Standardization of a Call-By-Value Lambda-Calculus

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Abstract

We study an extension of Plotkin’s call-by-value lambda-calculus by means of two commutation rules (sigma-reductions). Recently, it has been proved that this extended calculus provides elegant characterizations of many semantic properties, as for example solvability. We prove a standardization theorem for this calculus by generalizing Takahashi’s approach of parallel reductions. The standardization property allows us to prove that our calculus is conservative with respect to the Plotkin’s one. In particular, we show that the notion of solvability for this calculus coincides with that for Plotkin’s call-by-value lambda-calculus.

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1 Introduction

The $\lambda_v$-calculus ($\lambda_v$ for short) has been introduced by Plotkin in [15], in order to give a formal account of the call-by-value evaluation, which is the most commonly used parameter passing policy for programming languages. $\lambda_v$ shares the syntax with the classical, call-by-name, $\lambda$-calculus ($\lambda$ for short), but its reduction rule, $\beta_v$, is a restriction of $\beta$, firing only in case the argument is a value (i.e., a variable or an abstraction). While $\beta_v$ is enough for evaluation, it turned out to be too weak to study operational properties of terms. For example, in $\lambda$, the $\beta$-reduction is sufficient to characterize solvability and (using extensionality) separability, but, in order to characterize similar properties for $\lambda_v$, it has been necessary to introduce different notions of reduction unsuitable for a correct call-by-value evaluation (see [13, 14]): this is disappointing and requires complex reasoning. In this paper we study $\lambda^*_v$, the extension of $\lambda_v$ proposed in [3]. It keeps the $\lambda_v$ (and $\lambda$) syntax and it adds to the $\beta_v$-reduction two commutation rules, called $\sigma_1$ and $\sigma_3$, which unblock $\beta_v$-redexes that are hidden by the “hyper-sequential structure” of terms. It is well-known (see [14, 1]) that in $\lambda_v$ there are normal forms that are unsolvable, e.g. $(\lambda y x . x x)(z z)(\lambda x . x x)$. The more evident benefit of $\lambda^*_v$
is that the commutation rules make all normal forms solvable (indeed \((\lambda y.x.x)(zz)(\lambda x.xx)\) is not a \(\lambda^\sigma_v\) normal form). More generally, the so obtained language, allows us to characterize operational properties, like solvability and potential valuability, in an internal and elegant way (see [3]). In this paper we prove a standardization property in \(\lambda^\sigma_v\), and some of its consequences, namely its soundness with respect to the semantics of \(\lambda_v\).

Let us recall the notion of standardization, which has been first studied in the ordinary \(\lambda\)-calculus (see, for example [5, 8, 2]). A reduction sequence is standard if its redexes are ordered in a given way, and the corresponding standardization theorem establishes that every reduction sequence can constructively be transformed into a standard one. Standardization is a key tool to grasp the way in which reductions work, that sheds some light on redexes relationships and their dependencies. It is useful for characterization of semantic properties through reduction strategies (the proof of operational semantics adequacy is a typical use).

In the \(\lambda_v\) setting standardization theorems have been proved by Plotkin [15], Paolini and Ronchi Della Rocca [14, 12] and Crary [4]. The definition of standard sequence of reductions considered by Plotkin and Crary coincides, and it imposes a partial order on redexes, while Paolini and Ronchi Della Rocca define a total order on them. All these proofs are developed by using the notion of parallel reduction introduced by Tait and Martin-Löf (see Takahashi [17] for details and interesting technical improvements). We emphasize that this method does not involve the notion of residual of a redex, on which many classical proofs for the \(\lambda\)-calculus are based (see for example [8, 2]). As in [15, 17, 14, 4], we use a suitable notion of parallel reduction for developing our standardization theorem for \(\lambda^\sigma_v\). In particular we consider two groups of redexes, head \(\beta_v\)-redexes and head \(\sigma\)-redexes (putting together \(\sigma_1\) and \(\sigma_3\)), and we induce a total order on head redexes of the two groups, without imposing any order on head \(\sigma\)-redexes themselves. More precisely, when \(\sigma\)-redexes are missing, this notion of standardization coincides with that presented in [14]. Moreover, we show that it is not possible to strengthen our standardization by (locally) ordering \(\sigma_1\)-reduction to \(\sigma_3\)-reduction (or viceversa).

As usual, our standardization proof is based on a sequentialization result: inner reductions can always be postponed after the head ones, for a non-standard definition of head reduction. Sequentialization has interesting consequences: it allows us to prove that fundamental operational properties in \(\lambda^\sigma_v\), like observational equivalence, potential valuability and solvability, are conservative with respect to the corresponding notions of \(\lambda_v\). This fully justifies the project in [3] where \(\lambda^\sigma_v\) has been introduced as a tool for studying the operational behaviour of \(\lambda_v\).

Other variants of \(\lambda_v\) have been introduced in the literature for modeling the call-by-value computation. We would like to cite here at least the contributions of Moggi [10], Felleisen and Sabry [16], Maraist et al. [9], Herbelin and Zimmerman [7], Accattoli and Paolini [1]. All these proposals are based on the introduction of new constructs to the syntax of \(\lambda_v\), so the comparison between them is not easy with respect to syntactical properties (some detailed comparison is given in [1]). We point out that the calculi introduced in [10, 16, 9, 7] present some variants of our \(\sigma_1\) and/or \(\sigma_3\) rules, often in a setting with explicit substitutions.

Outline. In Section 2 we introduce the language \(\lambda^\sigma_v\) and its operational behaviour; in Section 3 the sequentialization property is proved; Section 4 contains the main result, i.e., standardization ; in Section 5 some conservativity results with respect to Plotkin’s \(\lambda_v\)-calculus are proved. Section 6 concludes the paper, with some hints for future work.
2 The call-by-value lambda calculus with sigma-rules

In this section we present $\lambda^*_{cv}$, a call-by-value $\lambda$-calculus introduced in [3] that adds two $\sigma$-reduction rules to pure (i.e. without constants) call-by-value $\lambda$-calculus defined by Plotkin in [15].

The syntax of terms of $\lambda^*_{cv}$ [3] is the same as the one of ordinary $\lambda$-calculus and Plotkin’s call-by-value $\lambda$-calculus $\lambda_{cv}$ [15] (without constants). Given a countable set $V$ of variables (denoted by $x, y, z, \ldots$), the sets $\Lambda$ of terms and $\Lambda_v$ of values are defined by mutual induction:

\[
\begin{align*}
\Lambda_v & : = \{ \} \mid \lambda x. M \\
\Lambda & : = V \mid MN \\
\end{align*}
\]

Clearly, $\Lambda_v \subseteq \Lambda$. All terms are considered up to $\alpha$-conversion. The set of free variables of a term $M$ is denoted by $fv(M)$. Given $V_1, \ldots, V_n \in \Lambda_v$ and pairwise distinct variables $x_1, \ldots, x_n$, $M\{V_1/x_1, \ldots, V_n/x_n\}$ denotes the term obtained by the capture-avoiding simultaneous substitution of $V_i$ for each free occurrence of $x_i$ in the term $M$ (for all $1 \leq i \leq n$). Note that, for all $V, V_1, \ldots, V_n \in \Lambda_v$ and pairwise distinct variables $x_1, \ldots, x_n$, $V\{V_1/x_1, \ldots, V_n/x_n\} \in \Lambda_v$.

Contexts (with exactly one hole $\langle \rangle$), denoted by $C$, are defined as usual via the grammar:

\[
C : = \langle \rangle \mid \lambda x.C \mid CM \mid MC.
\]

We use $\mathcal{C}[M]$ for the term obtained by the capture-allowing substitution of the term $M$ for the hole $\langle \rangle$ in the context $C$.

**Notation.** From now on, we set $I = \lambda x.x$ and $\Delta = \lambda x.xx$.

The reduction rules of $\lambda^*_{cv}$ consist of Plotkin’s $\beta_{cv}$-reduction rule, introduced in [15], and two simple commutation rules called $\sigma_1$ and $\sigma_3$, studied in [3].

**Definition 1 (Reduction rules).** We define the following binary relations on $\Lambda$ (for any $M, N, L, \in \Lambda$ and any $V \in \Lambda_v$):

\[
\begin{align*}
(\lambda x.M)V & \rightarrow_{\beta_{cv}} M\{V/x\} \\
(\lambda x.M)NL & \rightarrow_{\sigma_1} (\lambda x.ML)N \quad \text{with} \ x \notin fv(L) \\
V((\lambda x.M)L)N & \rightarrow_{\sigma_3} (\lambda x.VL)N \quad \text{with} \ x \notin fv(V).
\end{align*}
\]

For any $r \in \{\beta_{cv}, \sigma_1, \sigma_3\}$, if $M \rightarrow_r M'$ then $M$ is a $r$-redex and $M'$ is its $r$-contractum. In this sense, a term of the shape $(\lambda x.M)N$ (for any $M, N \in \Lambda$) is a $\beta$-redex.

We set $\rightarrow_\sigma = \rightarrow_{\sigma_1} \cup \rightarrow_{\sigma_3}$ and $\rightarrow_\rho = \rightarrow_{\beta_{cv}} \cup \rightarrow_\sigma$.

The side conditions on $\rightarrow_{\sigma_1}$ and $\rightarrow_{\sigma_3}$ in Definition 1 can be always fulfilled by $\alpha$-renaming.

Obviously, any $\beta_{cv}$-redex is a $\beta$-redex but the converse does not hold: $(\lambda x.z)(yI)$ is a $\beta$-redex but not a $\beta_{cv}$-redex. Redexes of different kind may overlap: for example, the term $\Delta I \Delta$ is a $\sigma_1$-redex and it contains the $\beta_{cv}$-redex $\Delta I$; the term $\Delta(I\Delta)(xI)$ is a $\sigma_1$-redex and it contains the $\sigma_3$-redex $\Delta(I\Delta)$, which contains in turn the $\beta_{cv}$-redex $I\Delta$.

