Abstract

We study two clustering problems, STARFOREST EDITING, the problem of adding and deleting edges to obtain a disjoint union of stars, and the generalization BICLUSTER EDITING. We show that, in addition to being \( \text{NP} \)-hard, none of the problems can be solved in subexponential time unless the exponential time hypothesis fails.

Misra, Panolan, and Saurabh (MFCS 2013) argue that introducing a bound on the number of connected components in the solution should not make the problem easier: In particular, they argue that the subexponential time algorithm for editing to a fixed number of clusters (\( p \)-CLUSTER EDITING) by Fomin et al. (J. Comput. Syst. Sci., 80(7) 2014) is an exception rather than the rule. Here, \( p \) is a secondary parameter, bounding the number of components in the solution.

However, upon bounding the number of stars or bicliques in the solution, we obtain algorithms which run in time \( O(2^{\sqrt{p}k} + n + m) \) for \( p \)-STARFOREST EDITING and \( O(2^{O(p\sqrt{\log(pk)})} + n + m) \) for \( p \)-BICLUSTER EDITING. We obtain a similar result for the more general case of \( t \)-PARTITE \( p \)-CLUSTER EDITING. This is subexponential in \( k \) for a fixed number of clusters, since \( p \) is then considered a constant.

Our results even out the number of multivariate subexponential time algorithms and give reasons to believe that this area warrants further study.

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1 Introduction

Identifying clusters and biclusters has been a central motif in data mining research [22] and forms the cornerstone of algorithmic applications in, e.g., biology [25] and expression data analysis [7]. Cai [6] showed that clustering – among many other graph modification problems of similar flavor – is solvable in fixed-parameter tractable time. Parallel to these general results, some progress was made in the area of structurally sparse graphs: many problems are, when restricted to classes characterized by a finite set of forbidden minors, solvable in \textit{subexponential parameterized time}, i.e. they admit algorithms with time complexity \( 2^{o(k)} \cdot \text{poly}(n) \).

The complexity class of problems admitting such an algorithm is called \textsc{SUBEPT} and was defined by Flum and Grohe in the seminal textbook on parameterized complexity [14]. They simultaneously noticed that most natural problems did, in fact, \textit{not} live in this complexity class: The classical \textsc{NP}-hardness reductions paired with the \textit{exponential time hypothesis} of Impagliazzo, Paturi, and Zane [20] is enough to show that no \( 2^{o(k)} \cdot \text{poly}(n) \) algorithm exists.
In this context, Jianer Chen posed the following open problem in the field of parameterized algorithms [5]: Are there examples of natural problems on graphs, that do not have such a topological constraint, and also have subexponential parameterized running time? Alon, Lokshtanov, and Saurabh [1] partially answered this question in the positive by providing a subexponential time algorithm for **Feedback Arc Set** on tournament graphs. However, the aforementioned graph classes with topological constraints are sparse, and tournament graphs are extremely dense. Chen’s question is therefore not fully answered – are there problems which are in **SUBEPT** on general graphs?

This is indeed the case. Fomin and Villanger [16] showed that **Minimum Fill-In** was solvable in time $2^{O(\sqrt{k} \log k)} + \text{poly}(n)$. **Minimum Fill-In** is the problem of completing a graph into a chordal graph by adding as few edges as possible. Following this, a line of research was established investigating whether more graph modification problems admit such algorithms. It proved to be a fruitful area; Since the result by Fomin and Villanger (ibid.), we now know that several graph modification problems towards classes such as split graphs [17], threshold graphs [10], trivially perfect graphs [11], (proper) interval graphs [3, 4], and more admit subexponential time algorithms.

While these classes are rather “simple”, they certainly are much more complex than simple cluster or bicluster graphs. Therefore, the problems **Cluster Editing** and **Cluster Deletion** were a logical candidate for subexponential time algorithms. Surprisingly, we cannot expect that such algorithms exist. Komusiewicz and Uhlmann gave an elegant reduction proving that both parameterized and exact subexponential time algorithms were not achievable, unless ETH fails [21]. On the other hand, the problem $p$-**Cluster Editing**, where the number of components in the target class is fixed to be at most $p$ – rather surprisingly – does indeed admit a subexponential parameterized time algorithm; This was shown by Fomin et al. [15], who designed an algorithm solving this problem in time $2^{O(\sqrt{pk})} \cdot \text{poly}(n)$.

Misra, Panolan, and Saurabh explicitly stated their surprise about this result: In their opinion, bounding the number of components in the target graph should in general not facilitate subexponential time algorithms [23]: “We show that this sub-exponential time algorithm for the fixed number of cliques is rather an exception than a rule.”

