace the sequences $\mathbf{v}(u)$ of Section 3 by the linear map

$$\mathbf{v} : \left\{ \begin{array}{ccl} C_0(U) & \to & H_k(M)^m \\ \mu & \mapsto & ([f_\sharp(z(\sigma_1, \mu))], [f_\sharp(z(\sigma_2, \mu))], \ldots, [f_\sharp(z(\sigma_m, \mu))]) \end{array} \right.$$

Finding collisions. The following lemma takes advantage of the vector space structure of $H_k(M)^m$ to find disjoint multipoints $\mu_1, \mu_2, \ldots$ to route the σ_i's more effectively than by simple pigeonhole.

▶ **Lemma 9.** *For any $r \geq 1$, any $\mathbb{Z}_2$-vector space V, and any linear map $\psi\colon C_0(U) \to V$, if* $\mathrm{card}(U) \geq (\dim(\psi(\mathcal{M}))+1)(r-1)+1$ *then $\mathcal{M}$ contains r disjoint multipoints $\mu_1, \mu_2, \ldots, \mu_r$ such that $\psi(\mu_1) = \psi(\mu_2) = \cdots = \psi(\mu_r)$.*

Proof. Let us write $U = \{v_{s+2}, v_{s+3}, \ldots, v_{n+1}\}$ and $d = \dim(\psi(\mathcal{M}))$. We first prove by induction on r the following statement:

> If $\mathrm{card}(U) \geq (d+1)(r-1)+1$ there exist r pairwise disjoint subsets $I_1, I_2, \ldots, I_r \subseteq U$ whose image under ψ have affine hulls with non-empty intersection.

(This is, in a sense, a simple affine version of Tverberg's theorem.) The statement is obvious for $r = 1$, so assume that $r \geq 2$ and that the statement holds for $r-1$. Let A denote the affine hull of $\{\psi(v_{s+2}), \psi(v_{s+3}), \ldots, \psi(v_{n+1})\}$ and let I_r denote a minimal cardinality subset of U such that the affine hull of $\{\psi(v) : v \in I_r\}$ equals A. Since $\dim A \leq d$ the set I_r has cardinality at most $d+1$. The cardinality of $U \setminus I_r$ is at least $(d+1)(r-2)+1$ so we can apply the induction hypothesis for $r-1$ to $U \setminus I_r$. We thus obtain $r-1$ disjoint subsets $I_1, I_2, \ldots, I_{r-1}$ whose images under ψ have affine hulls with non-empty intersection. Since the affine hull of $\psi(U \setminus I_r)$ is contained in the affine hull of $\psi(I_r)$, the claim follows.

Now, let $a \in V$ be a point common to the affine hulls of $\psi(I_1), \psi(I_2), \ldots, \psi(I_r)$. Writing a as an affine combination in each of these spaces, we get

$$a = \sum_{u \in J_1} \psi(u) = \sum_{u \in J_2} \psi(u) = \cdots = \sum_{u \in J_r} \psi(u)$$

where $J_j \subseteq I_j$ and $|J_j|$ is odd for any $j \in [r]$. Setting $\mu_j = \sum_{u \in J_j} u$ finishes the proof. ◀

Computing the dimension of $\mathbf{v}(\mathcal{M})$. Having in mind to apply Lemma 9 with $V = H_k(M)^m$ and $\psi = \mathbf{v}$, we now need to bound from above the dimension of $\mathbf{v}(\mathcal{M})$. An obvious upper bound is $\dim H_k(M)^m$, which equals $bm = b\binom{s+1}{k+1}$. A better bound can be obtained by an argument analogous to the proof of Lemma 7. We first extend Claim 8 to multipoints.

▶ **Claim 10.** *Let τ be a $(k+1)$-face of Δ_s and let $\mu \in \mathcal{M}$. Let $\sigma_{i_1}, \ldots, \sigma_{i_{k+2}}$ be all the k-faces of τ. Then*

$$\partial\tau + z(\sigma_{i_1}, \mu) + z(\sigma_{i_2}, \mu) + \cdots + z(\sigma_{i_{k+2}}, \mu) = 0. \tag{7}$$

Proof. By Claim 8 we know that (7) is true for points. For a multipoint μ, we get (7) as a linear combination of equations for the points in $\sup(\mu)$ (using that $\mathrm{card}(\sup(\mu))$ is odd). ◀

▶ **Lemma 11.** $\dim(\mathbf{v}(\mathcal{M})) \leq b\binom{s}{k}$.