According to the Girard’s call-by-value “boring” translation $(\cdot)^\circ$ of terms into Intuitionistic Multiplicative Exponential Linear Logic proof-nets, defined by $(A \Rightarrow B)^\circ = !A^* \rightarrow !B^*$ (see [6]), the images under $(\cdot)^\circ$ of a $\sigma_1$-redex (resp. $\sigma_3$-redex) and its contractum are equal modulo some “bureaucratic” steps of cut-elimination.

**Notation.** Let $R$ be a binary relation on $\Lambda$. We denote by $R^+$ (resp. $R^*$; $R^\circ$) the reflexive-transitive (resp. transitive; reflexive) closure of $R$. 

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Definition 2 (Reductions). Let \( r \in \{ \beta_v, \sigma_1, \sigma_3, \sigma, \nu \} \).

The \( r \)-reduction \( \rightarrow_r \) is the contextual closure of \( \rightarrow_r \), i.e. \( M \rightarrow_r M' \) iff there is a context \( C \) and \( N, N' \in \Lambda \) such that \( M = C(N) \), \( M' = C(N') \) and \( N \rightarrow_r N' \).

The \( r \)-equivalence \( \equiv_r \) is the reflexive-transitive and symmetric closure of \( \rightarrow_r \).

Let \( M \) be a term: \( M \) is \( r \)-normal if there is no term \( N \) such that \( M \rightarrow_r N \); \( M \) is \( r \)-normalizable if there is a \( r \)-normal term \( N \) such that \( M \rightarrow_r N \); \( M \) is strongly \( r \)-normalizing if there is no sequence \( (N_i)_{i \in \mathbb{N}} \) such that \( M = N_0 \) and \( N_i \rightarrow_r N_{i+1} \) for any \( i \in \mathbb{N} \). Finally, \( \rightarrow_r \) is strongly normalizing if every \( N \in \Lambda \) is strongly \( r \)-normalizing.

Remark 3. For any \( r \in \{ \beta_v, \sigma_1, \sigma_3, \sigma, \nu \} \) (resp. \( r \in \{ \sigma_1, \sigma_3, \sigma \} \)), values are closed under \( r \)-reduction (resp. \( r \)-expansion): for any \( V \in \Lambda_\sigma \), if \( V \rightarrow_r M \) (resp. \( M \rightarrow_r V \)) then \( M \in \Lambda_\sigma \); more precisely, \( V = \lambda x.N \) and \( M = \lambda x.N' \) for some \( N, N' \in \Lambda \) with \( N \rightarrow_r N' \) (resp. \( N' \rightarrow_r N \)).

Proposition 4 (See [3]). The \( \sigma \)-reduction is confluent and strongly normalizing. The \( \nu \)-reduction is confluent.

The \( \lambda^\nu_\sigma \)-calculus, \( \lambda^\nu_\sigma \) for short, is the set \( \Lambda \) of terms endowed with the \( \nu \)-reduction \( \rightarrow_\nu \). The set \( \Lambda \) endowed with the \( \beta_v \)-reduction \( \rightarrow_{\beta_v} \) is the \( \lambda_v \)-calculus (\( \lambda_v \) for short), i.e. the Plotkin’s call-by-value \( \lambda \)-calculus [15] (without constants), which is thus a sub-calculus of \( \lambda^\nu_\sigma \).

Example 5. \( M = (\lambda y.\Delta)(xI)\Delta \rightarrow_{\sigma_1} (\lambda y.\Delta\Delta)(xI) \rightarrow_{\beta_v} (\lambda y.\Delta\Delta)(xI) \rightarrow_{\beta_v} \ldots \) and \( N = \Delta((\lambda y.\Delta)(xI)) \rightarrow_{\sigma_3} (\lambda y.\Delta\Delta)(xI) \rightarrow_{\beta_v} (\lambda y.\Delta\Delta)(xI) \rightarrow_{\beta_v} \ldots \) are the only possible \( \nu \)-reduction paths from \( M \) and \( N \) respectively: \( M \) and \( N \) are not \( \nu \)-normalizable, and \( M = \nu N \). Meanwhile, \( M \) and \( N \) are \( \beta_v \)-normal and different, hence \( M \neq_{\beta_v} N \) (by confluence of \( \rightarrow_{\beta_v} \), see [15]).

Informally, \( \sigma \)-rules unblock \( \beta_v \)-redexes which are hidden by the “hyper-sequential structure” of terms. This approach is alternative to the one in [1] where hidden \( \beta_v \)-redexes are reduced using rules acting at a distance (through explicit substitutions). It can be shown that the call-by-value \( \lambda \)-calculus with explicit substitution introduced in [1] can be embedded in \( \lambda^\nu_\sigma \).

### 3 Sequentialization

In this section we aim to prove a sequentialization theorem (Theorem 22) for the \( \lambda^\nu_\sigma \)-calculus by adapting Takahashi’s method [17, 4] based on parallel reductions.

Notation. From now on, we always assume that \( V, V' \in \Lambda_\sigma \).

Note that the generic form of a term is \( VM_1 \ldots M_m \) for some \( m \in \mathbb{N} \) (in particular, values are obtained when \( m = 0 \)). The sequentialization result is based on a partitioning of \( \nu \)-reduction between head and internal reduction.

Definition 6 (Head \( \beta_v \)-reduction). We define inductively the head \( \beta_v \)-reduction \( \beta_v \) by the following rules (\( m \in \mathbb{N} \) in both rules):

\[
(\lambda x.M)VM_1 \ldots M_m \xrightarrow{\beta_v} M(V/x)M_1 \ldots M_m \quad \quad N \xrightarrow{\beta_v} N' \xrightarrow{\nu} VN'M_1 \ldots M_m
\]

The head \( \beta_v \)-reduction \( \beta_v \) reduces exactly the same redexes (see also [13]) as the “left reduction” defined in [15, p.136] for \( \lambda_v \) and called “evaluation” in [16, 4]. If \( N \xrightarrow{\beta_v} N' \) then \( N' \) is obtained from \( N \) by reducing the leftmost-outermost \( \beta_v \)-redex, not in the scope of a \( \lambda \); thus, the head \( \beta_v \)-reduction is deterministic (i.e., it is a partial function from \( \Lambda \) to \( \Lambda \)) and does not reduce values.
\[\begin{align*}
\text{(\(\lambda x.M\))NLM}_1 \ldots M_m & \xrightarrow{\sigma} (\lambda x.ML)N\text{M}_1 \ldots M_m & \text{σ_1} \\
N & \xrightarrow{\sigma} N' & \text{right} \\
V & \xrightarrow{\sigma} N' & \text{VNM}_1 \ldots M_m \xrightarrow{\sigma} V N' M_1 \ldots M_m & \text{σ_3}
\end{align*}\]

The head \((\nu)-\text{reduction}\) is \(\xrightarrow{\nu} \). The \(\text{internal (\(\nu\))-reduction}\) is \(\xrightarrow{\nu} = \xrightarrow{\nu} \cup \xrightarrow{\nu}\).

Notice that \(\xrightarrow{\beta}\) and \(\xrightarrow{\nu}\) are not deterministic, for example the leftmost-outermost \(\sigma_1\)- and \(\sigma_3\)-redexes may overlap: if \(M = (\lambda y.y')((\Delta(xI)))\) then \(M \xrightarrow{\sigma} (\lambda y.y')((\Delta(xI))) = N_1\) by applying the rule \(\sigma_1\) and \(M \xrightarrow{\sigma} (\lambda z.((\lambda y.y')((z)))((xI)) = N_2\) by applying the rule \(\sigma_3\). Note that \(N_1\) contains only a head \(\sigma_3\)-redex and \(N_2 \xrightarrow{\sigma} (\lambda z.((\lambda y.y')((z)))((xI))) = N\) which is normal for \(\xrightarrow{\nu}\); meanwhile \(N_2\) contains only a head \(\sigma_1\)-redex and \(N_2 \xrightarrow{\sigma} (\lambda z.((\lambda y.y')((z)))((xI))) = N\) which is normal for \(\xrightarrow{\nu}\) if \(N \neq N'\), hence the head reduction \(\xrightarrow{\nu}\) is not confluent and a term may have several head-normal forms (this example does not contradict the confluence of \(\sigma\), most "mal forms for the head reduction, but the converse does not hold: \(xI \notin \Lambda\) is head-normal).

Informally, if \(N \xrightarrow{\sigma} N'\) then \(N'\) is obtained from \(N\) by reducing "one of the leftmost" \(\sigma_1\)- or \(\sigma_3\)-redexes, not in the scope of a \(\lambda\): in general, a term may contain several head \(\sigma_1\)- and \(\sigma_3\)-redexes. Indeed, differently from \(\xrightarrow{\beta}\), the head \(\sigma\)-reduction \(\xrightarrow{\sigma}\) is not deterministic, for example the leftmost-outermost \(\sigma_1\)- and \(\sigma_3\)-redexes may overlap:

\[\begin{align*}
V & \Rightarrow V' & M_i \Rightarrow M_i' & (m \in \mathbb{N}, \ 0 \leq i \leq m) \\
(\lambda x.M_0)\text{M}_1 \ldots \text{M}_m & \Rightarrow (\lambda x.M_0)\text{V}x\text{M}_1 \ldots \text{M}_m & N & \Rightarrow N' & L \Rightarrow L' & M_i \Rightarrow M_i' & (m \in \mathbb{N}, \ 0 \leq i \leq m) \\
(\lambda x.M_0) & \text{N}\text{M}_1 \ldots \text{M}_m & \Rightarrow (\lambda x.M_0)\text{N}\text{M}_1 \ldots \text{M}_m & (\lambda x.M_0) & \text{N}\text{M}_1 \ldots \text{M}_m & \Rightarrow (\lambda x.M_0)\text{N}\text{M}_1 \ldots \text{M}_m
\end{align*}\]

In Definition 8 the rule \(\var\) has no premises when \(m = 0\): this is the base case of the inductive definition of \(\Rightarrow\). The rules \(\sigma_1\) and \(\sigma_3\) have exactly three premises when \(m = 0\).