We show that the related problem **Bicluster Editing** and its generalization $t$-**Partite p-Cluster Editing** as well as the special case **Starforest Editing** also belong to this exceptional class of problems where a bound on the number of target components greatly improves their algorithmic tractability. Since **Bicluster Editing** is an important tool in molecular biology and biological data analysis, and the necessary second parameter is not outlandish in these settings, we feel that this is a noteworthy insight. We complement these results with **NP-completeness proofs for Bicluster Editing and t-Partite p-Cluster Editing** on subcubic graphs and further show that, unless ETH fails, no parameterized or exact subexponential algorithm is possible without the secondary parameter. That a bound on the maximal degree does not contribute towards making these problems more tractable contrasts many other graph modification problems (like modifications towards split and threshold graphs [24]) which are polynomial-time solvable in this setting.

Previously, it was known **Bicluster Editing** in general is **NP-complete** [2], and Guo, Hüffner, Komusiewicz, and Zhang [18] studied the problem from a parameterized point of view, giving a linear problem kernel with $6k$ vertices, and an algorithm solving the problem in time $O(3.24^k + m)$. 
Our contribution. In this paper, we study both the very general \( t \)-PARTITTE \( p \)-CLUSTER EDITING as well as editing to the aforementioned special cases. On the positive side, we show that

- \( p \)-STARFOREST EDITING is solvable in time \( O(2^{\sqrt{pk}} + n + m) \),
- both \( p \)-BICLUSTER EDITING and the more general \( t \)-PARTITTE \( p \)-CLUSTER EDITING are solvable in time \( 2^{O(t\sqrt{\log pk})} + O(n + m) \) facilitated by a kernel of size \( O(pk) \), where \( t = 2 \) in the case of \( p \)-BICLUSTER EDITING.

In many cases, \( p \) is considered a constant, and in this case our kernel has size linear in \( k \). We supplement these algorithms with hardness results; Specifically, we show that

- assuming ETH, STARFOREST EDITING and BICLUSTER EDITING cannot be solved in time \( 2^{o(k)} \cdot \text{poly}(n) \) and thus neither can \( t \)-PARTITTE CLUSTER EDITING, and finally,
- \( p \)-STARFOREST EDITING is \( \text{W}[1] \)-hard if parameterized by \( p \) alone.

Organization of the paper. In Section 3 we give a subexponential time parameterized algorithm for the STARFOREST EDITING problem when parameterized by the editing budget and the number of stars in the solution simultaneously. One ingredient for our subexponential algorithms is a polynomial kernel. A kernel for BICLUSTER EDITING exists already [18] and we provide one for the \( t \)-partite case in Section 4. In Section 5 we show that \( p \)-BICLUSTER EDITING is solvable in subexponential time in \( k \); We give a \( 2^{O(t\sqrt{\log pk})} + O(n + m) \) algorithm and generalize it to editing to \( t \)-partite \( p \)-cluster graphs. The parameter \( p \) is usually considered to be a fixed constant, hence the running time is truly subexponential, \( 2^{o(k)} + O(n + m) \) in the editing budget \( k \). However, for a more fine-grained complexity analysis and for lower bounds, we treat \( p \) as a parameter.

In Section 6 we show that we cannot expect such an algorithm without an exponential dependency on \( p \); The problem is not solvable in time \( 2^{o(k)} \cdot \text{poly}(n) \) unless ETH fails. Further, we show that STARFOREST EDITING is \( \text{W}[1] \)-hard if parameterized by \( p \) alone, before we conclude in Section 7. Due to page limits, some proofs have been deferred to the full version, available online [12].

## Preliminaries

We consider only finite simple graphs \( G = (V, E) \) and we use \( n \) and \( m \) to denote the size of the vertex set and edge set, respectively. We denote by \( N_G(v) \) the set of neighbors of \( v \) in \( G \), and let \( \deg_G(v) = |N_G(v)| \). We omit subscripts when the graph in question is clear from context. We refer to the monograph by Diestel [9] for graph terminology and notation not defined here. For information on parameterized complexity, we refer to the textbook by Flum and Grohe [14]. We consider an edge in \( E(G) \) to be a set of size two, i.e., \( e \in E(G) \) is of the form \( \{u, v\} \subseteq V(G) \) with \( u \neq v \). We denote by \( |V(G)|^2 \) the set of all size two subsets of \( G \). When \( F \subseteq |V(G)|^2 \), we write \( G \triangle F \) to denote \( G' = (V, E \triangle F) \), where \( \triangle \) is the symmetric difference, i.e., \( E \triangle F = (E \setminus F) \cup (F \setminus E) \). When the graph is clear from context, we will refer to \( F \) simply as a set of edges rather than \( F \subseteq |V(G)|^2 \).