Proof. Let τ be a $(k+1)$-face of Δ_s and let $\sigma_{i_1}, \ldots, \sigma_{i_{k+2}}$ denote its k-faces.

For any multipoint μ, Claim 10 implies

$$[f_\sharp(\partial\tau)] = \sum_{j=1}^{k+2} [f_\sharp(z(\sigma_i, \mu))] = \sum_{j=1}^{k+2} \mathbf{v}_{i_j}(\mu) \qquad \text{so} \qquad \mathbf{v}_{i_{k+2}}(\mu) = [f_\sharp(\partial\tau)] + \sum_{j=1}^{k+1} \mathbf{v}_{i_j}(\mu).$$

(Remember that homology is computed over $\mathbb{Z}_2$.) Each vector $\mathbf{v}(\mu)$ is thus determined by the values of the $\mathbf{v}_j(\mu)$'s where σ_j contains the vertex v_1. Indeed, the vectors $[f_\sharp(\partial\tau)]$ are independent of μ, and for any σ_i not containing v_1 we can eliminate $\mathbf{v}_i(\mu)$ by considering $\tau := \sigma_i \cup \{v_1\}$ (and setting $\sigma_{i_{k+2}} = \sigma_i$). For each of the $\binom{s}{k}$ faces σ_j that contain v_1, the vector $\mathbf{v}_j(\mu)$ takes values in $H_k(M)$ which has dimension at most b. It follows that $\dim \mathbf{v}(\mathcal{M}) \leq b\binom{s}{k}$. ◀

Coloring hypergraphs to reduce the number of multipoints used. We could now apply Lemma 9 with $r = m$ to obtain one multipoints per k-face, all pairwise disjoint, to proceed with our "routing". As mentioned above, however, we only need that φ is an almost-embedding, so we can use the same multipoint for several k-faces provided they pairwise intersect. Optimizing the number of multipoints used reformulates as the following hypergraph coloring problem:

> Assign to each k-face σ_i of Δ_s some color $c(i) \in \mathbb{N}$ such that $\mathrm{card}\{c(i) : 1 \leq i \leq m\}$ is minimal and disjoint faces use distinct colors.

This question is classically known as Kneser's hypergraph coloring problem and an optimal solution uses $s - 2k + 1$ colors [17, 18]. Let us spell out one such coloring (proving its optimality is considerably more difficult, but we do not need to know that it is optimal). For every k-face σ_i we let $\min \sigma_i$ denote the smallest index of a vertex in σ_i. When $\min \sigma_i \leq s-2k$ we set $c(i) = \min \sigma_i$, otherwise we set $c(i) = s - 2k + 1$. Observe that any k-face with color $c \leq s - 2k$ contains vertex v_c. Moreover, the k-faces with color $s - 2k + 1$ consist of $k + 1$ vertices each, all from a set of $2k + 1$ vertices. It follows that any two k-faces with the same color have some vertex in common.

Defining φ. We are finally ready to define the chain map $\varphi\colon C_*(\Delta_s^{(k)}) \to C_*(\Delta_n^{(k)})$. Recall that we assume that $n \geq n_0 = (\binom{s}{k}b+1)(r-1)+s+1$. Using the bound of Lemma 11 we can apply Lemma 9 with $r = s-2k+1$, obtaining $s-2k+1$ multipoints $\mu_1, \mu_2, \ldots, \mu_{s-2k+1} \in \mathcal{M}$. We set $\varphi(\vartheta) = \vartheta$ for any face ϑ of Δ_s of dimension less than k. We then "route" each k-face σ_i through the multipoint $\mu_{c(i)}$ by putting

$$\varphi(\sigma_i) := \sigma_i + z(\sigma_i, \mu_{c(i)}), \tag{8}$$

where $c(i)$ is the color of σ_i in the coloring of the Kneser hypergraph proposed above. We finally extend φ linearly to $C_*(\Delta_s)$.