Intuitively, \(M \Rightarrow M'\) means that \(M'\) is obtained from \(M\) by reducing a number of \(\beta\)-, \(\sigma_1\)- and \(\sigma_3\)-redexes (existing in \(M\)) simultaneously.

\[\begin{align*}
N & \Rightarrow N' & (\lambda x.M)N & \overset{\lambda}{\Rightarrow} (\lambda x.M')N' & (m \in \mathbb{N}, \ 0 \leq i \leq m) & V \Rightarrow V' & N \overset{\var}{\Rightarrow} N' & M_i \Rightarrow M_i' & (m \in \mathbb{N}, \ 0 \leq i \leq m) & \text{VNM}_1 \ldots M_m \overset{\var}{\Rightarrow} V' N' M_1' \ldots M_m'
\end{align*}\]

The \(\text{strong parallel reduction}\) \(\Rightarrow\) is defined by: \(M \Rightarrow N\) if \(M \Rightarrow N\) and there exist \(M', M'' \in \Lambda\) such that \(M \xrightarrow{\beta} M' \xrightarrow{\sigma} M'' \Rightarrow N\).

Notice that the rule \(\Rightarrow\) for \(\Rightarrow\) has exactly two premises when \(m = 0\).
Remark 10. The relations \(\Rightarrow, \Rightarrow\) and \(\Rightarrow\) are reflexive. The reflexivity of \(\Rightarrow\) follows immediately from the reflexivity of \(\Rightarrow\) and \(\Rightarrow\). The proofs of reflexivity of \(\Rightarrow\) and \(\Rightarrow\) are both by structural induction on a term: in the case of \(\Rightarrow\), recall that every term is of the form \((\lambda x. N)M_1 \ldots M_m\) or \(x M_1 \ldots M_m\) for some \(m \in \mathbb{N}\) and then apply the rule \(\lambda\) or \(\text{var}\) respectively, together with the inductive hypothesis; in the case of \(\Rightarrow\), recall that every term is of the form \(\lambda x. M\) or \(x M_1 \ldots M_m\) for some \(m \in \mathbb{N}\) and then apply the rule \(\lambda\) (together with the reflexivity of \(\Rightarrow\)) or \(\text{var}\) or \(\text{right}\) (together with the reflexivity of \(\Rightarrow\)) and the inductive hypothesis respectively.

One has \(\Rightarrow \subseteq \Rightarrow \subseteq \Rightarrow\) (first, prove that \(\Rightarrow \subseteq \Rightarrow\) by induction on the derivation of \(M \Rightarrow M'\), the other inclusions follow from the definition of \(\Rightarrow\)) and, since \(\Rightarrow\) is reflexive (Remark 10), \(\Rightarrow\) \(\Rightarrow\) and \(\Rightarrow\) \(\Rightarrow\). Observe that \(\Delta \Delta \ominus \Delta \Delta\) for any \(R \in \{\Rightarrow_{\beta_\iota}, \Rightarrow_{\beta_\iota}, \Rightarrow, \Rightarrow\}\), even if for different reasons: for example, \(\Delta \Delta \Rightarrow \Delta \Delta\) by reflexivity of \(\Rightarrow\) (Remark 10), whereas \(\Delta \Delta \Rightarrow \Delta \Delta\) by reducing the (leftmost-outermost) \(\beta_\iota\)-redex.

Next two further remarks collect many minor properties that can be easily proved.

Remark 11. 1. The head \(\beta_\iota\)-reduction \(\Rightarrow_{\beta_\iota}\) does not reduce a value (in particular, does not reduce under \(\lambda\)'s), i.e., for any \(M \in \Delta\) and any \(V \in \Lambda\), one has \(V \not\Rightarrow_{\beta_\iota} M\).

2. The head \(\sigma\)-reduction \(\Rightarrow_{\sigma}\) does neither reduce a value nor reduce to a value, i.e., for any \(M \in \Delta\) and any \(V \in \Lambda\), one has \(V \not\Rightarrow_{\sigma} M\) and \(M \not\Rightarrow_{\sigma} V\).

3. Variables and abstractions are preserved under \(\Rightarrow\) (\(\Rightarrow\)-expansion), i.e., if \(M \Rightarrow x\) (resp. \(M \Rightarrow \lambda x. N\)) then \(M = x\) (resp. \(M = \lambda x. N\) for some \(N \in \Delta\) such that \(N \Rightarrow N')\).

4. If \(M \Rightarrow M'\) then \(\lambda x. M \Rightarrow \lambda x. M'\) for any \(R \in \{\Rightarrow, \Rightarrow\}\). Indeed, for \(R \in \{\Rightarrow, \Rightarrow\}\) apply the rule \(\lambda\) to conclude, then \(\lambda x. M \Rightarrow \lambda x. M'\) according to the definition of \(\Rightarrow\).

5. For any \(V, V' \in \Lambda\), \(V \Rightarrow V'\) if \(V \Rightarrow V'\). The left-to-right direction holds because \(\Rightarrow \subseteq \Rightarrow\); conversely, assume \(V \Rightarrow V'\): if \(V\) is a variable then necessarily \(V = V'\) and hence \(V \Rightarrow V'\) by applying the rule \(\text{var}\) for \(\Rightarrow\); otherwise \(V = \lambda x. N\) for some \(N \in \Delta\), and then necessarily \(V' = \lambda x. N'\) with \(N \Rightarrow N'\), so \(V \Rightarrow V'\) by applying the rule \(\lambda\) for \(\Rightarrow\).

Remark 12. 1. If \(M \Rightarrow M'\) and \(N \Rightarrow N'\) then \(MN \Rightarrow M'N'\). For the proof, it is sufficient to consider the last rule of the derivation of \(M \Rightarrow M'\).

2. If \(R \in \{\Rightarrow_{\beta_\iota}, \Rightarrow_{\sigma}\}\) and \(M \Rightarrow M'\), then \(MN \Rightarrow M'N\) for any \(N \in \Delta\). For the proof, it is sufficient to consider the last rule of the derivation of \(M \Rightarrow M'\), for any \(R \in \{\Rightarrow_{\beta_\iota}, \Rightarrow_{\sigma}\}\).

3. If \(M \Rightarrow M'\) and \(N \Rightarrow N'\) where \(M' \not\in \Lambda\), then \(MN \Rightarrow M'N'\): indeed, the last rule in the derivation of \(M \Rightarrow M'\) can be neither \(\lambda\) nor \(\text{var}\) because \(M' \not\in \Lambda\). The hypothesis \(M' \not\in \Lambda\) is crucial: for example, \(x \Rightarrow x\) and \(I \Delta \Rightarrow \Delta\) but \(I \Delta \not\Rightarrow \Delta\) and thus \(x(I \Delta) \not\Rightarrow x\Delta\).

4. \(\Rightarrow_{\lambda} \subseteq \Rightarrow_{\sigma} \subseteq \Rightarrow_{\sigma}\). As a consequence, \(\Rightarrow = \Rightarrow_{\sigma}\) and (by Proposition 4) \(\Rightarrow\) is confluent.

5. \(\Rightarrow_{\sigma} \subseteq \Rightarrow \subseteq \Rightarrow_{\sigma}\), so \(\Rightarrow_{\sigma} = \Rightarrow_{\sigma}\). Thus, by Remark 11.1, variables and abstractions are preserved under \(\Rightarrow_{\sigma}\)-expansion, i.e., if \(M \Rightarrow_{\sigma} x\) (resp. \(M \Rightarrow_{\sigma} \lambda x. N\)) then \(M = x\) (resp. \(M = \lambda x. N\) with \(N \Rightarrow x\)).

6. For any \(R \in \{\Rightarrow_{\beta_\iota}, \Rightarrow_{\sigma}\}\), if \(M \Rightarrow M'\) then \(M\{V/x\} \Rightarrow M'\{V/x\}\) for any \(V \in \Lambda\). The proof is by straightforward induction on the derivation of \(M \Rightarrow M'\) for any \(R \in \{\Rightarrow_{\beta_\iota}, \Rightarrow_{\sigma}\}\).

As expected, a basic property of parallel reduction \(\Rightarrow\) is the following:

Lemma 13 (Substitution lemma for \(\Rightarrow\)). If \(M \Rightarrow M'\) and \(V \Rightarrow V'\) then \(M\{V/x\} \Rightarrow M'\{V'/x\}\).

Proof. By straightforward induction on the derivation of \(M \Rightarrow M'\).
Lemma 14 (Commutation of head reductions).
1. If \( M \xrightarrow{\beta_\sigma} L \xrightarrow{\beta_\sigma} N \) then there exists \( L' \in \Delta \) such that \( M \xrightarrow{\beta_\sigma} L' \xrightarrow{\beta_\sigma} N \).
2. If \( M \xrightarrow{\beta_\sigma} L \xrightarrow{\beta_\sigma} N \) then there exists \( L' \in \Delta \) such that \( M \xrightarrow{\beta_\sigma} L' \xrightarrow{\beta_\sigma} N \).
3. If \( M \xrightarrow{\beta_\sigma} M' \) then there exists \( N \in \Delta \) such that \( M \xrightarrow{\beta_\sigma} N \xrightarrow{\sigma} M' \).

