Head reduction and normalization in a call-by-value lambda-calculus

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Abstract

Recently, a standardization theorem has been proven for a variant of Plotkin’s call-by-value lambda-calculus extended by means of two commutation rules (sigma-reductions): this result was based on a partitioning between head and internal reductions. We study the head normalization for this call-by-value calculus with sigma-reductions and we relate it to the weak evaluation of original Plotkin’s call-by-value lambda-calculus. We give also a (non-deterministic) normalization strategy for the call-by-value lambda-calculus with sigma-reductions.

1 Introduction

The call-by-value $\lambda$-calculus ($\lambda_v$-calculus or $\lambda_v$ for short) and the operational machine for its evaluation has been introduced by Plotkin [15] inspired by Landin’s seminal work [9] on the programming language ISWIM and the SECD machine. The $\lambda_v$-calculus is a paradigmatic language able to capture two features of many functional programming languages: call-by-value parameter passing policy (parameters are evaluated before being passed) and weak evaluation (the body of a function is evaluated only when parameters are supplied).

The syntax of $\lambda_v$ is the same as that of the ordinary (i.e. call-by-name) $\lambda$-calculus ($\lambda$ for short), but the reduction rule for $\lambda_v$, called $\beta_v$, is a restriction of the $\beta$-rule for $\lambda$: $\beta_v$ allows the contraction of a redex $(\lambda x.M)N$ only in case the argument $N$ is a value, i.e. a variable or an abstraction. Unfortunately, the semantic analysis of the $\lambda_v$-calculus has turned out to be more elaborate than that of ordinary $\lambda$-calculus. This is due essentially to the “weakness” of (full) $\beta_v$-reduction, a fact widely recognized: indeed, there are many proposals of alternative call-by-value $\lambda$-calculi extending Plotkin’s one [11, 10, 8, 2, 1]. To have an example of the “weakness” of the rewriting rules of $\lambda_v$, it is sufficient to consider that it is impossible to have an internal operational characterization (i.e. one that uses the $\beta_v$-reduction) of the semantically meaningful notions of call-by-value solvability and potential valuability, as shown in [13, 14, 2].

In this paper we will study the $\lambda_v^\sigma$-calculus ($\lambda_v^\sigma$ for short), a call-by-value extension of $\lambda_v$ recently proposed in [4]: it keeps the $\lambda_v$ (and $\lambda$) syntax and it adds to the $\beta_v$-reduction two commutation rules, called $\sigma_1$ and $\sigma_3$, which unblock “hidden” $\beta_v$-redexes that are concealed by the “hyper-sequential structure” of terms. The $\lambda_v^\sigma$-calculus enjoy some basic properties we expect from a calculus, namely confluence (see [4]) and standardization (see [7]). Moreover, $\lambda_v^\sigma$ provides elegant characterizations of many semantic properties, e.g. solvability and potential...
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valuability (see [4]), and it is conservative with respect to Plotkin’s $\lambda_\sigma$: in particular, [7] shows that the notions of solvability and potential valuability for $\lambda_\sigma^v$ coincide with those for $\lambda_v$.

The $v$-reduction (i.e. the reduction for $\lambda_\sigma^v$) can be partitioned into head $v$-reduction and internal $v$-reduction; the head $v$-reduction is in turn decomposed into head $\beta_\sigma$- and head $\sigma$-reduction. The head $\beta_\sigma$-reduction is just the deterministic weak evaluation strategy for Plotkin’s $\lambda_\sigma$-calculus. According to a sequentialization theorem proven in [7, Theorem 22], any $v$-reduction sequence can be sequentialized in an initial head $\beta_\sigma$-reduction sequence followed by a head $\sigma$-reduction sequence followed by an internal $v$-reduction sequence. Similar well-known results hold for $\lambda$ and $\lambda_\sigma$, and starting from them one can define a normalization strategy for $\lambda$ and $\lambda_\sigma$, i.e. a deterministic reduction strategy that reaches a normal form if and only if one exists: for example the leftmost reduction, see [19, Theorem 2.8] and [3, Theorem 13.2.2].

Is there a normalization strategy for $\lambda_\sigma^v$? Theorem 24, one of the main results of this paper, proves that, starting from the sequentialization theorem mentioned above, a normalization strategy can be defined for $\lambda_\sigma^v$, based on the notions of head $\beta_\sigma$- and head $\sigma$-reductions.

A first difference appears here between $\lambda_\sigma^v$ and $\lambda_v$ (or $\lambda$): the normalization strategy for $\lambda_\sigma^v$ is not deterministic. Indeed, while the head $\beta_\sigma$-reduction (or the call-by-name head reduction) is deterministic (i.e. a partial function), the head $v$-reduction is non-deterministic and, still worse, non-confluent and there are terms having several head $v$-normal forms: this might appear disappointing. So, three natural questions arise:

- With respect to head $v$-reduction, do normalization and strong normalization coincide?\(^1\)
- Can we relate the termination of head $\beta_\sigma$-reduction and head $v$-reduction?
- Can we characterize the terms having a unique head $v$-normal form?

Our Theorem 21 gives a positive answer to the first two questions. Observe that the lack of any form of confluence for head $v$-reduction requires a more complex reasoning, passing through a syntactic characterization of head $\beta_\sigma$- and head $v$-normal forms. Theorem 21 not only shows that the head $v$-reduction and the head $\beta_\sigma$-reduction are deeply related (and hence, again, $\lambda_\sigma^v$ is conservative with respect to $\lambda_v$) but also that both enjoy good properties analogous to the ones of the (call-by-name) head reduction for ordinary $\lambda$-calculus.

Our Proposition 27 gives a partial answer to the third question above: it shows that in some cases (of interest) a head $v$-normalizable term has a unique head $v$-normal form; in particular, every closed head $v$-normalizable term has a unique head $v$-normal form.

So, $\lambda_\sigma^v$ appears as an extension of Plotkin’s $\lambda_\sigma$-calculus that enjoys many meaningful conservation properties with respect to $\lambda_v$ and therefore it is a useful tool for theoretical and semantic investigations about $\lambda_v$ and call-by-value setting. See also conclusions in Section 6 for further and more precise motivations for this paper and future work.

Related work. The $\lambda_\sigma^v$-calculus has been recently introduced in [4] and further investigated in [7]. It is an extension of Plotkin’s $\lambda_\sigma$-calculus inspired by the call-by-value translation of $\lambda$-terms into linear logic proof-nets [6]. Other variants of $\lambda_v$ have been introduced in the literature for modeling the call-by-value computation. We would like to cite here at least the contributions of Moggi [11, Felleisen and Sabry [18], Maraist et al. [10], Herbelin and Zimmerman [8], Accattoli and Paolini [2] (the latter is inspired by the call-by-value translation of $\lambda$-terms into linear logic proof-nets, see [1]). All these proposals are based on the introduction of new constructs to the syntax of $\lambda_v$, so the comparison between them is

\(^1\) The answer is trivially positive in the case of call-by-name head normalization (for $\lambda$) and head $\beta_\sigma$-normalization, since these reductions are deterministic.
not easy with respect to syntactical properties (some detailed comparison is given in [2]).
We point out that the calculi introduced in [11, 18, 10, 8] present some variants of our \( \sigma_1 \)
and/or \( \sigma_3 \) rules, often in a setting with explicit substitutions. Regnier [16, 17] used the rule
\( \sigma_1 \) (but not \( \sigma_3 \)) in ordinary (i.e. call-by-name) \( \lambda \)-calculus.
The head \( v \)-reduction investigated here has been introduced in [7]. Some results of
this paper are inspired by the Takahashi’s results [19] on the ordinary (i.e. call-by-name)
\( \lambda \)-calculus, partially adapted by Crary [5] for \( \lambda_v \).

**Outline.** In Section 2 we introduce the syntax and the reduction rules of the \( \lambda_v^\sigma \)-calculus.
In Section 3 we define the head \( v \)-reduction and the internal \( v \)-reduction, and we recall some
results already proven in [7] concerning them. Section 4 is devoted to proving the first main
result of our paper: Theorem 21, which studies the normalization for the head \( v \)-reduction
and relates it to the weak evaluation strategy for Plotkin’s \( \lambda_v \)-calculus. In Section 5 we show
that the head \( v \)-reduction can be used to define a normalization strategy for the \( \lambda_v^\sigma \)-calculus
(Theorem 24), and moreover in some cases the head \( v \)-normal form (if any) of a term is
unique (Proposition 27). In Section 6 we summarize the findings and suggest future work.