Let us fix the following terminology: A star graph is a tree of diameter at most two (a graph isomorphic to \( K_{1, t} \) for some \( t \)). The degree-one vertices are called leaves and the vertex of higher degree the center. A starforest is a forest whose connected components are stars or, equivalently, a graph that does not contain \( \{K_3, P_4, C_4\} \) as induced subgraphs. A biclique is a complete bipartite graph \( K_{a,b} \) for some \( a, b \in \mathbb{N} \), and a bicluster graph is a disjoint union of bicliques. A \( t \)-partite clique graph is a graph whose vertex set can be partitioned into at most \( t \) independent sets, all pairwise fully connected, and a \( t \)-partite cluster graph is a
disjoint union of \( t \)-partite cliques. The problem of editing towards a starforest (resp. bicluster and \( t \)-partite cluster) is the algorithmic problem of adding and deleting as few edges as possible to convert a graph \( G \) to a starforest (resp. bicluster and \( t \)-partite cluster). We write \( f(n) = \text{poly}(n) \) to mean \( f(n) = n^{O(1)} \), i.e., that there exists a \( c \in \mathbb{N} \) such that \( f(n) = O(n^c) \).

**Exponential time hypothesis.** To show that there is no subexponential time algorithm for **Starforest Editing** we give a linear reduction from 3Sat, that is, a reduction which constructs an instance whose parameter is bounded linearly in the size of the input formula. The constructed instance will also have size bounded linearly in the size of the formula, and we use this to also rule out an exact subexponential algorithm of the form \( 2^{o(n+m)} \). Pipelining such a reduction with an assumed subexponential parameterized algorithm for the problem would give a subexponential algorithm for 3Sat, contradicting the complexity hypothesis of Impagliazzo, Paturi, and Zane [20]. Their Sparsification Lemma shows that, unless the exponential time hypothesis (ETH) fails, 3Sat cannot be solved in time \( 2^{o(n+m)} \), where \( n \) and \( m \) refer to the number of variables and the number of clauses, respectively.

## 3 Editing to starforests in subexponential time

A first natural step in handling modification problems related to bicluster graphs is modification towards the subclass of bicluster graphs called starforest. Recall that a graph is a starforest if it is a bicluster where every biclique has one side of size exactly one, or equivalently, every connected component is a star.

**Starforest Editing parameterized by \( k \)**

**Input:** A graph \( G = (V, E) \) and a non-negative integer \( k \).

**Question:** Is there a set of at most \( k \) edges \( F \) such that \( G \oplus F \) is a disjoint union of stars?

The problem where we only allow to delete edges is referred to as **Starforest Deletion**. These two problems can easily be observed to be equivalent; Adding an edge to a forbidden induced subgraph will create one of the other forbidden subgraphs, or simply put, it never makes sense to add an edge.

In Section 6 we show that this problem is \( \text{NP} \)-hard, and that it is not solvable in time \( 2^{o(k) \text{ poly}(n)} \) unless the exponential time hypothesis fails.

**Multivariate analysis.** Since no subexponential algorithm is possible under ETH, we introduce a secondary parameter by \( p \) which bounds the number of connected components in a solution graph. This has previously been done with success in the **Cluster Editing** problem [15]. Hence, we define the following multivariate variant of the above problem.

**p-Starforest Editing parameterized by \( p, k \)**

**Input:** A graph \( G = (V, E) \) and a non-negative integer \( k \).

**Question:** Is there a set \( F \) of edges of size at most \( k \) such that \( G \oplus F \) is a disjoint union of exactly \( p \) stars?

Observe that this problem is not the same as **p-Starforest Deletion** since we might need to merge stars to achieve the desired value \( p \) for the number of connected components. In Section 6 we show that the problem is \( \text{W}[1] \)-hard parameterized by \( p \) alone, and that we therefore need to parameterize on both \( p \) and \( k \).
Lemma 1. Let \((G, k)\) be input to \(p\)-Starforest Editing. If \((G, k)\) is a yes-instance, there can be at most \(p + 2k\) vertices with degree at least 2.

The following bound will be key to obtain the subexponential running time.

Proposition 2 ([15]). If \(a\) and \(b\) are non-negative integers, then \((a + b)/a \leq 2^\sqrt[3]{ab}\).

Lemma 3. Given a graph \(G\) and a vertex set \(S\), we can compute in linear time \(O(n + m)\) an optimal editing set \(F\) such that \(G \triangle F\) is a starforest, when restricted to have \(S\) as the set of centers in the solutions.