Proof. 1. By induction on the derivation of \( M \xrightarrow{\beta_\sigma} L \). Let us consider its last rule \( r \).
   - If \( r = \alpha_1 \), then \( M = (\alpha_1 . M_0) N_0 L_0 M_1 . . . M_m \) and \( m \in \mathbb{N} \) and \( x \notin \text{fv}(L_0) \). Since \( L \xrightarrow{\beta_\sigma} N \), there are only two cases:
     - either \( N_0 \xrightarrow{\beta_\sigma} N' \) and \( N = (\lambda x . M_0) N'_0 L_0 M_1 . . . M_m \) (according to the rule right for \( \xrightarrow{\beta_\sigma} \)), then \( M \xrightarrow{\beta_\sigma} \lambda x . (\lambda x . M_0) N'_0 L_0 M_1 . . . M_m \xrightarrow{\sigma} N \);
     - or \( N_0 \in \Delta \) and \( N = M_0 (\Lambda_0 / x) L_0 M_1 . . . M_m \) (by the rule \( \beta_\sigma \), as \( x \notin \text{fv}(L_0) \)), so \( M \xrightarrow{\beta_\sigma} N \).
   - If \( r = \alpha_2 \), then \( M = V((\alpha_2 . L_0) N_0) M_1 . . . M_m \) and \( L = (\alpha_2 . V L_0) N_0 M_1 . . . M_m \) with \( m \in \mathbb{N} \) and \( x \notin \text{fv}(V) \). Since \( L \xrightarrow{\beta_\sigma} N \), there are only two cases:
     - either \( N_0 \xrightarrow{\beta_\sigma} N' \) and \( N = (\alpha_2 . V L_0) N'_0 M_1 . . . M_m \) (according to the rule right for \( \xrightarrow{\beta_\sigma} \)), then \( M \xrightarrow{\beta_\sigma} \alpha_2 . V ((\lambda x . L_0) N'_0) M_1 . . . M_m \xrightarrow{\sigma} N \);
     - or \( N_0 \in \Delta \) and \( N = V L_0 (\Lambda_0 / x) M_1 . . . M_m \) (by the rule \( \beta_\sigma \), as \( x \notin \text{fv}(V) \)), so \( M \xrightarrow{\beta_\sigma} N \).
Finally, if \( r = \text{right} \) then \( M = V N_0 M_1 . . . M_m \) and \( L = V N'_0 M_1 . . . M_m \) with \( m \in \mathbb{N} \) and \( N_0 \xrightarrow{\beta_\sigma} N' \). By Remark 11.2, \( N'_0 \notin \Delta \), and thus, since \( L \xrightarrow{\beta_\sigma} N \), the only possibility is that \( N''_0 \xrightarrow{\beta_\sigma} N'' \) and \( N = V N''_0 M_1 . . . M_m \) (according to the rule right for \( \beta_\sigma \)). By induction hypothesis, there exists \( N''_0 \in \Delta \) such that \( N_0 \xrightarrow{\beta_\sigma} N''_0 \xrightarrow{\beta_\sigma} N'' \). Therefore, \( M \xrightarrow{\beta_\sigma} V N''_0 M_1 . . . M_m \xrightarrow{\sigma} N \).

2. Immediate consequence of Lemma 14.1, using standard techniques of rewriting theory.
3. Immediate consequence of Lemma 14.2, using standard techniques of rewriting theory.

We are now able to travel over again Takehashi’s method [17, 4] in our setting with \( \beta_\sigma \)- and \( \sigma \)-reduction. The next four lemmas govern the strong parallel reduction and will be used to prove Lemma 19.

Lemma 15. If \( M \Rightarrow M' \) and \( N \Rightarrow N' \) and \( M' \notin A_\sigma \) , then \( MN \Rightarrow M'N' \).

Proof. One has \( MN \Rightarrow M'N' \) by Remark 12.1 and since \( M \Rightarrow M' \). By hypothesis, there exist \( m, n \in \mathbb{N} \) and \( M_0, . . . , M_m, N_0, . . . , N_n \) such that \( M = M_0, M_m = N_0, N_n \Rightarrow M' \), \( M_i \xrightarrow{\beta_\sigma} M_{i+1} \) for any \( 0 \leq i < m \) and \( N_j \xrightarrow{\beta_\sigma} N_{j+1} \) for any \( 0 \leq j < n \); by Remark 12.2, \( M_i N \xrightarrow{\beta_\sigma} M_{i+1} N \) for any \( 0 \leq i < m \) and \( N_j N \xrightarrow{\beta_\sigma} N_{j+1} N \) for any \( 0 \leq j < n \). As \( M' \notin A_\sigma \), one has \( N_0 N \xrightarrow{\beta_\sigma} M'N' \) by Remark 12.3. Therefore, \( MN \Rightarrow M'N' \).

Lemma 16. If \( M \Rightarrow M' \) and \( N \Rightarrow N' \) then \( MN \Rightarrow M'N' \).

Proof. If \( M' \notin A_\sigma \) then \( MN \Rightarrow M'N' \) by Lemma 15 and since \( N \Rightarrow N' \).

Assume \( M' \in A_\sigma : MN \Rightarrow M'N' \) by Remark 12.1, as \( M \Rightarrow M' \) and \( N \Rightarrow N' \). By hypothesis, there are \( m, m', n, n' \in \mathbb{N} \) and \( M_0, . . . , M_m, M_0', . . . , M_{m'}', N_0, . . . , N_n, N_0', . . . , N_{n'}' \) such that:
- \( M = M_0, M_m = M_0', M_{m'}' \Rightarrow M', M_i \xrightarrow{\beta_\sigma} M_{i+1} \) for any \( 0 \leq i < m \), and \( M' \xrightarrow{\beta_\sigma} M'_{m+1} \) for any \( 0 \leq i < m' \),
- \( N = N_0, N_n = N_0', N_{n'}' \Rightarrow N', N_j \xrightarrow{\beta_\sigma} N_{j+1} \) for any \( 0 \leq j < n \), and \( N'_{j'} \xrightarrow{\beta_\sigma} N'_{j'+1} \) for any \( 0 \leq j' < n' \).

By Remark 11.3, \( M_{m'}' \in A_\sigma \), since \( M' \in A_\sigma \), therefore \( m' = 0 \) by Remark 11.2, and thus \( M_m = M_0 \Rightarrow M' \) (and \( M_m \Rightarrow M' \) since \( \Rightarrow \subseteq \Rightarrow \)) and \( M_m \in A_\sigma \). Using the rules right for \( \xrightarrow{\beta_\sigma} \) and \( \xrightarrow{\beta_\sigma} \), one has \( M_0 N_j \xrightarrow{\beta_\sigma} M_0 N_{j+1} \) for any \( 0 \leq j < n \), and \( M_m N'_{j'} \xrightarrow{\beta_\sigma} M_m N'_{j'+1} \) for any \( 0 \leq j' < n' \). By Remark 12.2, \( M_i N_0 \xrightarrow{\beta_\sigma} M_{i+1} N_0 \) for any \( 0 \leq i < m \). By applying
the rule right for $\Rightarrow$, one has $M_mN_n \Rightarrow M'N'$. Therefore, $MN = M_0N_0 \Rightarrow \beta_s M_mN_n \Rightarrow \beta_s M_mN_n = M_mN_0 \Rightarrow M_mN_0 \Rightarrow M'N'$ and hence $MN \Rightarrow M'N'$.

**Lemma 17.** If $M \equiv M'$ and $V \Rightarrow V'$, then $M\{V/x\} \Rightarrow M'\{V'/x\}$.

**Proof.** By Lemma 13, one has $M\{V/x\} \Rightarrow M'\{V'/x\}$ since $M \Rightarrow M'$ and $V \Rightarrow V'$. We proceed by induction on $M \in \Lambda$. Let us consider the last rule $r$ of the derivation of $M \equiv M'$.

If $r = \text{var}$ then there are two cases: either $M = x$ and then $M\{V/x\} = V \Rightarrow V' = M'\{V'/x\}$; or $M = y \neq x$ and then $M\{V/x\} = y = M'\{V'/x\}$, so $M\{V/x\} \Rightarrow M'\{V'/x\}$ by Remark 10.

If $r = \lambda$ then $M = \lambda y.N$ and $M' = \lambda y.N'$ with $N \Rightarrow N'$; we can suppose without loss of generality that $y \notin \text{fv}(V) \cup \{x\}$. One has $N\{V/x\} \Rightarrow N'\{V'/x\}$ according to Lemma 13. By applying the rule $\lambda$ for $\Rightarrow$, one has $M\{V/x\} = \lambda y.N\{V/x\} \Rightarrow \lambda y.N'\{V'/x\} = M'\{V'/x\}$ and thus $M\{V/x\} \Rightarrow M'\{V'/x\}$.

Finally, if $r = \text{right}$ then $M = U\text{NM}_1\ldots\text{NM}_m$, and $M' = U'\text{NM}'_1\ldots\text{NM}'_m$ for some $m \in \mathbb{N}$ with $U, U' \in \Lambda_\nu, U \Rightarrow U', \mathbb{N} \Rightarrow \mathbb{N}'$ and $M_i \Rightarrow M'_i$ for any $1 \leq i \leq m$. By induction hypothesis, $U\{V/x\} \Rightarrow U'\{V'/x\} /x$ (indeed $U \equiv U'$ according to Remark 11.5) and $\mathbb{N}\{V/x\} \Rightarrow \mathbb{N}'\{V'/x\}$. By Lemma 13, $M_i\{V/x\} \Rightarrow M'_i\{V'/x\}$ for any $1 \leq i \leq m$. By Lemma 16, $U\{V/x\}\mathbb{N}\{V/x\} \Rightarrow U'\{V'/x\}\mathbb{N}'\{V'/x\}$ and hence, by applying Lemma 15 $m$ times since $U'\{V'/x\}\mathbb{N}'\{V'/x\} \notin \Lambda_\nu$, one has $M\{V/x\} = U\{V/x\}\mathbb{N}\{V/x\}M_1\{V/x\} \ldots \text{NM}_m\{V/x\} \Rightarrow U'\{V'/x\}\mathbb{N}'\{V'/x\}M'_1\{V'/x\} \ldots \text{NM}'_m\{V'/x\} = M'\{V'/x\}$.

In the proof of the two next lemmas, as well as in the proof of Corollary 21 and Theorem 22, our Lemma 14 plays a crucial role: indeed, since the head $\sigma$-reduction well interact with the head $\beta_c$-reduction, Takahashi’s method [17, 4] is still working when adding the reduction rules $\sigma_1$ and $\sigma_2$ to Plotkin’s $\beta_c$-reduction.