## 2 The call-by-value lambda calculus with sigma-rules

In this section we present \( \lambda_v^\sigma \), a call-by-value \( \lambda \)-calculus introduced in [4] that adds two \( \sigma \)-reduction
rules to pure (i.e. without constants) call-by-value \( \lambda \)-calculus defined by Plotkin in [15].
The syntax of terms of \( \lambda_v^\sigma \) [4] is the same as the one of ordinary \( \lambda \)-calculus and Plotkin’s
call-by-value \( \lambda \)-calculus \( \lambda_v \) [15] (without constants). Given a countable set \( V \) of variables
(denoted by \( x, y, z, \ldots \)), the sets \( \Lambda \) of terms and \( \Lambda_v \) of values are defined by mutual induction:

\[
\begin{align*}
(\Lambda_v) & \quad V, U :::= x \mid \lambda x. M & \text{values} \\
(\Lambda) & \quad M, N, L :::= V \mid MN & \text{terms}
\end{align*}
\]

Clearly, \( \Lambda_v \subseteq \Lambda \). All terms are considered up to \( \alpha \)-conversion. The set of free variables of a
term \( M \) is denoted by \( \text{fv}(M) \). Given \( V_1, \ldots, V_n \in \Lambda_v \) and pairwise distinct variables \( x_1, \ldots, x_n \),
\( M \{V_1/x_1, \ldots, V_n/x_n\} \) denotes the term obtained by the capture-avoiding simultaneous substitution
of \( V_i \) for each free occurrence of \( x_i \) in the term \( M \) (for all \( 1 \leq i \leq n \)). Note that, for all \( V, V_1, \ldots, V_n \in \Lambda_v \)
and pairwise distinct variables \( x_1, \ldots, x_n \), \( V \{V_1/x_1, \ldots, V_n/x_n\} \in \Lambda_v \).

**Contexts** (with exactly one hole \( \langle \rangle \)), denoted by \( C \), are defined as usual via the grammar:

\[ C ::= \langle \rangle \mid \lambda x.C \mid CM \mid MC. \]

We use \( C[M] \) for the term obtained by the capture-allowing substitution of the term \( M \)
for the hole \( \langle \rangle \) in the context \( C \).

**Notation.** From now on, we set \( I = \lambda x.x \) and \( \Delta = \lambda x.xx \).

The reduction rules of \( \lambda_v^\sigma \) consist of Plotkin’s \( \beta_v \)-reduction rule, introduced in [15], and
two simple commutation rules called \( \sigma_1 \) and \( \sigma_3 \), studied in [4, 7].

**Definition 1 (Reduction rules).** We define the following binary relations on \( \Lambda \) (for any
\( M, N, L \in \Lambda \) and any \( V \in \Lambda_v \)):

\[
\begin{align*}
(\lambda x. M)V & \Rightarrow_{\beta_v} M \{V/x\} \\
(\lambda x. M)NL & \Rightarrow_{\sigma_1} (\lambda x. ML)N \quad \text{with } x \notin \text{fv}(L) \\
V((\lambda x. L)N) & \Rightarrow_{\sigma_3} (\lambda x. VL)N \quad \text{with } x \notin \text{fv}(V).
\end{align*}
\]
We set \( \equiv = \equiv_{\lambda} \cup \equiv_{\sigma} \) and \( \rightarrow_{v} = \equiv_{\beta_{v}} \cup \equiv_{\sigma}. \)

For any \( r \in \{ \beta_{v}, \sigma_{1}, \sigma_{3}, \sigma, v \} \), if \( M \rightarrow_{r} M' \) then \( M \) is a \( r \)-redex and \( M' \) is its \( r \)-contractum.

In this sense, a term of the shape \((\lambda x. M) N\) (for any \( M, N \in \Lambda \)) is a \( \beta \)-redex.

The side conditions on \( \rightarrow_{\sigma} \) and \( \rightarrow_{v} \) in Definition 1 can be always fulfilled by \( \alpha \)-renaming.

Obviously, any \( \beta_{v} \)-redex is a \( \beta \)-redex but the converse does not hold: \((\lambda x. z)(y I)\) is a \( \beta \)-redex but not a \( \beta_{v} \)-redex.

\[\text{Example 2.}\] Redexes of different kind may overlap: for example, the term \( \Delta I \Delta \) is a \( \sigma_{1} \)-redex and it contains the \( \beta_{v} \)-redex \( I \); the term \( \Delta(I \Delta)(x I) \) is a \( \sigma_{1} \)-redex and it contains the \( \sigma_{3} \)-redex \( \Delta(I \Delta) \), which contains in turn the \( \beta_{v} \)-redex \( I \Delta \).

\[\text{Notation.}\] Let \( R \) be a binary relation on \( \Lambda \). We denote by \( R^{+} \) (resp. \( R^{-} \); \( R^{=} \)) the reflexive-transitive (resp. transitive; reflexive) closure of \( R \).

\[\text{Definition 3 (Reductions).}\] Let \( r \in \{ \beta_{v}, \sigma_{1}, \sigma_{3}, \sigma, v \} \).

The \( r \)-reduction \( \rightarrow_{r} \) is the contextual closure of \( \rightarrow_{r} \), i.e. \( M \rightarrow_{r} M' \) iff there is a context \( C \) and \( N, N' \in \Lambda \) such that \( M = C[N] \), \( M' = C[N'] \) and \( \forall \rightarrow_{r}, N \rightarrow_{r} N' \).

The \( r \)-equivalence \( \simeq_{r} \) is the reflexive-transitive and symmetric closure of \( \rightarrow_{r} \).

Let \( M \) be a term: \( M \) is \( r \)-normal if there is no \( r \)-term such that \( M \rightarrow_{r} N \); \( M \) is \( r \)-normalizable if there is a \( r \)-normal term \( N \) such that \( M \rightarrow_{r}^{*} N \); \( M \) is strongly \( r \)-normalizable if there is no sequence \( (N_{i})_{i \in \mathbb{N}} \) of terms such that \( M = N_{0} \) and \( N_{i} \rightarrow_{r} N_{i+1} \) for any \( i \in \mathbb{N} \).

Obviously, \( \rightarrow_{\sigma} \rightarrow_{\sigma_{1}} \cup \rightarrow_{\sigma_{3}} \subseteq \rightarrow_{v} \) and \( \rightarrow_{\sigma} \subseteq \rightarrow_{\sigma} \) and \( \rightarrow_{v} = \rightarrow_{\beta_{v}} \cup \rightarrow_{\sigma} \).

\[\text{Remark 4.}\] For any \( r \in \{ \beta_{v}, \sigma_{1}, \sigma_{3}, \sigma, v \} \) (resp. \( r \in \{ \sigma_{1}, \sigma_{3}, \sigma \} \)), values are closed under \( r \)-reduction (resp. \( r \)-expansion): for any \( V \in \Lambda_{v} \), if \( V \rightarrow_{r} M \) (resp. \( M \rightarrow_{r} V \)) then \( M \in \Lambda_{v} \); more precisely, \( V = \lambda x. N \) and \( M = \lambda x. N' \) for some \( N, N' \in \Lambda \) with \( N \rightarrow_{r} N' \) (resp. \( N' \rightarrow_{r} N \)).

For any \( r \in \{ \beta_{v}, v \} \), values are not closed under \( r \)-expansion: \( I \Delta \rightarrow_{\beta_{v}} \Delta \in \Lambda_{v} \) but \( I \Delta \notin \Lambda_{v} \).

\[\text{Proposition 5 (See [4]).}\] \( \sigma \)-reduction is confluent and strongly normalizing. The \( v \)-reduction is confluent.

The \( \lambda_{\sigma}^{v} \)-calculus, \( \lambda_{\sigma}^{v} \) for short, is the set \( \Lambda \) of terms endowed with the \( v \)-reduction \( \rightarrow_{v} \).

The set \( \Lambda \) endowed with the \( \beta_{v} \)-reduction \( \rightarrow_{\beta_{v}} \) is the \( \lambda_{v} \)-calculus (\( \lambda_{v} \) for short), i.e. the Plotkin’s call-by-value \( \lambda \)-calculus [15] (without constants), which is thus a sub-calculus of \( \lambda_{\sigma}^{v} \).

\[\text{Example 6.}\] \( M = (\lambda y. \Delta)(x I) \Delta \rightarrow_{\sigma_{1}} (\lambda y. \Delta)(x I) \rightarrow_{\beta_{v}} (\lambda y. \Delta)(x I) \rightarrow_{\beta_{v}} \ldots \) and \( N = \Delta((\lambda y. \Delta)(x I)) \rightarrow_{\sigma_{1}} (\lambda y. \Delta)(x I) \rightarrow_{\beta_{v}} (\lambda y. \Delta)(x I) \rightarrow_{\beta_{v}} \ldots \) are the only possible \( v \)-reduction paths from \( M \) and \( N \) respectively: \( M \) and \( N \) are not \( v \)-normalizable, and \( M \simeq_{v} N \). Meanwhile, \( M \) and \( N \) are \( \beta_{v} \)-normal and different, hence \( M \neq_{\beta_{v}} N \) (by confluence of \( \rightarrow_{\beta_{v}} \), see [15]).

Informally, \( \sigma \)-rules unlock \( \beta_{v} \)-redexes which are hidden by the “hyper-sequential structure” of terms. This approach is alternative to the one in [2, 1] where hidden \( \beta_{v} \)-redexes are reduced using rules acting at a distance (through explicit substitutions). It can be shown that the call-by-value \( \lambda \)-calculus with explicit substitution introduced in [2] can be embedded in \( \lambda_{\sigma}^{v} \).