We now describe an algorithm which solves \(p\)-Starforest Editing in time \(O(2^{\sqrt[3]{pk}} + n + m)\).

The algorithm. Let \((G, k)\) be an input instance for \(p\)-Starforest Editing. If the number of vertices of degree at least two is greater than \(p + 2k\), we say no in accordance with Lemma 1. Otherwise we split the graph into \(G_1\) and \(G_2\) as follows: Let \(X \subseteq V(G)\) be the collection of vertices contained in connected components of size one or two, i.e., \(G[X]\) is a collection of isolated vertices and edges. Let \(G_1 = G[X]\) and \(G_2 = G[V(G) \setminus X]\). Clearly, there are no edges going out of \(X\) in \(G\). We will treat \(G_1, G_2\) as (almost) independent subinstances by guessing the budgets \(k_1 + k_2 = k\) and the number of components in their respective solutions \(p_1 + p_2 = p\). The only time we cannot treat them as independent instances is when \(p_1\) or \(p_2\) is zero; Let \(p^*_1\) be the number of stars completely contained in \(G_1\) in an optimal solution. If both \(p^*_1 > 0\), then there always exists an optimal solution that does not add any edge between \(G_1\) and \(G_2\).

Solving \((G_1, k_1)\) with \(p_1\) components: Assume \(G_1\) contains \(s\) isolated edges and \(t\) isolated vertices, with \(p_1 > 0\). If \(|V(G_1)| < p_1\), we immediately say no, since we need exactly \(p_1\) connected components. Depending on the values of \(s\) and \(t\), we execute the following operations as long as the budget \(k_1\) is positive. If \(s \leq p_1\) and \(s + t \leq p_1\), we have too few stars, and we arbitrarily delete edges to increase the number of connected components to \(p_1\).

If \(s = 0\) we turn the isolated vertices arbitrarily into \(p_1\) stars. Otherwise, fix an arbitrary endpoint \(c\) of an isolated edge. Assume that \(s \leq p_1\): then we connect enough isolated vertices to \(c\) such that the number of stars is \(p_1\). Finally, if \(s > p_1\), we first dissolve \(s - p_1\) edges and continue as in the previous case. It is easy to check that the above solutions are optimal.

Solving \((G_2, k_2)\) with \(p_2\) components: By Lemma 1, the number of vertices of degree at least two is bounded by \(p_2 + 2k_2\). Every vertex of degree one in \(G_2\) is adjacent to a vertex of larger degree, thus it never makes sense to choose it as a center (its neighbor will always be cheaper). Hence, it suffices to enumerate every set \(S_2\) of \(p_2\) vertices of degree larger than one and test in linear time, as per Lemma 3, whether a solution inside the budget \(k_2\) is possible. Using Proposition 2 we can bound the running time by

\[
\frac{(p_2 + 2k_2)}{p_2} \cdot pk + O(n + m) = O(2^{\frac{\sqrt[3]{2pk}}{p}} \cdot pk + n + m) = O(2^{\sqrt[3]{pk}} + n + m).
\]

We are left with the cases where \(p_1\) or \(p_2\) are equal to zero: then the only possible solution is to remove all edges within \(G_1\) or \(G_2\), respectively, and connect all the resulting isolated vertices to an arbitrary center in the other instance. We either follow through with the operation, if within the respective budget, or deduce that the subinstance is not solvable. We conclude that the above algorithm will at some point guess the correct budgets for \(G_1\) and \(G_2\) and thus find a solution of size at most \(k\). The theorem follows.

Theorem 4. \(p\)-Starforest Editing is solvable in time \(O(2^{\sqrt[3]{pk}} + n + m)\).
A polynomial kernel for $t$-partite $p$-cluster editing

We show a simple $O(kt^p)$ kernel for the $t$-PARTITE $p$-CLUSTER EDITING problem – which will be the foundation of the subsequent subexponential algorithms – with a single rule, Rule 1, which can be exhaustively applied in time $O(n + m)$. The problem at hand is the following generalization of $p$-BICLUSTER EDITING:

\[
\text{Input: } \text{A graph } G = (V, E) \text{ and a non-negative integer } k. \\
\text{Question: } \text{Is there a set } F \subseteq [V]^2 \text{ of edges of size at most } k \text{ such that } G \triangle F \text{ is a disjoint union of exactly } p \text{ complete } t\text{-partite graphs?}
\]

For our rule, we say that a set $X \subseteq V(G)$ is a non-isolate twin class if for every $v$ and $v'$ in $X$, $N_G(v) = N_G(v') \neq \emptyset$. Note that this is by definition a false twin class, i.e., $vv' \notin E(G)$, or in other words, a non-isolate twin class is an independent set.