**Lemma 18 (Substitution lemma for $\Rightarrow$).** If $M \Rightarrow M'$ and $V \Rightarrow V'$ then $M\{V/x\} \Rightarrow M'\{V'/x\}$.

**Proof.** By Lemma 13, one has $M\{V/x\} \Rightarrow M'\{V'/x\}$ since $M \Rightarrow M'$ and $V \Rightarrow V'$. By hypothesis, there exist $m, n \in \mathbb{N}$ and $M_0, \ldots, M_m, N_0, \ldots, N_n$ such that $M = M_0, M_m = N_m, N_n \Rightarrow M', M_i \Rightarrow \beta_s M_{i+1}$ for any $0 \leq i < m$ and $N_j \Rightarrow \sigma N_{j+1}$ for any $0 \leq j < n$; by Remark 12.6, $M_i\{V/x\} \Rightarrow \beta_s M_{i+1}\{V/x\}$ for any $0 \leq i < m$ and $N_j\{V/x\} \Rightarrow \sigma N_{j+1}\{V/x\}$ for any $0 \leq j < n$. By Lemma 17, one has $N_n\{V/x\} \Rightarrow M'\{V'/x\}$, thus there exist $L, N \in \Lambda$ such that $M\{V/x\} \Rightarrow \beta_s N_0\{V/x\} \Rightarrow \sigma N_{j+1}\{V/x\} \Rightarrow \beta_s N \Rightarrow \sigma L \Rightarrow M'\{V'/x\}$. By Lemma 14.2, there exists $\mathbb{N}' \in \Lambda$ such that $M\{V/x\} \Rightarrow \beta_s N_0\{V/x\} \Rightarrow \beta_s N' \Rightarrow \sigma L \Rightarrow M'\{V'/x\}$, therefore $M\{V/x\} \Rightarrow M'\{V'/x\}$.

Now we are ready to prove a key lemma, which states that parallel reduction $\Rightarrow$ coincides with strong parallel reduction $\Rightarrow$ (the inclusion $\Rightarrow \subseteq \Rightarrow$ is trivial).

**Lemma 19 (Key Lemma).** If $M \Rightarrow M'$ then $M \Rightarrow M'$.

**Proof.** By induction on the derivation of $M \Rightarrow M'$. Let us consider its last rule $r$.

If $r = \text{var}$ then $M = M_1 \ldots M_m$ and $M' = M'_1 \ldots M'_m$ where $m \in \mathbb{N}$ and $M_i \Rightarrow M'_i$ for any $1 \leq i \leq m$. By reflexivity of $\Rightarrow$ (Remark 10), $x \Rightarrow x$. By induction hypothesis, $M_i \Rightarrow M'_i$ for any $1 \leq i \leq m$. Therefore, $M \Rightarrow M'$ by applying Lemma 16 $m$ times.

If $r = \lambda$ then $M = (\lambda x.M_0)M_1 \ldots M_m$ and $M' = (\lambda x.M'_0)M'_1 \ldots M'_m$ where $m \in \mathbb{N}$ and $M_i \Rightarrow M'_i$ for any $0 \leq i \leq m$. By induction hypothesis, $M_i \Rightarrow M'_i$ for any $1 \leq i \leq m$. According to Remark 11.4, $\lambda x.M_0 \Rightarrow \lambda x.M'_0$. So $M \Rightarrow M'$ by applying Lemma 16 $m$ times.

If $r = \beta$, then $M = (\lambda x.M_0)V M_1 \ldots M_m$ and $M' = M'_0\{V/x\}M'_1 \ldots M'_m$ where $m \in \mathbb{N}, V \Rightarrow V'$ and $M_i \Rightarrow M'_i$ for any $0 \leq i \leq m$. By induction hypothesis, $V \Rightarrow V'$ and $M_i \Rightarrow M'_i$
for any $0 \leq i \leq m$. By applying the rule $\beta_v$ for $\overset{\text{t}}{\beta_{v}}$, one has $M \overset{\text{t}}{\beta_{v}} M_0 \{V/x\} M_1 \ldots M_m$; moreover $M_0 \{V/x\} M_1 \ldots M_m \Rightarrow M'$ by Lemma 18 and by applying Lemma 16 $m$ times, there are $L, N \in \Lambda$ such that $M \overset{\text{t}}{\beta_{v}} M_0 \{V/x\} M_1 \ldots M_m \overset{\text{t}}{\beta_{v}} L \overset{\text{t}}{\beta_{v}} N \Rightarrow \Lambda M'$. So $M \Rightarrow \Lambda M'$.

If $r = \sigma_1$ then $M = (\lambda x.M_0)N_0 M_1 \ldots M_m$ and $M' = (\lambda x.M_0' L_0)N_0' M_1' \ldots M_m'$ where $m \in \mathbb{N}$, $L_0 \Rightarrow L_0'$, $N_0 \Rightarrow N_0'$ and $M_i \Rightarrow M_i'$ for any $0 \leq i \leq m$. By induction hypothesis, $N_0 \Rightarrow N_0'$ and $M_i \Rightarrow M_i'$ for any $1 \leq i \leq m$. By applying the rule $\sigma_1$ for $\overset{\text{t}}{\sigma}$, one has $M \overset{\text{t}}{\sigma} (\lambda x.M_0 L_0)N_0 M_1 \ldots M_m \Rightarrow (\lambda x.M_0' L_0)N_0' M_1' \ldots M_m'$ according to Remark 11.4. So $(\lambda x.M_0 L_0)N_0 M_1 \ldots M_m \Rightarrow (\lambda x.M_0' L_0)N_0' M_1' \ldots M_m'$ by applying Lemma 16 $m+1$ times, hence there are $L, N \in \Lambda$ such that $M \overset{\text{t}}{\sigma} (\lambda x.M_0 L_0)N_0 M_1 \ldots M_m \overset{\text{t}}{\sigma} L \overset{\text{t}}{\sigma} N \Rightarrow \Lambda M'$. By Lemma 14.2, there is $L' \in \Lambda$ such that $M \overset{\text{t}}{\sigma} L' \overset{\text{t}}{\sigma} L \overset{\text{t}}{\sigma} N \Rightarrow \Lambda M'$ and thus $M \Rightarrow \Lambda M'$.

Finally, if $r = \sigma_3$ then $M = V((\lambda x.L_0)N_0) M_1 \ldots M_m$ and $M' = (\lambda x.V' L_0')N_0' M_1' \ldots M_m'$ with $m \in \mathbb{N}$, $V \Rightarrow V'$, $L_0 \Rightarrow L_0'$, $N_0 \Rightarrow N_0'$ and $M_i \Rightarrow M_i'$ for any $1 \leq i \leq m$. By induction hypothesis, $N_0 \Rightarrow N_0'$ and $M_i \Rightarrow M_i'$ for any $1 \leq i \leq m$. By the rule $\sigma_3$ for $\overset{\text{t}}{\sigma}$, one has $M \overset{\text{t}}{\sigma} (\lambda x.V L_0)N_0 M_1 \ldots M_m$. By Remark 12.1, $V L_0 \Rightarrow V' L_0'$ and thus $\lambda x.V L_0 \Rightarrow \lambda x.V' L_0'$ according to Remark 11.4. So $(\lambda x.V L_0)N_0 M_1 \ldots M_m \Rightarrow \Lambda M'$ by applying Lemma 16 $m+1$ times, hence there are $L, N \in \Lambda$ such that $M \overset{\text{t}}{\sigma} (\lambda x.V L_0)N_0 M_1 \ldots M_m \overset{\text{t}}{\sigma} L \overset{\text{t}}{\sigma} N \Rightarrow \Lambda M'$. By Lemma 14.2, there is $L' \in \Lambda$ such that $M \overset{\text{t}}{\sigma} L' \overset{\text{t}}{\sigma} L \overset{\text{t}}{\sigma} N \Rightarrow \Lambda M'$, so $M \Rightarrow \Lambda M'$.

Next Lemma 20 and Corollary 21 show that internal parallel reduction can be shifted after head reductions.

\begin{lemma}[Postponement] If $M \overset{\text{t}}{\beta_{v}} L$ and $L \overset{\text{t}}{\beta_{v}} N$ (resp. $L \overset{\text{t}}{\beta_{v}} N$) then there exists $L' \in \Lambda$ such that $M \overset{\text{t}}{\beta_{v}} L'$ (resp. $M \overset{\text{t}}{\beta_{v}} L'$) and $L' \Rightarrow N$.
\end{lemma}

\begin{proof}
By induction on the derivation of $M \overset{\text{t}}{\beta_{v}} L$. Let us consider its last rule $r$.

If $r = \text{var}$, then $M = x = L$ which contradicts $L \overset{\text{t}}{\beta_{v}} N$ and $L \overset{\text{t}}{\beta_{v}} N$ by Remarks 11.1-2.

If $r = \lambda$ then $M = \lambda x.L'$ for some $L' \in \Lambda$, which contradicts $L \overset{\text{t}}{\beta_{v}} N$ and $L \overset{\text{t}}{\beta_{v}} N$ by Remarks 11.1-2.

Finally, if $r = \Rightarrow$ then $M = VM_0 M_1 \ldots M_m$ and $L = V' L_0 L_1 \ldots L_m$ where $m \in \mathbb{N}$, $V \Rightarrow V'$ (so $V \overset{\text{t}}{\beta_{v}} V'$ by Remark 11.5), $M_0 \overset{\text{t}}{\beta_{v}} L_0$ (thus $M_0 \Rightarrow L_0$ since $\overset{\text{t}}{\beta_{v}} \subseteq \Rightarrow$) and $M_i \Rightarrow L_i$ for any $1 \leq i \leq m$.