We need the following analogue of Lemma 7.

▶ **Lemma 12.** *φ is a chain map and $\left[f_\sharp\big(\varphi(\partial\tau)\big)\right] = 0$ for every $(k+1)$-face $\tau \in \Delta_s$.*

The proof of Lemma 12 is very similar to the proof of Lemma 7; it just replaces points with multipoints and Claim 8 with Claim 10.

We next argue that φ behaves like an almost embedding.

▶ **Lemma 13.** *For any two disjoint faces ϑ, η of $\Delta_s^{(k)}$, the supports of $\varphi(\vartheta)$ and $\varphi(\eta)$ use disjoint sets of vertices.*

Proof. Since φ is the identity on chains of dimension at most $(k-1)$, the statement follows if neither face has dimension k. For any k-chain σ_i, the support of $\varphi(\sigma_i)$ uses only vertices from σ_i and from the support of $\mu_{c(i)}$. Since each $\mu_{c(i)}$ has support in U, which contains no vertex of Δ_s, the statement also holds when exactly one of ϑ or η has dimension k. When both ϑ and η are k-faces, their disjointness implies that they use distinct μ_j's, and the statement follows from the fact that distinct μ_j's have disjoint supports. ◀

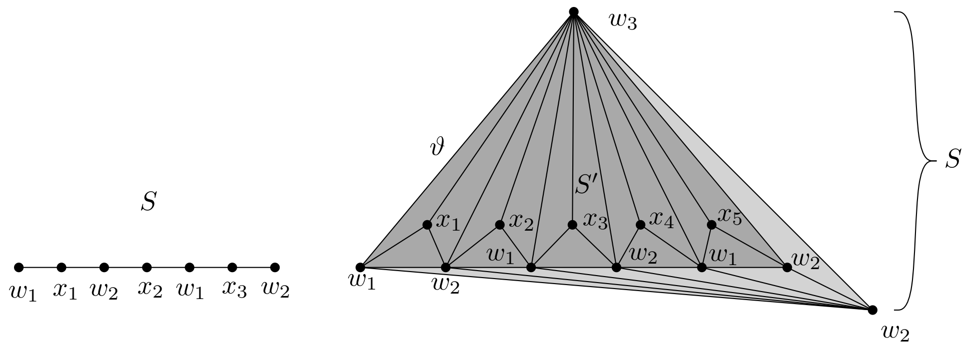


Figure 1 Examples of subdivisions for $k = 1$ and $\ell = 3$ (left) and for $k = 2$ and $\ell = 5$ (right).

4.2 Construction of D and g

We define D and g similarly as in Section 3, but the switch from points to multipoints requires to replace stellar subdivisions by a slightly more complicated decomposition.

The subdivision D. We define D so that it coincides with Δ_s on the faces of dimension at most $(k-1)$ and decomposes each face of dimension k independently. The precise subdivision of a k-face σ_i depends on the cardinality of the support of the multipoint $\mu_{c(i)}$ used to "route" σ_i under φ, but the method is generic and spelled out in the next lemma; refer to Figure 1.

▶ **Lemma 14.** *Let $k \geq 1$ and $\sigma = \{w_1, w_2, \ldots, w_{k+1}\}$ be a k-simplex. For any odd integer $\ell \geq 1$ there exists a subdivision S of σ in which no face of dimension $k-1$ or less is subdivided, and a labelling of the vertices of S by $\{w_1, w_2, \ldots, w_{k+1}, x_1, x_2, \ldots, x_\ell\}$ (some labels may appear several times) such that:*

1. *Every vertex in S corresponding to an original vertex w_i of σ is labelled by w_i,*
2. *no k-face of S has its vertices labelled $w_1, w_2, \ldots, w_{k+1}$,*
3. *for every $(i, j) \in [k+1] \times [\ell]$ there exists a unique k-face of S that is labelled by $w_1, w_2, \ldots, w_{i-1}, w_{i+1}, \ldots, w_{k+1}, x_j$,*
4. *no edge of S has its two vertices labelled in $\{x_1, x_2, \ldots, x_\ell\}$,*

Proof. This proof is done in the language of geometric simplicial complexes (rather than abstract ones).