\begin{itemize}
  \item Rule 1. If there is a non-isolate twin class $X \subseteq V(G)$ of size at least $2k + 2$, then delete all but $2k + 1$ of them.
  \item Lemma 5. Rule 1 is sound and can be exhaustively applied in linear time.
\end{itemize}

The following result is an immediate consequence of the above rule and its correctness.

\begin{itemize}
  \item Theorem 6. The problem $t$-PARTITE $p$-CLUSTER EDITING admits a kernel where the number of vertices is bounded by $pt(2k + 1) + 2k = O(ptk)$.
\end{itemize}

**Proof.** We now count the number of vertices we can have in a yes instance after the rule above has been applied. We claim that if $G$ has more than $pt(2k + 1) + 2k$ vertices, it is a no instance. Let $(G, k)$ be the reduced instance according to Rule 1 and let $F$ be a solution of size at most $k$. At most $2k$ vertices can be touched by $F$, so the rest of the graph remains as it is, and is a disjoint union of at most $p$ complete $t$-partite graphs, each of which has at most $t$ non-isolate twin classes. It follows that in a yes instance, $G$ has at most $pt(2k + 1) + 2k = O(ptk)$ vertices.

5 Editing to bicluster graphs in subexponential time

In this section we lift the result of Section 3 by showing that the following problem is solvable in time $2^{O(p\sqrt{k}\log(pk))} + O(n + m)$. Observe that we lose the subexponential dependence on $p$, however, contrary to the result of Misra et al. [23], for fixed (or small, relative to $k$) $p$, this still is truly subexponential parameterized by $k$.

\[
\text{Input: } \text{A graph } G = (V, E) \text{ and a non-negative integer } k. \\
\text{Question: } \text{Is there a set } F \subseteq [V]^2 \text{ of edges of size at most } k \text{ such that } G \triangle F \text{ is a disjoint union of exactly } p \text{ complete bipartite graphs?}
\]

We denote a biclique of $G$ as $C = (A, B)$ and call the sets $A, B$ the sides of $C$. Before describing the algorithm for the general problem, we show that the following simpler problem is solvable in linear time using a greedy algorithm:
We now show that the problem with respect to cost minimizes where deg(C) is minimal but v is placed by a solution F in a biclique C’ = (A_j, B_j) with cost_i(v) > cost_j(v). Deleting from F all edges E_j between v and A_j and adding all edges E_i between v and A_i creates a new solution F’ = (F \ E_j) ∪ E_i. Since cost_j(v) > cost_i(v), we have that |F| > |F’| hence F is not optimal. This concludes the proof of the claim and the lemma.

\textbf{▶ Claim 8. An optimal solution will always have v ∈ B in a biclique C_i = (A_i, B_i) which minimizes cost_i(v).}

Let G = (A, B, E), A = \{A_1, \ldots, A_p\}, k be an instance of Annotated Bicluster Editing. Consider a vertex v ∈ B and define cost_i(v) to be the cost of placing v in B_i where C_i = (A_i, B_i) is the ith biclique of the solution, i.e.,

\[ \text{cost}_i(v) = |A_i| - 2 \deg_{A_i}(v) + \deg(v), \]

where \( \deg_{A_i}(v) = |N(v) \cap A_i| \). We prove the following claim which implies that we can greedily assign each vertex v ∈ B to a biclique of minimum cost.

\textbf{Lemma 7. Annotated Bicluster Editing is solvable in time } O(n + m). \textbf{ ▶}

\textbf{5.1 Subexponential time algorithm}

We now show that the problem p-Bicluster Editing is solvable in subexponential time by using the kernel from Theorem 6, guessing the annotated sets and applying the polynomial time algorithm for the annotated version of the problem. The important ingredient will be cheap vertices, by which we mean vertices that are known to receive very few edits. Intuitively, a cheap vertex is a “pin” that in subexponential time reveals for us its neighborhood in the solution, and thus can be leveraged to uncover parts of said solution.

We adopt the following notation and vocabulary. For an instance \( (G, k) \) of p-Bicluster Editing, and a solution F, we call \( H = G \triangle F \) the target graph. A vertex v is called cheap with respect to F if it receives at most \( \sqrt{k} \) edits. Observe that any set X of size larger than \( 2\sqrt{k} \) has a cheap vertex. We call such a set large and all sets that contain at most \( 2\sqrt{k} \) vertices small. We will further classify the bicliques in the target graph into two different classes: A biclique is small if its vertex set is small and large otherwise.

The algorithm now works as follows. Given an input instance \( (G, k) \) of p-Bicluster Editing, we try all combinations of \( p_s + p_t = p \), with the intended meaning that \( p_s \) is the number of small bicliques and \( p_t \) is the number of large bicliques in the target graph.