It is well-known that the \( \beta_{v} \)-reduction can be simulated by linear logic cut-elimination via the call-by-value translation \( \cdot^{v} \) of \( \lambda \)-terms into proof-nets, called by Girard [6, pp. 81-82] “boring” and defined by \( (A \Rightarrow B)^{v} = !A^{v} \rightarrow !B^{v} \) (see also [1]). The images under \( \cdot^{v} \) of a \( \sigma \)-redex and its \( \sigma \)-contractum are equal modulo some non-structural cut-elimination steps.
3 Head and internal reductions

In this section we introduce the definitions of head v-reduction (which is decomposed in head βv and head σ-reductions) and internal v-reduction, then we recall some results proven in [7].

**Notation.** From now on, we always assume that \( V, V' \in \Lambda_v \).

Note that the generic form of a term is \( V M_1 \ldots M_m \) for some \( m \in \mathbb{N} \) (in particular, values are obtained when \( m = 0 \)). The sequentialization result is based on a partitioning of v-reduction between head v-reduction and internal v-reduction.

**Definition 7 (Head βv-reduction).** The head βv-reduction \( \beta_v \) is the binary relation on \( \Lambda \) defined inductively by the following rules (\( m \in \mathbb{N} \) in both rules):

\[
(\lambda x.M)V M_1 \ldots M_m \xrightarrow{\beta_v} M(V/x) M_1 \ldots M_m \\
V N M_1 \ldots M_m \xrightarrow{\beta_v} N' V N M_1 \ldots M_m
\]

The head \( \beta_v \) is exactly the (pure) “left reduction” defined in [15, p.136] for \( \lambda \), and called “(weak) evaluation” in [18, 5]. If \( N \xrightarrow{\beta_v} N' \) then \( N' \) is obtained from \( N \) by reducing the leftmost-outermost \( \beta_v \)-redex, not in the scope of a \( \lambda \): thus, the head \( \beta_v \) is deterministic (i.e. it is a partial function from \( \Lambda \) to \( \Lambda \)) and does not reduce values.

**Definition 8 (Head σ- and head v-reductions).** The head σ-reduction \( \sigma \) is the binary relation on \( \Lambda \) defined inductively by the following rules (\( m \in \mathbb{N} \) in all the rules, \( x \notin \text{fv}(L) \) in the rule \( \sigma_1 \), \( x \notin \text{fv}(V) \) in the rule \( \sigma_3 \)):

\[
(\lambda x.M)N L M_1 \ldots M_m \xrightarrow{\sigma_1} (\lambda x.M L) N M_1 \ldots M_m \\
V N M_1 \ldots M_m \xrightarrow{\sigma_3} (\lambda x.V L) N M_1 \ldots M_m
\]

The head v-reduction is \( \xrightarrow{v} = \xrightarrow{\beta_v} \cup \xrightarrow{\sigma} \).

Let \( t \in \{ \beta_v, \sigma, v \} \) and \( N \in \Lambda \): \( N \) is head r-normal if there is no \( N' \in \Lambda \) such that \( N \xrightarrow{\beta_v} N' \); \( N \) is head r-normalizable if there is a r-normal term \( N' \) such that \( N \xrightarrow{\beta_v} N' \); \( N \) is strongly head r-normalizable if there is no \((N_i)_{i \in \mathbb{N}} \) such that \( N = N_0 \) and \( N_i \xrightarrow{\beta_v} N_{i+1} \) for any \( i \in \mathbb{N} \).

Notice that \( \xrightarrow{\beta_v} \subseteq \xrightarrow{\beta_v} \subseteq \xrightarrow{\sigma} \) and \( \xrightarrow{\sigma} \subseteq \xrightarrow{v} \subseteq \xrightarrow{\sigma} \).

Informally, if \( N \xrightarrow{\sigma} N' \) then \( N' \) is obtained from \( N \) by reducing “one of the leftmost” \( \sigma_1 \)- or \( \sigma_3 \)-redexes, not in the scope of a \( \lambda \): in general, a term may contain several head \( \sigma_1 \)- and \( \sigma_3 \)-redexes. Indeed, differently from \( \xrightarrow{\beta_v} \), the head σ-reduction \( \xrightarrow{\sigma} \) is not deterministic, for example the leftmost-outermost \( \sigma_1 \)- and \( \sigma_3 \)-redexes may overlap: if \( M = (\lambda y.y')(\Delta(xI))I \) then \( M \xrightarrow{\sigma} (\lambda y.y'I)(\Delta(xI)) = N_1 \) by applying the rule \( \sigma_1 \) and \( M \xrightarrow{\sigma} (\lambda z.(\lambda y.y')(zz))(xI) = N_2 \) by applying the rule \( \sigma_3 \). Note that \( N_1 \) contains only a head \( \sigma_3 \)-redex and \( N_1 \xrightarrow{\sigma} (\lambda z.(\lambda y.y')(zz))(xI) = N \) which is head v-normal; meanwhile \( N_2 \) contains only a head \( \sigma_1 \)-redex and \( N_2 \xrightarrow{\sigma} (\lambda z.(\lambda y.y')(zz))(xI) = N' \) which is head v-normal: \( N \neq N' \), so the head \( \sigma \)- and head v-reductions are not (locally) confluent and a term may have several head v-normal forms (this example does not contradict the confluence of σ-reduction because \( N' \xrightarrow{\sigma} N \) but by performing an internal v-reduction step, see next Definition 9).

The head v-reduction \( \xrightarrow{v} \) is non-deterministic not only because the head σ-reduction \( \xrightarrow{\sigma} \) is non-deterministic, but also because the leftmost-outermost \( \beta_v \)-redex of a term may overlap with “one of its leftmost” \( \sigma_1 \)- or \( \sigma_3 \)-redexes, as seen in Example 2.
Definition 9 (Internal $\nu$-reduction). The internal $\nu$-reduction $\overset{\nu}{\rightarrow}$ is the binary relation on $\Lambda$ defined inductively by the following rules:

\[
\begin{align*}
(m \in \mathbb{N}) & \quad N \rightarrow_\nu N' \\
(\lambda x. N)M_1 \ldots M_m & \overset{\nu}{\rightarrow} (\lambda x. N')M_1 \ldots M_m \\
VN M_1 \ldots M_m & \overset{\nu}{\rightarrow} VN' M_1 \ldots M_m \\
VN M_1 \ldots M_m & \overset{\nu}{\rightarrow} VN M_1 \ldots M_m, \quad \text{for some } 1 \leq i \leq m.
\end{align*}
\]


The following fact collects many minor properties which can be easily proved by inspection of the rules of Definitions 7-9.

Fact 10.

1. The head $\beta_0$-reduction $\beta_0 \rightarrow$ does not reduce a value (in particular, does not reduce under $\lambda$’s), i.e., for any $M \in \Lambda$ and any $V \in \Lambda_\nu$, one has $V \not\rightarrow \beta_0 M$.

2. The head $\sigma$-reduction $\rightarrow_\sigma$ does neither reduce a value nor reduce to a value, i.e., for any $M \in \Lambda$ and any $V \in \Lambda_\nu$, one has $V \not\rightarrow \beta_0 M$ and $M \not\rightarrow \beta_0 V$.

3. Values are closed under $\overset{\nu}{\rightarrow}$-conversion, i.e., for all $M \in \Lambda$ and $V \in \Lambda_\nu$, if $M \overset{\nu}{\rightarrow} V$ then $M \in \Lambda_\nu$; more precisely, $M = \lambda x. N$ and $V = \lambda x. N'$ for some $N, N' \in \Lambda$ where $N \rightarrow_\nu N'$.

4. If $\mathcal{R} \in \{\not\rightarrow_\beta_0, \not\rightarrow_\sigma, \not\rightarrow_\nu, \not\rightarrow_\sigma \nu\}$ and $M \mathcal{R} M'$, then $MN \mathcal{R} M'N$ for any $N \in \Lambda$.

Clearly, $\overset{\nu}{\rightarrow} \subseteq \not\rightarrow_\nu$. Next Proposition 11 (whose proof uses Fact 10.4) relates $\overset{\nu}{\rightarrow}$ and $\not\rightarrow_\nu$.

Proposition 11. One has $\overset{\nu}{\rightarrow} = \not\rightarrow_\nu \setminus \not\rightarrow_\beta_0$.