The case $\ell = 1$ can be done by a stellar subdivision and labelling the added apex x_1. The case $k = 1$ is easy, as illustrated in Figure 1 (left). We therefore assume that $k \geq 2$ and build our subdivision and labelling in four steps:

- We start with the boundary of our simplex σ where each vertex w_i is labelled by itself. Let ϑ be the $(k-1)$-face of $\partial\sigma$ opposite vertex w_2, *ie* labelled by $w_1, w_3, w_4, \cdots w_{k+1}$. We create a vertex in the interior of σ, label it w_2, and construct a new simplex σ' as the join of ϑ and this new vertex; this is the dark simplex in Figure 1 (right).
- We then subdivide σ' by considering $\ell - 1$ distinct hyperplanes passing through the vertices of σ' labelled $w_3, w_4, \ldots, w_{k+1}$ and through an interior points of the edge of σ' labelled w_1, w_2. These hyperplanes subdivide σ' into ℓ smaller simplices. We label the new interior vertices on the edge of σ' labelled w_1, w_2 by alternatively, w_1 and w_2; since ℓ is odd we can do so in a way that every sub-edge is bounded by two vertices labelled w_1, w_2.
- We operate a stellar subdivision of each of the ℓ smaller simplices subdividing σ', and label the added apices $x_1, x_2, \ldots, x_\ell$. This way we obtain a subdivision S' of σ'.

- We finally consider each face η of S' subdividing $\partial\sigma'$ and other than ϑ and add the simplex formed by η and the (original) vertex w_2 of σ. These simplices, together with S', form the desired subdivision S of σ.

It follows from the construction that no face of $\partial\sigma$ was subdivided.

Property 1 is enforced in the first step and preserved throughout. We can ensure that Property 2 holds in the following way. First, we have that any k-simplex of S' contains a vertex x_j for some $j \in [\ell]$. Next, if we consider a k-simplex of S which is not in S' it is a join of a certain $(k-1)$-simplex η of S', with $\eta \subset \partial\sigma'$, and the vertex w_2 of σ. However, the only such $(k-1)$-simplex labelled by $w_1, w_3, w_4, \dots, w_{k+1}$ is ϑ, but the join of ϑ and w_2 does not belong to S.

Properties 3 and 4 are enforced by the stellar subdivisions of the third step, and no other step creates, destroys or modifies any simplex involving a vertex labelled x_i. ◀

The subdivision D of $\Delta_s^{(k)}$ is now defined as follows. First, we leave the $(k-1)$-skeleton untouched. Next, for each k-simplex σ_i we let ℓ_i denote the number of points in the support of $\mu_{c(i)}$; since we work with $\mathbb{Z}_2$ coefficients, ℓ_i is odd. We then compute some subdivision $S(i)$ of σ_i using Lemma 14 with $\ell := \ell_i$.

We let $\rho\colon C_*(\Delta_s^{(k)}) \to C_*(D)$ denote the map that is the identity on $\Delta_s^{(k-1)}$ and that maps each σ_i to the sum of the k-dimensional simplices of $S(i)$. This maps induces an isomorphism ρ_* in homology.

The simplicial map g_{simp}. We now define a simplicial map $g_{\mathrm{simp}}\colon D \to \Delta_n^{(k)}$. We first set $g_{\mathrm{simp}}(v) = v$ for every vertex v of Δ_s. Consider next some k-face $\sigma_i = \{w_1(i), w_2(i), \dots, w_{k+1}(i)\}$. We denote by $v_1(i), v_2(i) \dots, v_{k+1}(i)$ the vertices on the boundary of $S(i)$, being understood that each $v_j(i)$ is labelled by w_j, and let $u_1(i), u_2(i), \dots, u_{\ell(i)}(i)$ denote the vertices of the support of $\mu_{c(i)}$. We map each interior vertex of $S(i)$ to either some $w_j(i)$ if its label, as given by Lemma 14, is $w_j(i)$, or some $u_j(i)$ if that label is x_j.

▶ **Lemma 15.** $(g_{\mathrm{simp}})_\sharp \circ \rho = \varphi$.