\textbf{Handling small bicliques.} We enumerate a set of \( p_s \) sets \( A_s \subseteq 2^V \) with the property that they are pairwise disjoint, and each of size at most \( 2\sqrt{k} \). Furthermore, \( G[\bigcup A_s] \) contains at most \( k \) edges. Delete all edges in \( A_s \) and reduce the budget accordingly. These are going to be all the left sides in small bicliques. This enumeration takes time

\[ (2\sqrt{k})^{p_s} \left( \frac{n}{2\sqrt{k}} \right)^{p_s} \leq (2\sqrt{k})^p \left( \frac{pk + k^2}{2\sqrt{k}} \right)^p = 2^{O(pv \tau \log(pk))}. \]
Handling large bicliques. The large bicliques have the following nice property. Since the vertex set of each such biclique is large, every biclique contains a cheap vertex. We guess a set $B_i$ of size $p_i$. For the biclique $C_i$, the vertex $v_i$ of $B_i$ will be a cheap vertex in $B_i$. Now, we enumerate all combinations of $p_i$ sets $\mathcal{N} = \{N_1, N_2, \ldots, N_{p_i}\}$, each of size at most $2\sqrt{k}$ which will be the edited neighborhood of each cheap vertex, and we conclude that $A_i = N_H(v_i) = N_G(v_i) \triangle N_i$. The enumeration of this asymptotically takes time
\[
\left(\frac{n}{p_i}\right) \cdot (2\sqrt{k})^{p_i} \leq \left(\frac{pk + k^2}{p}\right) \cdot (2\sqrt{k})^p \left(\frac{pk}{2\sqrt{k}}\right)^p = 2^{O\left(p\sqrt{k}\log(pk)\right)}.
\]

Putting things together. With the above two steps, in time $2^{O\left(p\sqrt{k}\log(pk)\right)}$ we obtained all the left sides $\mathcal{A}$, partitioned into $\mathcal{A}_s$ and $\mathcal{A}_t$. Using this information, we can in polynomial time compute whether the Annotated Bicluster Editing instance $(G, k, A)$ is a yes-instance. If so, we conclude yes, otherwise, we backtrack.

\textbf{Theorem 9.} \textit{$p$-Bicluster Editing is solvable in time} $2^{O\left(p\sqrt{k}\log(pk)\right)} + O(n+m)$.

\textbf{Proof.} We now show that the algorithm described above correctly decides \textit{$p$-Bicluster Editing} given an instance $(G, k)$. Suppose that the algorithm above concludes that $(G, k)$ is a yes instance. The only time it outputs yes, is when Annotated Bicluster Editing for a given set $\mathcal{A}$ and a given budget $k'$ outputs yes. Since this budget is the leftover budget from making $A$ an independent set, it is clear that any Annotated Bicluster Editing solution of size at most $k'$ gives a yes instance for \textit{$p$-Bicluster Editing}.

Suppose now for the other direction that $(G, k)$ is a yes instance for \textit{$p$-Bicluster Editing} and let $F$ be a solution. Consider the left sides $A_1, \ldots, A_p$ of $G \triangle F$ with the restriction that the smaller of the two sides in $C_i$ is named $A_i$. First we observe that during our subexponential time enumeration of sets, all the $A_i$s that are of size at most $2\sqrt{k}$ will be enumerated in one of the branches where $p_i$ is set to the number of small bicliques. Furthermore, if $A_i$ is large, then both are large, and then, for each of the large bicliques, there is a branch where we selected exactly one cheap vertex for each of the largest sides. Given these cheap vertices, there is a branch where we guess exactly the edits affecting each of the cheap vertices, hence we can conclude that in some branch, we know the entire partition $\mathcal{A}$. From Lemma 7, we can conclude that the algorithm described above concludes correctly that we are dealing with a yes-instance.

\textbf{5.2 The $t$-partite case}

We can in fact obtain similar (we treat $t$ here as a constant so the results are up to some constant factors in the exponents) results for the more general case of \textit{$t$-Partite} \textit{$p$-Cluster Editing}. Again we need the polynomial kernel described in Theorem 6. The only difference now to the biclique case is that we define a cluster to be small if \textit{every side} is small. In this case, we can enumerate $\binom{n}{k}^p$ sets, which will form the small clusters.