Proof.

\[\subseteq: \text{The proof that } \overset{\nu}{\rightarrow} \subseteq \not\rightarrow_\nu \text{ is trivial. The proof that } M \overset{\nu}{\rightarrow} M' \text{ implies } M \not\rightarrow_\beta_0 M' \text{ by induction on the derivation of } M \overset{\nu}{\rightarrow} M'. \text{ Let us consider its last rule } r. \text{ If } r \in \{\lambda, \ominus\}, \text{ then it is evident that there is no last rule to derive } M \not\rightarrow_\nu M'. \text{ If } r = \text{ right} \text{ then } M = VN M_1 \ldots M_m \text{ and } M' = VN' M_1 \ldots M_m \text{ with } m \in \mathbb{N} \text{ and } N \overset{\nu}{\rightarrow} N'; \text{ by induction hypothesis, } N \not\rightarrow_\nu N' \text{ and hence there is no last rule to derive } M \rightarrow_\nu M'.\]

\[\supseteq: \text{We show that } M \rightarrow_\nu M' \text{ and } M \not\rightarrow_\beta_0 M' \text{ implies } M \overset{\nu}{\rightarrow} M', \text{ for all } M, M' \in \Lambda. \text{ Since } M \rightarrow_\nu M', \text{ there exist a context } C \text{ and terms } N \text{ and } N' \text{ such that } M = C\langle N \rangle, M' = C\langle N' \rangle \text{ and } N \not\rightarrow_\beta_0 N'. \text{ We proceed by induction on } C. \]

If $C = \langle \rangle$ then $M = N \rightarrow_\beta_0 N' = M'$ and thus $M \not\rightarrow_\beta_0 M'$ since $\not\rightarrow_\beta_0 \subseteq \not\rightarrow_\nu$, which contradicts the hypothesis.

If $C = \lambda x. C'$ for some context $C'$, then $M \overset{\nu}{\rightarrow} M'$ by applying the rule $\lambda$ for $\overset{\nu}{\rightarrow}$, since $C\langle N \rangle \rightarrow_\nu C'\langle N \rangle$.

If $C = C'L$ for some context $C'$ and term $L$, then $C'\langle N \rangle \rightarrow_\nu C'\langle N' \rangle$ and $C'\langle N' \rangle \not\rightarrow_\nu C'\langle N' \rangle$ (by Fact 10.4, since $C'\langle N \rangle L \not\rightarrow_\nu C'\langle N' \rangle L$). By induction hypothesis, $C'\langle N \rangle \overset{\nu}{\rightarrow} C'\langle N' \rangle$, then $M = C'\langle N \rangle L \overset{\nu}{\rightarrow} C'\langle N' \rangle L = M'$ by Fact 10.4.

If $C = V C'$ for some context $C'$ and value $V$, then $C'\langle N \rangle \rightarrow_\nu C'\langle N' \rangle$. There are two cases:

- either $C'\langle N \rangle \not\rightarrow_\nu C'\langle N' \rangle$, hence $M = V C'\langle N \rangle \not\rightarrow_\nu V C'\langle N' \rangle = M'$ by the rule $\beta_0$ for $\not\rightarrow_\beta_0$, which contradicts the hypothesis;

- or $C'\langle N \rangle \not\rightarrow_\nu C'\langle N' \rangle$, hence $C'\langle N \rangle \overset{\nu}{\rightarrow} C'\langle N' \rangle$ by induction hypothesis, thus $M = V C'\langle N \rangle \overset{\nu}{\rightarrow} V C'\langle N' \rangle = M'$ by applying the rule $\beta_0$ for $\not\rightarrow_\beta_0$.

Finally, if $C = LC'$ for some context $C'$ and term $L \not\in \Lambda_\nu$, then $L = VN_0 \ldots N_n$ for some $n \in \mathbb{N}$, thus $M = VN_0 \ldots N_n C'\langle N \rangle \overset{\nu}{\rightarrow} VN_0 \ldots N_n C'\langle N' \rangle = M'$ by the rule $\ominus$ for $\overset{\nu}{\rightarrow}$.
We end this section by recalling three results proven in [7] concerning head \( v \)-reduction and internal \( v \)-reduction: they will be used to prove the main results in Sections 4-5.

The following lemma (proven in [7, Lemma 14]) shows that a head \( \sigma \)-reduction step can be postponed after a head \( \beta_v \)-reduction step, and hence every head \( v \)-reduction sequence can be rearranged into a head \( \beta_v \)-reduction sequence followed by a head \( \sigma \)-reduction sequence.

\textbf{Lemma 12} (Commutation of head \( \beta_v \) and head \( \sigma \)-reductions, see [7]).
1. If \( M \xrightarrow{h_v} L \xrightarrow{h_{\beta_v}} N \) then there exists \( L' \in \Lambda \) such that \( M \xrightarrow{h_{\beta_v}} L' \xrightarrow{h_v} N \).
2. If \( M \xrightarrow{h_v} N \) then there exists \( N' \in \Lambda \) such that \( M \xrightarrow{h_{\beta_v}} N' \xrightarrow{h_v} N \).

Next Lemma 13 (proven in [7, Corollary 21]) says that internal \( v \)-reduction can be shifted after head \( v \)-reductions.\(^2\)

\textbf{Lemma 13} (Postponement, see [7]). If \( M \xrightarrow{v} L \) and \( L \xrightarrow{h_v} N \) (resp. \( L \xrightarrow{h_v} N \)), then there exist \( L', L'' \in \Lambda \) such that \( M \xrightarrow{h_v} L' \xrightarrow{h_v} L'' \xrightarrow{h_v} N \) (resp. \( M \xrightarrow{h_v} L' \xrightarrow{h_v} L'' \xrightarrow{h_v} N \)).

Next Theorem 14 is one of the main results proven in [7, Theorem 22] by adapting Takahashi’s method [19, 5]: any \( v \)-reduction sequence can be sequentialized into a head \( \beta_v \)-reduction sequence followed by a head \( \sigma \)-reduction sequence, followed by an internal \( v \)-reduction sequence. In ordinary \( \lambda \)-calculus, the well-known result corresponding to our Theorem 14 states that a \( \beta \)-reduction sequence can be factorized in a head reduction sequence followed by an internal reduction sequence (see for example [19, Corollary 2.6]).

\textbf{Theorem 14} (Sequentialization, see [7]). If \( M \xrightarrow{v} M' \) then there exist \( L, N \in \Lambda \) such that \( M \xrightarrow{h_v} L \xrightarrow{h_v} N \xrightarrow{h_v} M' \).

The sequentialization of Theorem 14 imposes no order between head \( \sigma \)-reductions. Indeed, the example in [7, p. 10] shows that it is impossible to sequentialize them by giving way to head \( \sigma_1 \)- or head \( \sigma_2 \)-redexes: a head \( \sigma_1 \)-reduction step can create a head \( \sigma_2 \)-redex, and vice versa.

In [7, Definition 27 and Corollary 29] it has also been proven that the \( v \)-equivalence (and in particular the \( \sigma \)-equivalence) is contained in the call-by-value observational equivalence.

\section{Head normalization}

In this section we prove the first main result of our paper: Theorem 21, which studies the normalization for head \( v \)-reduction and relates it to the head \( \beta_v \)-reduction (i.e. the weak evaluation strategy for Plotkin’s \( \lambda_v \)-calculus). Let us start with a preliminary remark.

\textbf{Remark 15.} According to Facts 10.1-2, every \( V \in \Lambda_v \) is head \( \beta_v \)- and head \( \sigma \)-normal, and hence is head \( v \)-normal. The converse does not hold: \( xI \) is head \( v \)-normal but \( xI \notin \Lambda_v \).

First, we give a syntactic characterization of head \( v \)- and head \( \beta_v \)-normal forms.

\textbf{Definition 16.} We define the subsets \( \Lambda_a, \Lambda_b \) and \( \Lambda_c \) (whose elements are denoted by \( A, B \) and \( C \) respectively) of \( \Lambda \) as follows (for any variable \( x \), any \( V \in \Lambda_v \) and any \( N \in \Lambda \)):

\[
\begin{align*}
(\Lambda_a) & \quad A := xV \mid xA \mid AN \\
(\Lambda_b) & \quad B := (\lambda x.N)A \\
(\Lambda_c) & \quad C := xV \mid VC \mid CN
\end{align*}
\]

\(^2\) In [7, Corollary 21] there is a more informative statement of our Lemma 13, involving a notion of internal parallel reduction \( \parallel \). Our Lemma 13 follows immediately from [7, Corollary 21] since \( \xrightarrow{h_v} \subseteq \parallel \subseteq \xrightarrow{h_v} \).
Notice that $\Lambda_a \cup \Lambda_b \subseteq \Lambda_c$ and $M, N \in \Lambda_c \setminus (\Lambda_a \cup \Lambda_b)$ where $M = (\lambda y. \Delta)(xI)\Delta$ and $N = \Delta((\lambda y. \Delta)(xI))$ (as in Example 6). Moreover, $\Lambda_c \cap \Lambda_a = \Lambda_c \cap \Lambda_b = \Lambda_c \cap \Lambda_a \cap \Lambda_b = \emptyset$ and all terms in $\Lambda_a \cup \Lambda_b \cup \Lambda_c$ are not closed. All terms in $\Lambda_b$ are $\beta$-redexes that are not $\beta_v$-redexes; all terms in $\Lambda_a$ have a free “head variable” and are neither a value nor a $\beta$-redex.

**Proposition 17** (Characterization of head $\beta_v$-normal forms). Let $M$ be a term.

1. $M$ is head $\beta_v$-normal and is not a $\lambda$-value if and only if $M \in \Lambda_c$.
2. $M$ is head $\beta_v$-normal if and only if $M \in \Lambda_v \cup \Lambda_c$.