Proof. All three maps are the identity on $\Delta_s^{(k-1)}$ so let us focus on the k-faces. Since ρ maps σ_i to the formal sum of the k-faces of $S(i)$. Each k-face of $S(i)$ is mapped, under g_{simp}, to a k-face with labels $v_1(i), v_2(i), \dots, v_{j-1}(i), v_{j+1}(i), \dots, v_{k+1}(i), u_{j'}(i)$ for some $(j, j') \in [k+1] \times [\ell(i)]$. Although tedious, it is elementary to check that the chain $(g_{\mathrm{simp}})_\sharp \circ \rho(\sigma_i)$ has the same support as $\varphi(\sigma_i)$. Since we are working with $\mathbb{Z}_2$ coefficients, the chains are therefore equal. ◀

The continuous map g. Since D is a subdivision of $\Delta_s^{(k)}$, we have $|\Delta_s^{(k)}| = |D|$ and the simplicial map $g_{\mathrm{simp}}\colon D \to \Delta_n^{(k)}$ induces a continuous map $g\colon |\Delta_s^{(k)}| \to |\Delta_n^{(k)}|$. All that remains to do is check that g satisfies the two conditions of Lemma 4. Condition 1 follows from a direct translation of Lemma 13. Condition 2 can be verified by a computation in the same way as in Section 3. Specifically, in homology we have

$$f_* \circ \varphi_* = f_* \circ (g_{\mathrm{simp}})_* \circ \rho_*$$

and we know that $f_* \circ \varphi_*$ is trivial on $\Delta_s^{(k)}$ by Lemma 12. As ρ_* is an isomorphism, this implies that $f_* \circ (g_{\mathrm{simp}})_*$ is trivial. Lemma 6 then implies that $(f \circ g)_*$ is trivial. This concludes the proof of Lemma 4.

5 Related Questions and Outlook

While we consider Conjecture 1 natural and interesting in its own right, there are a number of connections to other problems that are worth mentioning and provide additional motivation.

5.1 Topological Helly-Type Theorems for subsets of Manifolds

In [6, Thm. 1], we use the Van Kampen–Flores Theorem to prove the following topological *Helly-type theorem* for finite families $\mathcal{F}$ of subsets of $\mathbb{R}^d$, under the assumption that for every proper subfamily $\mathcal{G} \subsetneq \mathcal{F}$, the $\mathbb{Z}_2$-Betti numbers $b_i(\bigcup \mathcal{G})$, $0 \leq i \leq \lceil d/2 \rceil - 1$, are bounded.

More precisely, our proof heavily relies the fact that the Van Kampen–Flores Theorem also applies to the following generalization of almost-embeddings: We define a *homological almost-embedding* of a finite simplicial complex K into a topological space X as a chain map φ from the simplicial chain complex $C_*(K; \mathbb{Z}_2)$ to the singular chain complex $C_*(X; \mathbb{Z}_2)$ with the properties that (i) for every vertex of K, $\varphi(v)$ consists of an odd number of points in X and (ii) for any pair σ, τ of disjoint simplices of K, the image chains $\varphi(\sigma)$ and $\varphi(\tau)$ have disjoint underlying point sets.

One can show that Volovikov's theorem, and consequently Theorem 2 extend to homological almost-embeddings; we plan to discuss this in more detail in the full version of the present paper. Thus, (3) holds whenever $\Delta_n^{(k)}$ homologically almost-embeds into M.

As a consequence, the Helly-type result in [6, Thm. 1] generalizes (with an appropriate change in the constants) to families of subsets of an arbitrary d-dimensional manifold.

5.2 Extremal Problems for Embeddings

Closely related to the classical Heawood inequality is the well-known fact that for a (simple) graph embedded into a surface M, the number of edges of G is at most linear in the number of vertices of G (see, e.g., [22, Thm. 4.2]). More specifically, if G embeds into a surface M with first $\mathbb{Z}_2$-Betti number $b_1(M)$, and if $f_1(G)$ and $f_0(G)$ denote the number of vertices and of edges of G, respectively, then

$$f_1(G) \leq 3f_0(G) - 6 + 3b_1(M).$$

Note that this immediately implies (1) when applied to $G = K_n$.