In the other case a cluster $C = \{A_1, A_2, \ldots, A_t\}$ is divided into $A_1, A_2, \ldots, A_t$, small sides and $A_{t+1}, A_{t+2}, \ldots, A_t$ large sides. For this case, we guess all the small sides and for each of the large sides we guess a cheap vertex. Guessing the neighborhoods $N_{t+1}, N_{t+2}, \ldots, N_t$ for the cheap vertices $v_{t+1}, v_{t+2}, \ldots, v_t$ gives us complete information on $C$; To compute what $A_j$ is, if $j > t$, we simply take the intersection $\bigcap_{i < j, i \neq j} N_i$ and remove $\bigcup_{i < t} A_i$. We arrive at the following lemma whose proof is directly analogous to that of Theorem 9.

\textbf{Theorem 10.} \textit{The problem $t$-Partite $p$-Cluster Editing is solvable in subexponential time} $2^{O\left(p\sqrt{k}\log(pk)\right)} + O(n+m)$.
6 Lower bounds

We show that (a) STARFOREST EDITING is NP-hard and that we cannot expect a subexponential algorithm unless the ETH fails; and (b) that $p$-STARFOREST EDITING is $\mathcal{W}[1]$-hard parameterized only by $p$.

6.1 Starforest editing

In the following we describe a linear reduction from 3SAT to STARFOREST EDITING. Furthermore, the instance we reduce to has maximal degree three, thus not only showing that STARFOREST EDITING is NP-hard on graphs of bounded degree, but also not solvable in subexponential time on subcubic graphs.

\begin{itemize}
  \item \textbf{Theorem 11.} The problem STARFOREST EDITING is NP-complete and, assuming ETH, does not admit a subexponential parameterized algorithm when parameterized by the solution size $k$, i.e., it cannot be solved in time $2^{o(k)} \cdot \text{poly}(n)$, nor in exact exponential time $2^{o(n+m)}$, even when restricted to subcubic graphs.
\end{itemize}

To prove the theorem above we will reduce from 3SAT. But to obtain the result, it is crucial that in our reduction, both the parameter $k$, and the size of the instance $G$ are bounded in linearly in $n$ and $m$. Such results have been shown earlier, in particular by Komusiewicz and Uhlmann for CLUSTER EDITING [21] and Drange and Pilipczuk for TRIVIALLY PERFECT EDITING [13]. Thus we resort to similar reductions as used there, however, the reductions here are tweaked to work for the problem at hand. We also achieve lower bounds for subcubic graphs. See Figures 1a and 1b for figures of the gadgets.

\textbf{Variable gadget.} Let $\varphi$ be an input instance of 3SAT, and denote its variable set and clause set as $V(\varphi)$ and $C(\varphi)$, respectively. We construct for $x \in V(\varphi)$ a graph $G_x \cong C_{6p_x}$ where $p_x$ is the number of clauses in $\varphi$ which $x$ appears in. The vertices of $G_x$ are labeled, consecutively, $\top_i^x, \bot_i^x, A_i^x, B_i^x, C_i^x, D_i^x$ for $i \in [0, p_x - 1]$.
There are exactly three ways of deleting $G_x$ into a starforest using at most $k_x = 6p_x$ edges. Clearly a collection of $P_8$ is a starforest and is our target graph. We will define the $\top$-deletion for $G_x$ as the deletion set $S^\top_x = \{C^\top_x D^\top_x, \bot^\top_x A^\top_x \mid i \leq p_x - 1\}$ and the $\bot$-deletion for $G_x$ as the deletion set $S^\bot_x = \{A^\bot_x B^\bot_x, D^\bot_x \top^\bot_x \mid i \leq p_x - 1\}$, taking the $i + 1$ in the index of $\top^\bot_x$ modulo $p_x$. In other words, in the gadget $G_x$, we are keeping the edges

- $D^\top_x \bot^\top_x \bot^\bot_x, A^\bot_x B^\bot_x C^\bot_x$, when $x$ is set to true, and
- $\top^\top_x \bot^\bot_x A^\bot_x, B^\bot_x C^\bot_x D^\bot_x$, when $x$ is set to false.  

Observe that when $x$ is set to true, we will have paths on three vertices, where $\top^\top_x$ is the middle vertex, and if $x$ is set to false, we will have paths on three vertices with $\bot^\bot_x$ being the middle vertex. Later, we will see that if $x$ satisfies a clause $c$, the $i$th clause $x$ appears in, then either $\top^\top_x$ or $\bot^\bot_x$ will be the middle vertex of a claw, depending on whether $x$ appears positively or negatively in $c$.

Observation 12. In an optimal edge edit of a cycle of length divisible by 6, no edge is added and exactly every third consecutive edge is deleted.