**Proof.** Statement (2) is an immediate consequence of statement (1) and Remark 15.

$\Rightarrow$: We prove the left-to-right direction of statement (1), by induction on $M \in \Lambda$.

The case where $M \in \Lambda_v$ is impossible by hypothesis.

If $M = M_1M_2$ (for some $M_1, M_2 \in \Lambda$) is head $\beta_v$-normal then $M$ is not a $\lambda$-value and $M_1$ and $M_2$ are head $\beta_v$-normal, moreover either $M_1 \notin \lambda x. N$ (for any $N \in \Lambda$) or $M_2 \notin \Lambda_v$ (otherwise $M$ would be a head $\beta_v$-redex). Therefore, there are only three cases:

- either $M_1 \notin \Lambda_v$, thus $M_1 \in \Lambda$, by induction hypothesis, and hence $M \in \Lambda_c$;
- or $M_1 \in \Lambda_v$ and $M_2 \notin \Lambda_v$, so $M_2 \in \Lambda$, by induction hypothesis, and thus $M \in \Lambda_c$;
- or $M_1$ is a variable and $M_2 \in \Lambda_v$, hence $M \in \Lambda_c$ (this is the base case).

$\Leftarrow$: The right-to-left direction of statement (1) can easily be proved by induction on $M \in \Lambda_v$.

A consequence of Proposition 17 is that all closed head $\beta_v$-normal forms are abstractions.

**Proposition 18** (Characterization of head $\beta_v$-normal forms). Let $M \in \Lambda$.

1. $M$ is head $\beta_v$-normal and is neither a $\lambda$-value nor a $\beta$-redex if and only if $M \in \Lambda_a$.
2. $M$ is head $\beta_v$-normal and is a $\beta$-redex if and only if $M \in \Lambda_b$.
3. $M$ is head $\beta_v$-normal if and only if $M \in \Lambda_v \cup \Lambda_a \cup \Lambda_b$.

**Proof.** Statement (3) is an immediate consequence of statements (1)-(2) and Remark 15.

$\Rightarrow$: We prove simultaneously the left-to-right direction of statements (1) and (2), by induction on $M \in \Lambda$. The case where $M \in \Lambda_a$ is impossible by hypothesis.

If $M = M_1M_2$ (for some $M_1, M_2 \in \Lambda$) is head $\beta_v$-normal then $M$ is not a $\lambda$-value and $M_1$ and $M_2$ are head $\beta_v$-normal, moreover $M_1$ is not a $\beta$-redex (otherwise $M$ would be a head $\sigma_1$-redex), and either $M_1 \notin \lambda x. N$ (for any $N \in \Lambda$) or $M_2 \notin \Lambda_v$ (otherwise $M$ would be a head $\beta_v$-redex), and either $M_1 \notin \Lambda_v$, or $M_2$ is not a $\beta$-redex (otherwise $M$ would be a head $\sigma_3$-redex). There are only three cases:

- either $M_1$ is a variable and $M_2$ is not a $\beta$-redex, so $M$ is not a $\beta$-redex; if $M_2 \in \Lambda_v$ then $M \in \Lambda_a$ (this is the base case); moreover $M_2 \in \Lambda_v$ by induction hypothesis, so $M \in \Lambda_a$;
- or $M_1 \notin \Lambda_v$, thus $M$ is not a $\beta$-redex and $M_1 \in \Lambda_v$ by induction hypothesis, so $M \in \Lambda_a$;
- or $M_1 = \lambda x. N$ for some $N \in \Lambda$ and $M_2$ is neither a $\lambda$-value nor a $\beta$-redex, so $M$ is a $\beta$-redex, furthermore $M_2 \in \Lambda_v$ by induction hypothesis, and thus $M \in \Lambda_b$.

$\Leftarrow$: The right-to-left direction of statement (1) can easily be proved by induction on $M \in \Lambda_a$.

Let us prove the right-to-left direction of statement (2): if $M \in \Lambda_b$ then $M = (\lambda x. N)A$ for some $N \in \Lambda$ and $A \in \Lambda_a$, thus $M$ is a $\beta$-redex. For any $M' \in \Lambda$, the last rule of the derivation of $M \rightarrow \sigma_1 M'$ might be neither $\sigma_1$ nor $\sigma_3$ (because $A$ is not a $\beta$-redex by statement 1) nor $\beta_v$ (because $A \notin \Lambda_v$ by statement 1 again) nor $right$ (because $A$ is head $\beta_v$-normal, by statement 1 again). Therefore, $M$ is head $\beta_v$-normal.

As a consequence of Proposition 18, all closed head $\beta_v$-normal forms are abstractions.

The sets of terms $\Lambda_a$, $\Lambda_b$ and $\Lambda_c$ of Definition 16 enjoy the closure properties summarized in Lemma 19 below. Together with the syntactic characterizations of head $\beta_v$-normal forms...
(Proposition 17) and head ν-normal forms (Proposition 18), these closure properties allow one to reason about head ν-reduction in spite of its non-confluence: they will be used to prove our main results, Theorems 21 and 24 and Proposition 27.

Lemma 19 (Closure properties).
1. The set $\Lambda_\sigma$ is closed under $\nu$-internal reduction and expansion, i.e., for any $N' \in \Lambda$ and $N \in \Lambda_\sigma$, if $N' \overset{\nu}{\rightarrow}_\sigma N$ or $N \overset{\nu}{\rightarrow}_\sigma N'$ then $N' \in \Lambda_\sigma$.
2. The set $\Lambda_\sigma$ is closed under $\nu$-internal reduction and expansion, i.e., for any $N' \in \Lambda$ and $N \in \Lambda_\sigma$, if $N' \overset{\nu}{\rightarrow}_\sigma N$ or $N \overset{\nu}{\rightarrow}_\sigma N'$ then $N' \in \Lambda_\sigma$.
3. Head ν-normal forms are closed under ν-internal reduction and expansion, i.e., for any $N, N' \in \Lambda$ where $N$ is head ν-normal, if $N' \overset{\nu}{\rightarrow}_\sigma N$ or $N \overset{\nu}{\rightarrow}_\sigma N'$ then $N'$ is head ν-normal.
4. Head $\beta$-normal forms are closed under head $\sigma$-reduction and expansion, i.e., for any $N, N' \in \Lambda$ where $N$ is head $\beta$-normal, if $N' \overset{\beta}{\rightarrow}_\sigma N$ or $N \overset{\beta}{\rightarrow}_\sigma N'$ then $N'$ is head $\beta$-normal.

Proof.
1. We show that if $N \in \Lambda_\sigma$ and $N' \overset{\nu}{\rightarrow}_\sigma N$ (resp. $N \overset{\nu}{\rightarrow}_\sigma N'$) then $N' \in \Lambda_\sigma$, by induction on the derivation of $N' \overset{\nu}{\rightarrow}_\sigma N$ (resp. $N \overset{\nu}{\rightarrow}_\sigma N'$). Let us consider its last rule $r$.
   - Since $N \in \Lambda_\sigma$ (see Definition 16), $N = xLN_1 \ldots N_n$ for some $n \in \mathbb{N}$, some variable $x$, some $L \in \Lambda_\sigma$ and some $N_1, \ldots, N_n \in \Lambda$, thus $r \neq \lambda$ and hence either $r = \text{right}$ or $r = \otimes$.
   - If $r = \text{right}$ then $N' = xLN_1 \ldots N_n$ where $L' \overset{\nu}{\rightarrow}_\sigma L$ (resp. $L \overset{\nu}{\rightarrow}_\sigma L'$). Since $L \in \Lambda_\sigma \cup \Lambda_\sigma$, there are two cases:
     - either $L \in \Lambda_\sigma$ and then $L' \in \Lambda_\sigma$ by induction hypothesis, so $N' = xLN_1 \ldots N_n \in \Lambda_\sigma$;
     - or $L \in \Lambda_\sigma$ and then $L' \in \Lambda_\sigma$ by Fact 10.3 (resp. Remark 4, since $\overset{\nu}{\rightarrow}_\sigma \subseteq \rightarrow_\sigma$), therefore $N' = xLN_1 \ldots N_n \in \Lambda_\sigma$.
   - Finally, if $r = \otimes$ then $n \in \mathbb{N}^+$ and $N' = xLN_1 \ldots N_i N_{i+1} \ldots N_n$ for some $1 \leq i \leq n$ with $N_i' \overset{\nu}{\rightarrow}_\sigma N_i$ (resp. $N_i \overset{\nu}{\rightarrow}_\sigma N_i'$), hence $N' \in \Lambda_\sigma$ because $xL \in \Lambda_\sigma$.