This question also naturally generalizes to higher dimensions:

▶ **Conjecture 16.** *Let M be a $2k$-dimensional manifold with kth $\mathbb{Z}_2$-Betti number $b_k(M)$. If K is a finite k-dimensional simplicial complex that embeds into M then*

$$f_k(K) \leq C \cdot f_{k-1}(K),$$

where f_i denotes the number of i-dimensional faces of K, $-1 \leq i \leq k$, and C is a constant that depends only on k and on $b_k(M)$.[5]

The special case $M = \mathbb{R}^{2k}$ of the problem was first raised by Grünbaum [7] more than forty years ago, and has since then been rediscovered and posed independently by a number of authors (see, e.g., Dey [4], where the problem is motivated by the question of counting

[5] In the spirit of the bound for graphs on surfaces, it is also natural to wonder if there might be a bound of the form $f_k(K) \leq C \cdot f_{k-1}(K) + B$, with C depending only on the dimension k and the additive term B depending on k and $b_k(M)$.

triangulations of higher-dimensional point sets), and the problem remains wide open even in that case. Moreover, there is a beautiful conjecture, due to Kalai and Sarkaria [11, Conjecture 27] that gives a necessary condition for embeddability into $\mathbb{R}^{2k}$ in terms of *algebraic shifting* and would, in particular, imply that the constant C in Conjecture 16 can be taken to be $k+2$ if $M = \mathbb{R}^{2k}$.

The aforementioned extension of Theorem 2 to homological almost-embeddings together with [25, Thm. 7] imply the following result for random complexes:

▶ **Corollary 17.** *Let $X^k(n,p)$ denote the Linial–Meshulam model [16, 19] of k-dimensional random complexes on n vertices.*[6] *Given integers $k \geq 1$ and $b \geq 0$, there exists a constant $C = C(k,b)$ with the following property: If M is a $2k$-dimensional manifold with $\mathbb{Z}_2$-Betti number $b_k(M) \leq b$ and if $p \geq C/n$ then asymptotically almost surely, $X^k(n,p)$ does not embed into M.*

This generalizes [25, Thm. 2] and can be viewed as evidence for Conjecture 16 (in a sense, it shows that the conjecture holds for "almost all complexes").

The arguments in [25, Thm. 7] are based on the following notion closely related to homological almost-embeddings: If K and L are simplicial complexes, we say that K is a homological minor of K if there is a chain map φ from the simplicial chain complex $C_*(K;\mathbb{Z}_2)$ into the simplicial chain complex of $C_*(K;\mathbb{Z}_2)$ that satisfies conditions (i) and (ii) in the definition of a homological almost-embedding (one might call φ a *simplicial* homological almost-embedding). In [25, Conj. 6], we propose a conjectural generalization of *Mader's theorem* to the extent that a finite k-dimensional simplicial complex K contains $\Delta_t^{(k)}$ as a homological minor provided that $f_k(K) \geq C \cdot f_{k-1}(K)$ for some suitable constant $C = C(k,t)$. If true, this conjecture, together with the extension of Theorem 2 to homological almost-embeddings, would imply Conjecture 16.

We remark that Conjecture 1 is also closely related to the combinatorial theory of *face numbers* of triangulated spheres and manifolds, in particular the *Generalized Lower Bound Theorem* for polytopes (which is the main ingredient in Kühnel's proof of his special case) and conjectured generalizations thereof to triangulated spheres and manifolds. A detailed discussion of these questions goes beyond the scope of this extended abstract, and we refer the reader to [12] and [13, Ch. 4].

Acknowledgement. U. W. learned about Conjecture 1 from Wolfgang Kühnel when attending the *Mini Symposia on Discrete Geometry and Discrete Topology* at the *Jahrestagung der Deutschen Mathematiker-Vereinigung* in München in 2010. He would like to thank the organizers Frank Lutz and Achill Schürmann for the invitation, and Prof. Kühnel for stimulating discussions.

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[6] By definition, $X^k(n,p)$ has n vertices, a complete $(k-1)$-skeleton, and every subset of $k+1$ vertices is chosen independently with probability p as a k-simplex.

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