- If $L \overset{\text{t}}{\beta_{v}} N$ then there are two cases, depending on the last rule $r'$ of the derivation of $L \overset{\text{t}}{\beta_{v}} N$.
  - If $r' = \beta_v$ then $V' = \lambda x.N_0'$, $L_0 \in \Lambda_v$ and $N = N_0' \{L_0/x\} L_1 \ldots L_m$, thus $M_0 \in \Lambda_v$ and $V = \lambda x.N_0$ with $N_0 \Rightarrow N_0'$ by Remark 11.3. By Lemma 13, one has $N_0 \{M_0/x\} \Rightarrow N_0' \{L_0/x\}$. Let $L' = N_0 \{M_0/x\} M_1 \ldots M_m$; so $M = (\lambda x.N_0) M_0 M_1 \ldots M_m \overset{\text{t}}{\beta_{v}} L'$ (apply the rule $\beta_v$ for $\overset{\text{t}}{\beta_{v}}$ and $L' \Rightarrow N$ by applying Remark 12.1 $m$ times).
  - If $r' = \Rightarrow$ then $V' N_0 L_1 \ldots L_m$ with $L_0 \overset{\text{t}}{\beta_{v}} N_0$. By induction hypothesis, there exists $L_0' \in \Lambda$ such that $M_0 \overset{\text{t}}{\beta_{v}} L_0 \Rightarrow N_0$. Let $L' = V' L_0 M_1 \ldots M_m$; so $M \overset{\text{t}}{\beta_{v}} L'$ (apply the rule $\Rightarrow$ for $\overset{\text{t}}{\beta_{v}}$) and $L' \Rightarrow N$ by applying Remark 12.1 $m+1$ times.

- If $L \overset{\text{t}}{\sigma} N$ then there are three cases, depending on the last rule $r'$ of the derivation of $L \overset{\text{t}}{\sigma} N$.
  - If $r' = \sigma_1$ then $m > 0$, $V' = \lambda x.N_0'$ and $N = (\lambda x.N_0' L_1) L_2 \ldots L_m$, thus $V = \lambda x.N_0$ with $N_0 \Rightarrow N_0'$ by Remark 11.3. Using Remarks 12.1 and 11.4, one has $\lambda x.N_0 M_1 \Rightarrow \lambda x.N_0' L_1$. Let $L' = (\lambda x.N_0 M_1) M_2 \ldots M_m$; so $M = (\lambda x.N_0) M_0 M_1 \ldots M_m \overset{\text{t}}{\beta_{v}} L'$ (apply the rule $\sigma_1$ for $\overset{\text{t}}{\sigma}$) and $L' \Rightarrow N$ by applying Remark 12.1 $m$ times.
  - If $r' = \sigma_3$ then $L_0 = (\lambda x.L_0) L_0' L_0$ and $N = (\lambda x.V' L_0') L_0' L_0 \ldots L_m$. Since $M_0 \overset{\text{t}}{\beta_{v}} (\lambda x.\Lambda_0) L_0'$, necessarily $M_0 = (\lambda x.M_0) M_0' \overset{\text{t}}{\beta_{v}} L_0$ (so $M_0 \Rightarrow L_0$). Using Remarks 12.1 and 11.4, one has $\lambda x.V M_0 \Rightarrow \lambda x.V' L_0$. Let
\end{proof}
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\[ L' = (\lambda x. VM01)M_{02}M_1 \ldots M_m \] therefore \( M = V((\lambda x. M_{01})M_{02})M_1 \ldots M_m \stackrel{\sigma}{\rightarrow} L' \) (apply the rule \( \sigma_3 \) for \( \beta \sigma \)) and \( L' \Rightarrow N \) by applying Remark 12.1 \( m + 1 \) times.

1. If \( r' = \text{right} \) then \( N = V'N_0L_1 \ldots L_m \) with \( L_0 \stackrel{\beta}{\rightarrow} N_0 \). By induction hypothesis, there exists \( \lambda L_0 \in \Lambda \) such that \( M_0 \stackrel{\beta}{\rightarrow} L_0' \Rightarrow N_0 \). Let \( L' = V'L_1M_1 \ldots M_m \) so \( M \stackrel{\beta}{\rightarrow} L' \) (apply the rule \textit{right} for \( \beta \sigma \)) and \( L' \Rightarrow N \) by applying Remark 12.1 \( m + 1 \) times.

\[ \text{\textbf{Corollary 21.}} \text{ If } M \xrightarrow{\beta} L \text{ and } L \xrightarrow{\beta} N \text{ (resp. } L \xrightarrow{\beta} N \text{), then there exist } L', L'' \in \Lambda \text{ such that } M \xrightarrow{\beta} L' \xrightarrow{\beta} L'' \xrightarrow{\beta} N \text{ (resp. } M \xrightarrow{\beta} L' \xrightarrow{\beta} L'' \xrightarrow{\beta} N \text{).} \]

\[ \text{\textbf{Proof.}} \text{ Immediate by Lemma 20 and Lemma 19, applying Lemma 14.2 if } L \xrightarrow{\beta} N. \]

Now we obtain our first main result (Theorem 22): any \( v \)-reduction sequence can be sequentialized in a head \( \beta \)-reduction sequence followed by an internal reduction sequence. In ordinary \( \lambda \)-calculus, the well-known result corresponding to our Theorem 22 says that a \( \beta \)-reduction sequence can be factorized in a head reduction sequence followed by an internal reduction sequence (see for example [17, Corollary 2.6]).

\[ \text{\textbf{Theorem 22 (Sequentialization).}} \text{ If } M \xrightarrow{\ast} M' \text{ then there exist } L, N \in \Lambda \text{ such that } M \xrightarrow{\ast} L \xrightarrow{\ast} N \xrightarrow{\ast} M'. \]

\[ \text{\textbf{Proof.}} \text{ By Remark 12.4, } M \Rightarrow^* M' \text{ and thus there are } m \in \mathbb{N} \text{ and } M_0, \ldots, M_m \in \Lambda \text{ such that } M = M_0, \text{ } M_m = M' \text{ and } M_i \Rightarrow M_{i+1} \text{ for any } 0 \leq i < m. \text{ We prove by induction on } m \in \mathbb{N} \text{ that there are } L, N \in \Lambda \text{ such that } M \Rightarrow^*_L L \Rightarrow^*_N M'. \text{ We prove by induction hypothesis, there exist } L', N' \in \Lambda \text{ such that } M_1 \Rightarrow^*_L L' \Rightarrow^*_N N' \Rightarrow^* M'. \text{ By applying Lemma 19 to } M, \text{ there exist } L_0, N_0 \in \Lambda \text{ such that } M \xrightarrow{\beta} L_0 \xrightarrow{\beta} N_0 \xrightarrow{\beta} M_1. \text{ By applying Corollary 21 repeatedly, there exists } N \in \Lambda \text{ such that } N_0 \xrightarrow{\beta} N \xrightarrow{\beta} N' \text{ and hence } M \xrightarrow{\beta} N \xrightarrow{\beta} M'. \text{ According to Lemma 14.3, there exists } L \in \Lambda \text{ such that } M \xrightarrow{\beta} L \xrightarrow{\beta} N \xrightarrow{\beta} M'. \]

It is worth noticing that in Definition 7 there is no distinction between head \( \sigma_1 \) and head \( \sigma_3 \)-reduction steps, and, according to it, the sequentialization of Theorem 22 imposes no order between head \( \sigma \)-reductions. We denote by \( \rightarrow_{\sigma_1} \) and \( \rightarrow_{\sigma_3} \) respectively the reduction relations \( \rightarrow_{\sigma_1} \cap \rightarrow_{\sigma_3} \) and \( \rightarrow_{\sigma_3} \cap \rightarrow_{\sigma_3} \). So, a natural question arises: is it possible to sequentialize them? The answer is negative, as proved by the next two counterexamples.

- Let \( M = \lambda y.(\lambda y. z)(z I) \Delta \) and \( N = (\lambda y. z)(z I) \Delta \) where \( M \xrightarrow{\ast}_{\sigma} (\lambda y. z)(z I) \Delta \xrightarrow{\ast}_{\sigma} N \), but there exists no \( L \) such that \( M \xrightarrow{\ast}_{\sigma} L \xrightarrow{\ast}_{\sigma} N \). In fact \( M \) contains only a head \( \sigma_3 \)-redex \( (\lambda y. z)(z I) \Delta \) contains only a head \( \sigma_3 \)-redex.

- Let \( M = x \cdot ((\lambda y. z')(z I) \Delta) \) and \( N = (\lambda y. z')(z I) \Delta \) where \( M \xrightarrow{\ast}_{\sigma} x \cdot ((\lambda y. z')(z I) \Delta) \xrightarrow{\ast}_{\sigma} N \), but there is no \( L \) such that \( M \xrightarrow{\ast}_{\sigma} L \xrightarrow{\ast}_{\sigma} N \). In fact \( M \) contains only a head \( \sigma_1 \)-redex \( x \cdot ((\lambda y. z')(z I) \Delta) \) contains only a head \( \sigma_3 \)-redex.

So, the imposibility of sequentializing a head \( \sigma \)-reduction sequence is due to the fact that a head \( \sigma_1 \)-reduction step can create a head \( \sigma_3 \)-redex, and viceversa. This is not a problem, since head \( \sigma \)-reduction is strongly normalizing (by Proposition 4 and since \( \rightarrow_{\sigma} \subseteq \rightarrow_{\sigma_3} \)). Our approach does not force a strict order of head \( \sigma \)-reductions.
4 Standardization

Now we are able to prove the main result of this paper, i.e., a standardization theorem for $\lambda^v_\sigma$ (Theorem 25). In particular we provide a notion of standard reduction sequence that avoids any auxiliary notion of residual redexes, by closely following the definition given in [15].

- **Notation.** For any $k, m \in \mathbb{N}$ with $k \leq m$, we denote by $[M_0, \ldots, M_k, \ldots, M_m]^{\text{head}}$ a sequence of terms such that $M_i \beta_{\beta_v} \to M_{i+1}$ when $0 \leq i < k$, and $M_i \beta_{\sigma} \to M_{i+1}$ when $k \leq i < m$.