2. We show that if $N \in \Lambda_\sigma$ and $N' \overset{\nu}{\rightarrow}_\sigma N$ (resp. $N \overset{\nu}{\rightarrow}_\sigma N'$) then $N' \in \Lambda_\sigma$, by induction on the derivation of $N' \overset{\nu}{\rightarrow}_\sigma N$ (resp. $N \overset{\nu}{\rightarrow}_\sigma N'$). Let us consider its last rule $r$. Since $N \in \Lambda_\sigma$, then $N = (\lambda x.M)A$ for some $M \in \Lambda$ and $A \in \Lambda_\sigma$, hence $r \neq \otimes$ because $N$ has not the shape $VLM_1 \ldots M_m$ for any $m \in \mathbb{N}^+$; therefore either $r = \lambda$ or $r = \text{right}$:
   - if $r = \lambda$, then $N' = (\lambda x.M')A$ where $M' \rightarrow_\sigma M$ (resp. $M \rightarrow_\sigma M'$), hence $N' \in \Lambda_\sigma$;
   - if $r = \text{right}$, then $N' = (\lambda x.M')A'$ where $A' \overset{\nu}{\rightarrow}_\sigma A$ (resp. $A \overset{\nu}{\rightarrow}_\sigma A'$), thus $A' \in \Lambda_\sigma$ by Lemma 19.1, hence $N' \in \Lambda_\sigma$.

3. Thanks to Proposition 18.3, it is sufficient to show that if $N \in \Lambda_\sigma \cup \Lambda_\sigma \cup \Lambda_\sigma$ and $N' \overset{\nu}{\rightarrow}_\sigma N$ (resp. $N \overset{\nu}{\rightarrow}_\sigma N'$) then $N' \in \Lambda_\sigma \cup \Lambda_\sigma \cup \Lambda_\sigma$. If $N \in \Lambda_\sigma$, then $N' \in \Lambda_\sigma$ by Fact 10.3 (resp. Remark 4, since $\overset{\nu}{\rightarrow}_\sigma \subseteq \rightarrow_\sigma$). If $N \in \Lambda_\sigma$ then $N' \in \Lambda_\sigma$ by Lemma 19.1. Finally, if $N \in \Lambda_\sigma$ then $N' \in \Lambda_\sigma$ by Lemma 19.2.

4. By Proposition 17.2, $N \in \Lambda_\sigma \cup \Lambda_\sigma$. Since $M \overset{\beta}{\rightarrow}_\sigma N$ or $N \overset{\beta}{\rightarrow}_\sigma M$, $N \not\in \Lambda_\sigma$ by Fact 10.2. We prove by induction on $N \in \Lambda_\sigma$ that $M \in \Lambda_\sigma$. By Definition 16, there are only two cases:
   - either $N = xVN_1 \ldots N_n$ for some $n \in \mathbb{N}$, variable $x$, $V \in \Lambda_\sigma$ and $N_1, \ldots, N_n \in \Lambda$, but this is impossible since the last rule of the derivation of $M \overset{\beta}{\rightarrow}_\sigma N$ or $N \overset{\beta}{\rightarrow}_\sigma M$ can be neither $\sigma_1$ nor $\sigma_3$ (because of the subterm $xV$) nor $\otimes$ (because of Fact 10.2);
   - or $N = VLN_1 \ldots N_n$ for some $n \in \mathbb{N}$, $V \in \Lambda_\sigma$, $L \in \Lambda_\sigma$ and $N_1, \ldots, N_n \in \Lambda$, and then there are three sub-cases, depending on the last rule $r$ of the derivation of $M \overset{\beta}{\rightarrow}_\sigma N$ (resp. $N \overset{\beta}{\rightarrow}_\sigma M$):
     - if $r = \sigma_1$ then $V = \lambda x.N'N_0$ (resp. $\lambda x.N'$) and $M = (\lambda x.N')LN_0 \ldots N_n$ (resp. $M = (\lambda x.N'N_1)LN_2 \ldots N_n$ with $n > 0$) for some $N_0, N_1 \in \Lambda$, hence $M \in \Lambda_\sigma$;
     - if $r = \sigma_3$ then $V = \lambda x.V'N'$ (resp. $L = (\lambda x.N')L'$) and $M = V'((\lambda x.N')L)N_1 \ldots N_n$ (resp. $M = (\lambda x.V'N)LN_1 \ldots N_n$) for some $V' \in \Lambda_\sigma$ (resp. $L' \in \Lambda_\sigma$) and $N' \in \Lambda$, thus $(\lambda x.N')L \in \Lambda_\sigma$ (resp. $(\lambda x.V'N')L' \in \Lambda_\sigma$) and hence $M \in \Lambda_\sigma$.\)
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Lemma 19.4 is a formalization of the two following facts: (a) a head $\sigma$-reduction step may create a new $\beta_v$-redex but in this case it is not a head $\beta_v$-redex; (b) when $M \xrightarrow{\beta_v} N$, the head $\beta_v$-redex of $M$ (if any) has a residual in $N$ which is the head $\beta_v$-redex of $N$.

Lemma 20. There exists no infinite head $\nu$-reduction sequence with finitely many head $\nu$-reduction steps.

Proof. Suppose the opposite holds: then there would exist $M \in N$ and an infinite sequence of terms $(M_i)_{i \in N}$ such that $M_i \xrightarrow{\nu} M_{i+1}$ for any $1 \leq i \leq m$, $M_m \xrightarrow{\beta_v} M_{m+1}$ and $M_{i+1} \xrightarrow{\sigma} M_i$ for any $i > m$ (since $\xrightarrow{\nu} = \xrightarrow{\beta_v} \cup \xrightarrow{\sigma}$). But this is impossible because $\xrightarrow{\sigma}$ is strongly normalizing (by Proposition 5 and since $\xrightarrow{\beta_v} \subseteq \xrightarrow{\sigma}$). Contradiction.

Now we can state and prove our main result about head $\beta_v$- and head $\nu$-normalization.

Theorem 21 (Head normalization). Let $M \in \Lambda$. The following are equivalent:

1. there exists a head $\beta_v$-normal form $N$ such that $M \xrightarrow{\beta_v} N$;
2. there exists a head $\nu$-normal form $N$ such that $M \xrightarrow{\nu} N$;
3. $M$ is head $\nu$-normalizable;
4. $M$ is head $\beta_v$-normalizable;
5. there is no $\nu$-reduction sequence from $M$ with infinitely many head $\beta_v$-reduction steps;
6. $M$ is strongly head $\nu$-normalizable.

Proof.

(1)$\Rightarrow$(2) By hypothesis, there exists a head $\beta_v$-normal $N \in \Lambda$ such that $M \xrightarrow{\beta_v} N$, thus $M \xrightarrow{\nu} N$. Since $\xrightarrow{\beta_v}$ is strongly normalizing (by Proposition 5 and because $\xrightarrow{\beta_v} \subseteq \xrightarrow{\sigma}$), there exists a head $\sigma$-normal $N' \in \Lambda$ such that $N \xrightarrow{\beta_v} N'$, therefore $M \xrightarrow{\nu} N'$ since $\xrightarrow{\beta_v} \subseteq \xrightarrow{\sigma}$. By Lemma 19.4, $N'$ is also head $\beta_v$-normal and hence head $\nu$-normal.

(2)$\Rightarrow$(3) Since $M \xrightarrow{\nu} N$, there is $L \in \Lambda$ such that $M \xrightarrow{\nu} L$ and $N \xrightarrow{\nu} L$, by confluence of $\xrightarrow{\nu}$ (Proposition 5). By Theorem 14, there are $M_1, M_2, N_1, N_2 \in \Lambda$ such that $M \xrightarrow{\beta_v} M_1 \xrightarrow{\beta_v} M_2 \xrightarrow{\nu} L$ and $N \xrightarrow{\nu} N_1 \xrightarrow{\beta_v} N_2 \xrightarrow{\nu} L$. As $N$ is head $\nu$-normal, $N = N_1 = N_2 \xrightarrow{\nu} L$.

By Lemma 19.3, $L$ and $M_2$ are $\nu$-head normal. So, $M \xrightarrow{\beta_v} M_2$ with $M_2$ head $\nu$-normal.

(3)$\Rightarrow$(4) By hypothesis, there is $N \in \Lambda$ head $\nu$-normal such that $M \xrightarrow{\beta_v} N$. By Lemma 12.2, there is $L \in \Lambda$ such that $M \xrightarrow{\beta_v} L \xrightarrow{\beta_v} N$. Since $N$ is head $\nu$-normal and in particular head $\beta_v$-normal, $L$ is head $\beta_v$-normal according to Lemma 19.4. So $M$ is head $\beta_v$-normalizable.

(4)$\Rightarrow$(5) Lemma 12.1 says that if $N \xrightarrow{\beta_v} L \xrightarrow{\beta_v} N'$ then there exists $L' \in \Lambda$ such that $N \xrightarrow{\beta_v} L' \xrightarrow{\beta_v} N'$; Lemma 13 and Fact 10.3 show that if $N \xrightarrow{\beta_v} L \xrightarrow{\beta_v} N'$ then there exist $L', L'' \in \Lambda$ such that $N \xrightarrow{\beta_v} L' \xrightarrow{\beta_v} L'' \xrightarrow{\beta_v} N'$. Since $\xrightarrow{\nu} = \xrightarrow{\beta_v} \cup \xrightarrow{\nu} \cup \xrightarrow{\beta_v}$, this means that if there is an infinite $\nu$-reduction sequence from $M$ with infinitely many head $\beta_v$-reduction steps, then for any $n \in N$ there is a head $\nu$-reduction sequence from $M$ whose length is at least $n$. Therefore, $M$ is not head $\beta_v$-normalizable, since the head $\beta_v$-reduction is deterministic.