Clause gadget. A clause gadget simply consists of one vertex, i.e., for a clause $c \in C(\varphi)$, we construct the vertex $v_c$. This vertex will be connected to $G_x$, $G_y$, and $G_z$, for $x, y, z$ being its variables, in appropriate places, depending on whether or not the variable occurs negated in $c$. In fact, it will be connected to $\top^\top_x$ if $c$ is the $i$th clause $x$ appears in, and $x$ appears positively in $c$, and it is connected to $\bot^\bot_x$ if $c$ is the $i$th clause $x$ appears in, and $x$ appears negatively in $c$.

Let $k_x = 2|C| + 2 \sum p_x = 2|C| + 3 \cdot 2|C| = 8|C|$ be the budget for a formula $\varphi$. We now observe that the budget is tight.

Lemma 13. The graph $G_\varphi$ has no starforest editing set of size less than $k_\varphi$, and if the editing set has size $k_\varphi$, it contains only deletions.

We now continue to the main lemma, from which Theorem 11 follows.

Lemma 14. A 3Sat instance $\varphi$ is satisfiable if and only if $(G_\varphi, k_\varphi)$ is a yes instance for Starforest Editing.

Observing that the maximum degree of $G_\varphi$ is three – the clause vertices have exactly degree three, and the variable gadgets are cycles with some vertices connected to at most one clause vertex – this concludes the proof of Theorem 11. From the discussions above, the following result is an immediate consequence:

Corollary 15. The problem Starforest Deletion is NP-complete and not solvable in subexponential time under ETH, even on subcubic graphs.

Before going into parameterized lower bounds of Starforest Editing, we show that the exact same reduction above simultaneously proves similar results for the bicluster case. We note that the NP-hardness was shown by Amit [2], but their reduction suffers a quadratic blowup and is therefore not suitable for showing subexponential lower bounds.

Corollary 16. The problems Bicluster Editing and Bicluster Deletion are NP-complete and not solvable in subexponential time under ETH, even on subcubic graphs.
6.2 \(W[1]\)-hardness parameterized by \(p\)

In this section we show that the parameterization by \(k\) is necessary, even for the case of \(p\)-Starforest Editing. That is, we show that when we parameterize by \(p\) alone, the problem becomes \(W[1]\)-hard, and we can thus not expect any algorithms of the form \(f(p)\cdot\text{poly}(n)\) for any function \(f\) solving \(p\)-Starforest Editing. We reduce from the problem Multicolored Regular Independent Set. An instance of this problem consists of a regular graph colored into \(p\) color classes, each color class inducing a complete graph, and we are asked to find an independent set of size \(p\).

\[\triangleright\text{Proposition 17}\ (\cite{8}).\] The problem Multicolored Regular Independent Set is \(W[1]\)-complete.

Since each color class is complete, any independent set will be of size at most \(p\) and any independent set of size \(p\) is maximum. The reduction is direct; In fact we have that given a budget \(k = (n-p)(d-1)\), where \(d\) is the regularity degree, the following direct translation between the two problems holds:

\[\triangleright\text{Lemma 18.}\] Let \(G\) be a \(d\)-regular graph on \(n\) vertices, \(p \leq n\) and \(k = (n-p)(d-1)\). Then \((G,p)\) is a yes instance for Multicolored Regular Independent Set if and only if \((G,k)\) is a yes instance for \(p\)-Starforest Editing.

Combining Proposition 17 with Lemma 18 yields the following result:

\[\triangleright\text{Theorem 19.}\] \(p\)-Starforest Editing is \(W[1]\)-hard when parameterized by \(p\) alone.

7 Conclusion

We presented subexponential time algorithms for editing problems towards bicluster graphs, and more generally, \(t\)-partite cluster graphs when the number of connected components in the target graph is bounded. We supplemented these findings with lower bounds, showing that this dual parameterization is indeed necessary.

As an interesting open problem, we pose the question of whether \(t\)-Partite \(p\)-Cluster Editing can be solved in time \(2^{O(\sqrt{kp})}\) poly\((n)\), i.e., in subexponential time with respect to both parameters. It is known that Bicluster Editing admits a linear kernel, but when introducing the secondary parameter, we only obtain a kernel whose size is bounded by the product of both parameters; Recall that we got a \(tp(2k+1) + 2k\) kernel, which in the bicluster case is \(p(4k+2) + 2k\). Does Bicluster Editing admit a truly linear kernel, i.e., a kernel with \(O(p+k)\) vertices?

Finally, in many practical applications of biclustering problems, the input can often be considered bipartite. The proof of the NP-completeness and subexponential algorithm lower bounds is highly non-bipartite, hence a natural question is whether it is possible to get similar lower bounds for the problem Bipartite Bicluster Editing, the problem where you are given a bipartite graph and asked to respect the bipartition.

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