It is easy to check that $[M]^{\text{head}}$ for any $M \in \Lambda$. The notion of standard sequence of terms is defined by using the previous notion of head-sequence. Our notion of standard reduction sequence is mutually defined together with the notion of inner-sequence of terms (Definition 23). This definition allows us to avoid non-deterministic cases remarked in [7] (we provide more details at the end of this section). We denote by $[M_0, \ldots, M_m]^{\text{std}}$ (resp. $[M_0, \ldots, M_m]^{\text{in}}$) a standard (resp. inner) sequence of terms.

- **Definition 23** (Standard and inner sequences). Standard and inner sequences of terms are defined by mutual induction as follows:
  1. if $[M_0, \ldots, M_m]^{\text{head}}$ and $[M_0, \ldots, M_{m+n}]^{\text{in}}$ then $[M_0, \ldots, M_{m+n}]^{\text{std}}$, where $m, n \in \mathbb{N}$;
  2. $[M]^{\text{in}}$, for any $M \in \Lambda$;
  3. if $[M_0, \ldots, M_m]^{\text{std}}$ then $[\lambda z. M_0, \ldots, \lambda z. M_m]^{\text{in}}$, where $m \in \mathbb{N}$;
  4. if $[V_0, \ldots, V_n]^{\text{std}}$ and $[N_0, \ldots, N_n]^{\text{in}}$ then $[V_0 N_0, \ldots, V_n N_n]^{\text{in}}$, where $h, n \in \mathbb{N}$;
  5. if $[N_0, \ldots, N_m]^{\text{in}}$ and $N_0 \notin \Lambda$, then $[N_0 M_0, \ldots, N_m M_m]^{\text{in}}$, where $m, n \in \mathbb{N}$.

For instance, let $M = (\lambda y.Ix)(z(\Delta I))(II) : M \to (\lambda y.Ix)(z(\Delta I))I \to (\lambda y.x)(z(\Delta I))I$ and $M \to (\lambda y.Ix(II))(z(\Delta I)) \to (\lambda y.Ix(II))(z(\Delta I))I$ are not standard sequences; $M \to (\lambda y.Ix)(z(\Delta I))I$ and $M \to (\lambda y.Ix)(z(\Delta I))I \to (\lambda y.Ix)(z(\Delta I))I \to (\lambda y.Ix)(z(\Delta I))I \to (\lambda y.Ix)(z(\Delta I))I \to (\lambda y.Ix)(z(\Delta I))I$ are standard sequences.

- **Remark 24.** For any $n \in \mathbb{N}$, if $[N_0, \ldots, N_n]^{\text{in}}$ (resp. $[N_0, \ldots, N_n]^{\text{head}}$) then $[N_0, \ldots, N_n]^{\text{std}}$. Indeed, $[N_0]^{\text{head}}$ (resp. $[N_n]^{\text{in}}$ by Definition 23.2), so $[N_0, \ldots, N_n]^{\text{std}}$ by Definition 23.1.

  In particular, $[N]^{\text{std}}$ for any $N \in \Lambda$: apply Definition 23.2 and Remark 24 for $n = 0$.

- **Theorem 25** (Standardization).
  1. If $M \to^*_v M'$ then there is a sequence $[M, \ldots, M']^{\text{std}}$.
  2. If $M \to^*_v M'$ then there is a sequence $[M, \ldots, M']^{\text{in}}$.

Proof. Both statements are proved simultaneously by induction on $M' \in \Lambda$.

  1. If $M = z$ then, by Theorem 22, $M \to^*_\beta L \to^*_\sigma N \to^*_v z$ for some $L, N \in \Lambda$. By Remarks 12.5 and 11.2, $L = N = z$; therefore $M \to^*_v z$ and hence there is a sequence $[M, \ldots, z]^{\text{head}}$. Thus, $[M, \ldots, z]^{\text{std}}$ by Remark 24.
  2. If $M' = \lambda z. N$ then, by Theorem 22, $M \to^*_\beta L \to^*_\sigma L' \to^*_v \lambda z. N'$ for some $L, L' \in \Lambda$. By Remarks 12.5 and 11.2, $L = L' = \lambda z. N$ with $N \to^*_v N'$. So $M \to^*_\beta \lambda z. N$ and hence there is a sequence $[M, \ldots, \lambda z. N]^{\text{head}}$. By induction on (1), there is a sequence $[N, \ldots, N']^{\text{std}}$, thus $[\lambda z. N, \ldots, \lambda z. N']^{\text{in}}$ by Definition 23.3. Therefore $[M, \ldots, \lambda z. N, \ldots, \lambda z. N']^{\text{std}}$ by Definition 23.1.
  3. If $M' = N'L'$ then, by Theorem 22, $M \to^*_\beta M'' \to^*_\sigma M_0 \to^*_v N'L'$ for some $M''$, $M_0 \in \Lambda$. By Remark 3, $M_0 = NL$ for some $N, L \in \Lambda$, since $\to^*_v \subseteq \to^*_\sigma$ and $M' \notin \Lambda$. Thus there is a sequence $[M, \ldots, M'', \ldots, NL]^{\text{head}}$. By Remark 12.5, $NL \to^*_v N'L'$; clearly, each step of $\to^*_v$ is an instance of the rule right of Definition 9. There are two sub-cases.
If $N \in \Lambda_v$ then $N \Rightarrow^* N'$ and $L \xrightarrow{\ast} L'$, so $N \Rightarrow^* N'$ and $L \xrightarrow{\ast} L'$ by Remarks 12.4.5. By induction respectively on (1) and (2), there are sequences $[N, \ldots, N']_{std}$ and $[L, \ldots, L']_{in}$, thus $[NL, \ldots, NL', N'L']_{in}$ by Definition 23.4. Therefore $[M, \ldots, M', \ldots, NL, \ldots, N'L']_{std}$ by Definition 23.1.

If $N \notin \Lambda_v$ (i.e., $N = VM_1 \ldots M_m$ with $m > 0$) then $N \xrightarrow{\ast} N'$ and $L \Rightarrow^* L'$, so $N \xrightarrow{\ast} N'$ and $L \Rightarrow^* L'$ by Remarks 12.4-5. By induction respectively on (2) and (1), there are sequences $[N, \ldots, N']_{in}$ and $[L, \ldots, L']_{std}$. Hence $[NL, \ldots, NL', N'L']_{in}$ by Definition 23.5. Thus $[M, \ldots, M', \ldots, NL, \ldots, N'L']_{std}$ by Definition 23.1.

Due to non-sequentialization of head $\sigma_\gamma$- and head $\sigma_\delta$-reductions, several standard sequences may have the same starting term and ending term: for instance, if $M = I(\Delta I)I$ and $N = (\lambda z. (\lambda x. I)(zz))I$ then $M \Rightarrow^* N'$ and $L \Rightarrow^* L'$, thus $N \subseteq \Rightarrow^* N'$ and $L \Rightarrow^* L'$ by Remarks 12.4-5. By induction respectively on (2) and (1), there are sequences $[N, \ldots, N']_{in}$ and $[L, \ldots, L']_{std}$ by Definition 23.5.

5 Some conservativity results

The sequentialization result (Theorem 22) has some interesting semantic consequences. It allows us to prove that (Corollary 29) the $\lambda^*_{\sigma}$-calculus is sound with respect to the call-by-value observational equivalence introduced by Plotkin in [15] for $\Lambda_v$. Moreover we can prove that some notions, like that of potential valuability and solvability, introduced in [13] for $\lambda^*_{\sigma}$, coincide with the respective notions for $\lambda^*_{\sigma}$ (Theorem 31). This justifies the idea that $\lambda^*_{\sigma}$ is a useful tool for studying the properties of $\lambda^*_{\sigma}$. Our starting point is the following corollary.

**Corollary 26.**

1. If $M \Rightarrow^*_v V \in \Lambda_v$ then there exists $V' \in \Lambda_v$ such that $M \Rightarrow^*_v V' \xrightarrow{\ast} V'$.

2. For every $V \in \Lambda_v$, $M \xrightarrow{\ast} V$ if and only if $M \xrightarrow{\ast} V$.

**Proof.** The first point is proved by observing that, by Theorem 22, there are $N, L \in \Lambda$ such that $M \xrightarrow{\ast} N \xrightarrow{\ast} L \xrightarrow{\ast} V$. By Remark 12.5, $N \in \Lambda_v$ and thus $L = N$ according to
Remark 11.2. Concerning the second point, the right-to-left direction is a consequence of Lemma 14.3 and Remark 11.2; the left-to-right direction follows from $\rightarrow_{\beta_v} \subseteq \rightarrow_{\lambda_v}$. ▶

Let us recall the notion of observational equivalence defined by Plotkin [15] for $\lambda_v$.

**Definition 27** (Halting, observational equivalence). Let $M \in \Lambda$. We say that (the evaluation of) $M$ halts if there exists $V \in \Lambda_v$ such that $M \xrightarrow{\beta_v} V$.

The (call-by-value) observational equivalence is an equivalence relation $\equiv$ on $\Lambda$ defined by: $M \equiv N$ if, for every context $C$, one has that $C[M]$ halts iff $C[N]$ halts.$^1$

Clearly, similar notions can be defined for $\lambda_v^n$ using $\beta_v^n$ instead of $\beta_v$. Head $\sigma$-reduction plays no role neither in deciding the halting problem for evaluation (Corollary 26.1), nor in reaching a particular value (Corollary 26.2). So, we can conclude that the notions of halting and observational equivalence in $\lambda_v^n$ coincide with the ones in $\lambda_v$, respectively.

Now we compare the equational theory of $\lambda_v^n$ with Plotkin’s observational equivalence.

**Theorem 28** (Adequacy of $\nu$-reduction). If $M \rightarrow^* \nu M'$ then: $M$ halts iff $M'$ halts.

Proof. If $M' \xrightarrow{\beta_v} \nu V \in \Lambda_v$ then $M \rightarrow^* \nu M' \xrightarrow{\beta_v} \nu V$ since $\beta_v \subseteq \nu$. By Corollary 26.1, there exists $V' \in \Lambda_v$ such that $M \xrightarrow{\beta_v} V'$. Thus $M$ halts.