(5)$\Rightarrow$(6) If $M$ is not strongly head $\nu$-normalizable then there exists an infinite head $\nu$-reduction sequence. By Lemma 20, this head $\nu$-reduction (and hence $\nu$-reduction, since $\xrightarrow{\nu} \subseteq \xrightarrow{\nu}$) sequence has infinitely many head $\beta_v$-reduction steps.
As $M$ is strongly head $v$-normalizable, in particular is head $v$-normalizable, hence there exists $N \in \Lambda$ head $v$-normal and in particular head $\beta_v$-normal such that $M \rightarrow^*_{\lambda_v} N$. By Lemma 12.2, there exists $L \in \Lambda$ such that $M \rightarrow^*_{\beta_v} L \rightarrow^*_{\sigma} N$. Therefore $M \simeq_{\beta_v} L$ since $\rightarrow^*_{\beta_v} \subseteq \rightarrow_{\beta_v}$. According to Lemma 19.4, $L$ is head $\beta_v$-normal.

In Theorem 21, the equivalence (3)$\Leftrightarrow$(6) means that (weak) normalization and strong normalization are equivalent for head $v$-reduction (for head $\beta_v$-reduction they are trivially equivalent since the head $\beta_v$-reduction is deterministic), therefore if one is interested in studying the termination of head $v$-reduction, no difficulty arises from its non-determinism. The equivalence (4)$\Leftrightarrow$(3) or (4)$\Leftrightarrow$(6) says that the weak evaluation process defined for Plotkin’s $\lambda_v$-calculus (the head $\beta_v$-reduction) terminates if and only if the weak evaluation process defined for $\lambda^0_v$ (the head $v$-reduction) terminates: $\sigma$-rules play no role in deciding the termination of a head $v$-reduction sequence. The equivalence (3)$\Leftrightarrow$(2) (resp. (4)$\Leftrightarrow$(1)) is the version for $\lambda^0_v$ (resp. $\lambda_v$) of a well-known theorem for ordinary $\lambda$-calculus (see for example [3, Theorem 8.3.11]): in some sense, it claims that the head $v$-reduction (resp. head $\beta_v$-reduction) is complete with respect to the $v$-equivalence (resp. $\beta_v$-equivalence). The equivalence (5)$\Leftrightarrow$(2) (resp. (5)$\Leftrightarrow$(1)) can be seen as the version for $\lambda^0_v$ (resp. $\lambda_v$) of the Quasi-Head Reduction Theorem [19, Theorem 2.10] stated by Takahashi for ordinary $\lambda$-calculus.

5 Normalization strategy and other results

Theorems 14 and 21 strengthen the idea that, in spite of non-determinism and non-confluence of head $v$-reduction and non-sequentiability of head $\sigma$-reduction steps, the head $v$-reduction can be used to define a normalization strategy for the $\lambda^0_v$-calculus, as proven in next Theorem 24, the second main result of our paper: given a term $M$, one starts the (unique) head $\beta_v$-head reduction sequence from $M$ as long as a head $\beta_v$-normal form $N$ is reached (recall that, according to Theorem 21, a term is (strongly) head $v$-normalizable if and only if it is head $\beta_v$-normalizable); then, one starts a head $\sigma$-reduction sequence from $N$ (where head $\sigma_1$- and head $\sigma_3$-reduction steps can be performed in whatever order) as long as a head $\sigma$-normal form $N'$ is reached (such a $N'$ always exists because $\rightarrow^*_{\sigma}$ is strongly normalizing, and it is head $v$-normal by Lemma 19.4); finally, one performs the internal $v$-reduction steps starting from $N'$ by iteratively the head $\beta_v$-reduction sequences and then the head $\sigma$-reduction sequences as above on the subterms of $N'$, from the left to the right. More precisely:

Definition 22 (Successors path). Let $M \in \Lambda$.

A successor of $M$ is a $M' \in \Lambda$ defined by induction on $M \in \Lambda$ as follows:

- if $M$ is not head $\beta_v$-normal, then $M'$ is such that $M \rightarrow_{\beta_v} M'$;
- if $M$ is head $\beta_v$-normal but not head $\sigma$-normal, then $M'$ is such that $M \rightarrow_{\sigma} M'$;
- if $M$ is head $v$-normal then:
  - if $M$ is a variable then $M' = M$,
  - if $M = \lambda x. N$ for some $N \in \Lambda$, then $M' = \lambda x. N'$ for some successor $N'$ of $N$,
  - if $M = NL$ for some $N, L \in \Lambda$, then either $N$ is not $v$-normal and $M' = N' L$ where $N'$ is a successor of $N$, or $N$ is $v$-normal and $M' = NL'$ where $L'$ is a successor of $L$.

A successors path of $M$ is an infinite sequence $(M_i)_{i \in \mathbb{N}}$ of terms such that $M_0 = M$ and $M_{i+1}$ is a successor of $M_i$, for any $i \in \mathbb{N}$.

Clearly, for every term $M$ there is at least one successor $M'$ of $M$; moreover, this successor $M'$ is unique when $M$ is not head $\beta_v$-normal, since the head $\beta_v$-reduction is deterministic, and $M = M'$ when $M$ is $v$-normal.
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**Remark 23.** Let $M \in \Lambda$ and let $(M_i)_{i \in \mathbb{N}}$ be a successors path of $M$.
1. For every $i \in \mathbb{N}$, there exist $0 \leq j \leq k \leq i$ such that $M \xrightarrow{\beta_i} M_j \xrightarrow{\sigma} M_k \xrightarrow{\nu} M_i$.
2. For every $i \in \mathbb{N}$, if $M_i$ is $\nu$-normal then $M_j$ is $\nu$-normal for any $j \geq i$.

A successors path of a term $M$ is a call-by-value left-to-right $\nu$-evaluation strategy starting from $M$ that can reduce under a $\lambda$ only when a head $\nu$-normal from is reached. Due to the non-determinism of the head $\sigma$-reduction, a term $M$ may have several successors paths. We cannot get rid of the non-determinism of the successors path of $M$ because of the non-sequentiability of head $\sigma$-reductions, see p. 9 and [7, p. 10].

**Theorem 24 (Normalization strategy).** Let $M \in \Lambda$. Every successors path $(M_i)_{i \in \mathbb{N}}$ of $M$ is a normalization strategy for $M$, i.e. if $M$ is $\nu$-normalizable then there exists $j,k,\ell \in \mathbb{N}$ such that $j \leq k \leq \ell$, $M_j$ is head $\beta_v$-normal, $M_k$ is head $\nu$-normal and $M_\ell$ is $\nu$-normal.

**Proof.** Let $(M_i)_{i \in \mathbb{N}}$ be a successors path of $M$ and $N \in \Lambda$ be such that $N$ is $\nu$-normal and $M \xrightarrow{\nu} N$; we prove by induction on $N \in \Lambda$ that there exist $j,k,\ell \in \mathbb{N}$ such that $M_j$ is head $\beta_v$-normal, $M_k$ is head $\nu$-normal and $M_\ell$ is $\nu$-normal.

Since $M$ is $\nu$-normalizable, then it is head $\beta_v$-normalizable (because $\beta_v \subseteq \beta_\sigma$), thus there exists $j \in \mathbb{N}$ such that $M_j$ is head $\beta_v$-normal because $\beta_\sigma$ is deterministic. As $\beta_\sigma$ is strongly normalizing (by Proposition 5, since $\beta_\sigma \subseteq \beta_\nu$), there exists $k \in \mathbb{N}$ with $j \leq k$ such that $M_k$ is head $\sigma$-normal. According to Lemma 19.4, $M_k$ is also head $\beta_v$-normal, hence $M_k$ is head $\nu$-normal. Certainly, $M_k = VN_1 \ldots N_n$ for some $n \in \mathbb{N}$, $V \in \Lambda_0$ and $N_1, \ldots, N_n \in \Lambda$. By confluence of $\nu$ (Proposition 5) and since $N$ is $\nu$-normal and $M_k$ is head $\nu$-normal, one has $M_k \xrightarrow{\nu} N$ and hence $N = V'N'_1 \ldots N'_n$ for some $\nu$-normal $V'$, and some $\nu$-normal $N'_1, \ldots, N'_n \in \Lambda$ such that $V \rightarrow_\nu V'$ and $N_r \rightarrow_\nu N'_r$ for any $1 \leq r \leq n$. By induction hypothesis, for every successors path $(V_i)_{i \in \mathbb{N}}$ of $V$ and, for any $1 \leq r \leq n$, for every successors path $(N'_i)_{i \in \mathbb{N}}$ of $N'_r$ there exist $p_1, \ldots, p_n \in \mathbb{N}$ such that $V'_p, L^1_{p_1}, \ldots, L^n_{p_n}$ are $\nu$-normal: by confluence of $\nu$ (Proposition 5), $V'_p = V'$ and $N'_r = L^r_{p_r}$ for any $1 \leq r \leq n$.

Let us consider the infinite sequence of terms $s = (M=M_0, \ldots, M_k=VN_1 \ldots N_n = V_0N_1 \ldots N_n, \ldots, V_pN_1 \ldots N_n = V'L^1_{p_1}N_2 \ldots N_n, \ldots, V'L^1_{p_1}N_2 \ldots N_n = V'N'_1 \ldots N_n, \ldots, V'N'_1 \ldots N_n = N, N, \ldots)$: this is a successors path of $M$ and, for an opportune choice of the successors paths $(V_i)_{i \in \mathbb{N}}, (L^1_i)_{i \in \mathbb{N}}, \ldots, (L^n_i)_{i \in \mathbb{N}}$, one has that $s = (M_i)_{i \in \mathbb{N}}$, in particular there exists $\ell \in \mathbb{N}$ such that $j \leq k \leq \ell$ and $M_\ell = N$.

In ordinary $\lambda$-calculus, the well-known theorem corresponding to our Theorem 24 is the Leftmost Reduction Theorem, see [19, Theorem 2.8] or [3, Theorem 13.2.2]. Differently from the leftmost reduction of ordinary $\lambda$-calculus, our normalization strategy is not deterministic, i.e., our Theorem 24 provides a family of normalization strategies.

Finally, we have shown at p. 7 that the head $\sigma$- and head $\nu$-reductions are not (locally) confluent and a term may have several head $\nu$-normal forms. Nevertheless, the characterization of head $\nu$-normal forms given by Proposition 18 allows us to claim that (see next Proposition 27) in some cases (of interest), more precisely when a term has a head $\nu$-normal form which is a value or an element of $\Lambda_0$, the head $\nu$-normal form is unique (Proposition 27.1): all terms having several head $\nu$-normal forms are such that all their head $\nu$-normal forms are in $\Lambda_0$. In particular, every head $\nu$-normalizable closed term has a unique head $\nu$-normal form, which is an abstraction and coincides with its head $\beta_v$-normal form (Proposition 27.2).

**Remark 25.** By inspection on the rules of Definition 8, it easy to check that the head $\sigma$-reduction does not reduce to a term in $\Lambda_0$, i.e., for any $M \in \Lambda$ and $N \in \Lambda_0$, one has $M \not\xrightarrow{\sigma} N$.

Remark 25 does not hold if we replace $\xrightarrow{\sigma}$ with $\xrightarrow{\beta_v}$; for instance, $x(I) \xrightarrow{\beta_v} xI \in \Lambda_0$. 

Fact 26. For every \( N \in \Lambda_v \cup \Lambda_a \), one has \( M \xrightarrow{\beta_v} N \) if and only if \( M \xrightarrow{\beta_v^*} N \).

Proof. The left-to-right direction follows from \( \xrightarrow{\beta_v} \subseteq \xrightarrow{\beta_v^*} \). The right-to-left direction is a consequence of Lemma 12.2 and either Fact 10.2 (if \( N \in \Lambda_a \)) or Remark 25 (if \( N \in \Lambda_v \)).

Fact 26 means that, given a head \( v \)-reduction sequence, the head \( \sigma \)-reduction plays no role not only in deciding its termination (as stated in Theorem 21), but also in reaching a particular value or term in \( \Lambda_v \). Fact 26 will be used in the proof of Proposition 27.

Proposition 27 (Uniqueness of “some” head \( v \)-normal forms). Let \( M \in \Lambda \) and \( M \xrightarrow{\beta_v^*} N \).

1. If \( N \notin \Lambda_v \cup \Lambda_a \) then, for every head \( v \)-normal \( L \in \Lambda \), \( M \xrightarrow{\gamma_v^*} L \) implies \( N = L \).

2. If \( M \) is closed and \( N \) is head \( v \)-normal, then \( M \xrightarrow{\beta_v^*} N \) and \( N = \lambda x.N' \) for some \( N' \in \Lambda \) such that \( fv(N') \subseteq \{x\} \); moreover, for any head \( v \)-normal \( L \in \Lambda \), \( M \xrightarrow{\gamma_v^*} L \) implies \( N = L \).

Proof.

1. Since \( N \notin \Lambda_v \cup \Lambda_a \), \( M \xrightarrow{\gamma_v^*} N \) implies \( M \xrightarrow{\beta_v^*} N \) by Fact 26. According to Proposition 18.3, \( N \) is head \( v \)-normal.

Let \( L \in \Lambda \) be head \( v \)-normal and such that \( M \xrightarrow{\beta_v^*} L \); by Proposition 18.3, \( L \in \Lambda_v \cup \Lambda_a \cup \Lambda_b \).

We claim that \( L \notin \Lambda_b \). Otherwise, \( L \in \Lambda_b \), and then, by confluence of \( \xrightarrow{v} \), there would exist \( M' \in \Lambda \) such that \( N \xrightarrow{v} M' \) and \( L \xrightarrow{v} M' \). According to Proposition 11 and since \( N \) and \( L \) are head \( v \)-normal, \( N \xrightarrow{\beta_v^*} M' \) and \( L \xrightarrow{\beta_v^*} M' \). By Remark 4 (since \( \xrightarrow{v} \subseteq \xrightarrow{v} \)) and Lemma 19.1, \( M' \in \Lambda_v \cup \Lambda_a \). By Lemma 19.2, \( M' \in \Lambda_b \). But \( \Lambda_v \cap \Lambda_a = 0 = \Lambda_a \cap \Lambda_b \); contradiction, therefore \( L \notin \Lambda_b \).

So, \( L \in \Lambda_v \cup \Lambda_a \) and thus \( M \xrightarrow{\beta_v^*} L \) by Fact 26, hence \( N = L \) since \( \xrightarrow{\beta_v^*} \) is deterministic.

2. Since \( M \) is closed, \( N \) is closed too. Hence, by Proposition 18.3, \( N \in \Lambda_v \) (since the terms in \( \Lambda_v \cup \Lambda_b \) are not closed) and \( N \) is not a variable, therefore \( N = \lambda x.N' \) for some \( N' \in \Lambda \) such that \( fv(N') \subseteq \{x\} \). By Fact 26, \( M \xrightarrow{\beta_v^*} N \). According to Proposition 27.1, for every head \( v \)-normal \( L \in \Lambda \), \( M \xrightarrow{\gamma_v^*} L \) implies \( N = L \).

Recall that all head \( v \)-normal terms are head \( \beta_v \)-normal, since \( \xrightarrow{\beta_v} \subseteq \xrightarrow{\beta_v^*} \).

6 Conclusions and future work

In this paper, we have investigated the \( \lambda_v^* \)-calculus introduced in [4], an extension of Plotkin’s call-by-value \( \lambda \)-calculus \( \lambda_v \) [15] with the same syntax as \( \lambda_v \) (without constants) and ordinary (i.e. call-by-name) \( \lambda \)-calculus. The peculiarity of \( \lambda_v^* \) is in its reduction rules: the \( v \)-reduction adds to Plotkin’s \( \beta_v \)-reduction two commutation rules called \( \sigma_1 \) and \( \sigma_3 \) which unblock “hidden” \( \beta_v \)-redexes. We have studied the head \( v \)-reduction, a non-confluent sub-reduction of the \( v \)-reduction already introduced in [7]. We now summarize our main contributions:

1. Theorem 21 is about head \( v \)-normalization: it shows that:
   - for the head \( v \)-reduction, normalization coincides with strong normalization;
   - the head \( v \)-reduction is deeply related to Plotkin’s deterministic weak evaluation strategy for \( \lambda_v \) (the former terminates if and only if the latter terminates);
   - both head \( v \)-reduction and weak evaluation strategy for \( \lambda_v \) enjoy good properties analogous to the ones of the (call-by-name) head reduction for ordinary \( \lambda \)-calculus.

2. Theorem 24 is about \( v \)-normalization: it proves that a top-down extension of the head \( v \)-normalization provides a family of normalization strategies for the (full) \( v \)-reduction.

3. Proposition 27 is about the uniqueness of the head \( v \)-normal form: it shows that, even if there are terms having several head \( v \)-normal forms, in some case of interest (for instance, closed terms) the head \( v \)-normal form, if any, is unique.
These results, together with the results proven in [4, 7], shows that $\lambda_v^\pi$ is a useful tool to study some theoretical and semantic properties of Plotkin’s $\lambda_v$-calculus, for instance the notions of call-by-value solvability and potential valuability. This is hard (or impossible) to obtain directly in $\lambda_v$ because of the “weakness” of Plotkin’s $\beta_v$-reduction. In the case of ordinary (i.e. call-by-name) $\lambda$-calculus, head reduction and solvability are the starting point to investigate separability, semi-separability and Böhm’s trees. Hence, it may reasonably be supposed that we have all the ingredients for tackling the question of separability, semi-separability and Böhm’s trees in a call-by-value setting. In particular, one may reasonably hope to improve in $\lambda_v^\sigma$ the separability theorem already proven by Paolini [12] for $\lambda_v$.

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References