Conversely, if $M \xrightarrow{\beta_v} \nu V \in \Lambda_v$ then $M \xrightarrow{\beta_v} \nu V$ since $\beta_v \subseteq \nu$. By confluence of $\nu$ (Proposition 4, since $M \rightarrow^* \nu M'$) and Remark 3 (since $V \in \Lambda_v$), there is $V' \in \Lambda_v$ such that $V \xrightarrow{\beta_v} V'$ and $M' \rightarrow^* \nu V'$. By Corollary 26.1, there is $V'' \in \Lambda_v$ such that $M' \xrightarrow{\beta_v} V''$. So $M'$ halts. ▶

**Corollary 29** (Soundness). If $M \equiv_\nu N$ then $M \equiv N$.

Proof. Let $C$ be a context. By confluence of $\rightarrow_\nu$ (Proposition 4), $M \equiv_\nu N$ implies that there exists $L \in \Lambda$ such that $M \xrightarrow{\beta_v} L$ and $N \xrightarrow{\beta_v} L$, hence $C[M] \rightarrow^*_\nu C[L]$ and $C[N] \rightarrow^*_\nu C[L]$. By Theorem 28, $C[M]$ halts iff $C[N]$ halts. Therefore, $M \equiv N$. ▶

Plotkin [15, Theorem 5] has already proved that $M \equiv_{\beta_v} N$ implies $M \equiv N$, but our Corollary 29 is not obvious since our $\lambda_v^n$-calculus equates more than Plotkin’s $\lambda_v$-calculus ($=_{\beta_v} \equiv \equiv_\nu$ since $\rightarrow_{\beta_v} \subseteq \rightarrow_\nu$, and Example 5 shows that this inclusion is strict).

The converse of Corollary 29 does not hold since $\lambda x.x(\lambda y.xy) \equiv_\Delta \lambda x.x(\lambda y.xy)$ and $\Delta$ are different $\nu$-normal forms and hence $\lambda x.x(\lambda y.xy) \not\equiv_\nu \Delta$ by confluence of $\rightarrow_\nu$ (Proposition 4).

A further remarkable consequence of Corollary 26.1 is that the notions of potential valubility and solvability for $\lambda_v^n$-calculus (studied in [3]) can be shown to coincide with the ones for Plotkin’s $\lambda_v$-calculus (studied in [13, 14]), respectively. Let us recall their definition.

**Definition 30** (Potential valuability, solvability). Let $M$ be a term:

- $M$ is $\nu$-potentially valuable (resp. $\beta_v$-potentially valuable) if there are $m \in \mathbb{N}$, pairwise distinct variables $x_1, \ldots, x_m$ and $V, V_1, \ldots, V_m \in \Lambda_v$ such that $M\{V_1/x_1, \ldots, V_m/x_m\} \xrightarrow{\beta_v} V$ (resp. $M\{V_1/x_1, \ldots, V_m/x_m\} \xrightarrow{\beta_v} \nu V$);

- $M$ is $\nu$-solvable (resp. $\beta_v$-solvable) if there are $n, m \in \mathbb{N}$, variables $x_1, \ldots, x_m$ and $N_1, \ldots, N_n \in \Lambda$ such that $(\lambda x_1 \ldots x_m.M)N_1 \cdots N_n \xrightarrow{\beta_v} \nu I$ (resp. $(\lambda x_1 \ldots x_m.M)N_1 \cdots N_n \xrightarrow{\beta_v} \nu I$).

**Theorem 31.** Let $M$ be a term:

1. $M$ is $\nu$-potentially valuable if and only if $M$ is $\beta_v$-potentially valuable;

2. $M$ is $\nu$-solvable if and only if $M$ is $\beta_v$-solvable.

$^1$ Original Plotkin’s definition of call-by-value observational equivalence (see [15]) also requires that $C[M]$ and $C[N]$ are closed terms, according to the tradition identifying programs with closed terms.
Since M is v-potentially valuable, there are variables x_1, ..., x_m and V, V_1, ..., V_m ∈ Λ_v (with m ≥ 0) such that M{V_1/x_1, ..., V_m/x_m} →_βv V; then, there exists V' ∈ Λ_v such that M{V_1/x_1, ..., V_m/x_m} →_βv V' by Corollary 26.1 and because $\beta$→$\beta$ v-sub-reductions of $\beta$-potential valuability and $\beta$-solvability, there exist variables $\sigma_0$, $\sigma_1$, ..., $\sigma_n$ (for some $n, m ≥ 0$) such that $(\lambda x_1 ... x_m.M)N_1 ... N_n →_β^* I$; then, there exists V' ∈ Λ_v such that $(\lambda x_1 ... x_m.M)N_1 ... N_n →_β^* V' \langle \gamma \gamma \rangle v I$ by Corollary 26.1 and because $\beta$→$\beta$ v-sub-reductions of $\beta$-potential valuability and $\beta$-solvability, there exist variables $\sigma_0$, $\sigma_1$, ..., $\sigma_n$ (for some $n, m ≥ 0$) such that $(\lambda x_1 ... x_m.M)N_1 ... N_n →_β^* I$ by Corollary 26.1 and because $\beta$→$\beta$ v-sub-reductions of $\beta$-potential valuability and $\beta$-solvability, there exist variables $\sigma_0$, $\sigma_1$, ..., $\sigma_n$ (for some $n, m ≥ 0$) such that $(\lambda x_1 ... x_m.M)N_1 ... N_n →_β^* V' \langle \gamma \gamma \rangle v x$, hence V' = V by Remark 26.1 again. Since $\beta$→$\beta$ v-solvable, there exist variables $\sigma_0$, $\sigma_1$, ..., $\sigma_n$ (for some $n, m ≥ 0$) such that $(\lambda x_1 ... x_m.M)N_1 ... N_n →_β^* x$. By Corollary 26.1, there is V' ∈ Λ_v such that N $\beta$→$\beta$ v-solvable, there exist variables $\sigma_0$, $\sigma_1$, ..., $\sigma_n$ (for some $n, m ≥ 0$) such that $(\lambda x_1 ... x_m.M)N_1 ... N_n →_β^* I$, therefore M is $\beta$-v-solvable.

So, due to Theorem 31, the semantic (via a relational model) and operational (via two sub-reductions of $\gamma_\lambda$) characterization of $\gamma_\lambda$-potential valuability and $\gamma_\lambda$-solvability given in [3, Theorems 24-25] is also a semantic and operational characterization of $\beta_\lambda$-potential valuability and $\beta_\lambda$-solvability. The difference is that in $\lambda_\omega$ these notions can be studied operationally inside the calculus, while it has been proved in [13, 14] that the $\beta_\lambda$-reduction is too weak to characterize them: an operational characterization of $\beta_\lambda$-potential valuability and $\beta_\lambda$-solvability cannot be given inside $\lambda_\omega$. Hence, $\lambda_\omega^v$ is a useful, conservative and “complete” tool for studying semantic properties of $\lambda_\omega$.

6 Conclusions

In this paper we have proved a standardization theorem for the $\lambda_\omega^v$-calculus introduced in [3]. The used technique is a notion of parallel reduction. Let us recall that parallel reduction in $\lambda$-calculus has been defined by Tait and Martin-Löf in order to prove confluence of the $\beta$-reduction, without referring to the difficult notion of residuals. Takahashi in [17] has simplified this technique and showed that it can be successfully applied to standardization. We would like to remark that our parallel reduction cannot be used to prove confluence of $\gamma_\lambda$.

Indeed, take $M = (\lambda x.L)((\lambda y.N)((\lambda z.N')(\lambda z.N''))L')$, $M_1 = (\lambda x.LL')((\lambda y.N)((\lambda z.N')(\lambda z.N''))L')$ and $M_2 = ((\lambda y.(\lambda x.L))N)((\lambda z.N')(\lambda z.N''))L'$ then $M \rightarrow M_1$ and $M \Rightarrow M_2$ but there is no term $M'$ such that $M_1 \Rightarrow M'$ and $M_2 \Rightarrow M'$. To sum up, $\Rightarrow$ does not enjoy the Diamond Property.

The standardization result allows us to formally verify the correctness of $\lambda_\omega^v$ with respect to the semantics of $\lambda_\omega$, so we can use $\lambda_\omega^v$ as a tool for studying properties of $\lambda_\omega$. This is a remarkable result: in fact some properties, like potential valuability and solvability, cannot be characterized in $\lambda_\omega$ by means of $\beta_\lambda$-reduction (as proved in [13, 14]), but they have a natural operational characterization in $\lambda_\omega^v$ (via two sub-reductions of $\gamma_\lambda$).

We plan to continue to explore the call-by-value computation, using $\lambda_\omega^v$. As a first step, we would like to revisit and improve the Separability Theorem given in [11] for $\lambda_\omega$. Still the issue is more complex than in the call-by-name, indeed in ordinary $\lambda$-calculus different $\beta_\eta$-normal forms can be separated (by the Böhm Theorem), while in $\lambda_\omega$ there are different normal forms that cannot be separated, but which are only semi-separable (e.g. $I$ and $\lambda z.(\lambda u.z)(z)$). We hope to completely characterize separable and semi-separable normal forms in $\lambda_\omega^v$. This should be a first step aimed to define a semantically meaningful notion of approximants. Then, we should be able to provide a new insight on the denotational analysis of the call-by-value, maybe overcoming limitations as that of the absence of fully abstract filter models [14, Theorem 12.1.25]. Last but not least, an unexplored but challenging
research direction is the use of commutation rules to improve the call-by-value evaluation. We do not have concrete evidence supporting such possibility, but since $\lambda^v_\sigma$ is strongly related to the calculi presented in [7, 1], which are endowed with explicit substitutions, we are confident that a sharp use of commutations can have a relevant impact in the evaluation.

References