Simple Priced Timed Games are not That Simple

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Abstract
Priced timed games are two-player zero-sum games played on priced timed automata (whose locations and transitions are labeled by weights modeling the costs of spending time in a state and executing an action, respectively). The goals of the players are to minimise and maximise the cost to reach a target location, respectively. We consider priced timed games with one clock and arbitrary (positive and negative) weights and show that, for an important subclass of theirs (the so-called simple priced timed games), one can compute, in exponential time, the optimal values that the players can achieve, with their associated optimal strategies. As side results, we also show that one-clock priced timed games are determined and that we can use our result on simple priced timed games to solve the more general class of so-called reset-acyclic priced timed games (with arbitrary weights and one-clock).

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1 Introduction
The importance of models inspired from the field of game theory is nowadays well-established in theoretical computer science. They allow to describe and analyse the possible interactions of antagonistic agents (or players) as in the controller synthesis problem, for instance. This problem asks, given a model of the environment of a system, and of the possible actions of a controller, to compute a controller that constraints the environment to respect a given specification. Clearly, one can not, in general, assume that the two players (the environment and the controller) will collaborate, hence the need to find a controller strategy that enforces the specification whatever the environment does. This question thus reduces to computing a so-called winning strategy for the corresponding player in the game model.

In order to describe precisely the features of complex computer systems, several game models have been considered in the literature. In this work, we focus on the model of Priced Timed Games [17] (PTGs for short), which can be regarded as an extension (in several directions) of classical finite automata. First, like timed automata [2], PTGs have clocks, which are real-valued variables whose values evolve with time elapsing, and which can be

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play whose cost is as small as possible. Player Max has an antagonistic objective: he tries
to avoid the target locations, and, if not possible, to maximise the accumulated cost up to
the first visit of a target location. To reflect these objectives, we define the upper value
of the game as a mapping of the configurations of the PTG to the least cost that Min can
guarantee while reaching the target, whatever the choices of Max. Similarly, the lower value
Val returns the greatest cost that Max can ensure (letting the cost being +\infty in case the
target locations are not reached).

In this setting, the goal of Min is to reach a given set of target locations, following a
play whose cost is as small as possible. Player Max has an antagonistic objective: he tries
to avoid the target locations, and, if not possible, to maximise the accumulated cost up to
the first visit of a target location. To reflect these objectives, we define the upper value \Val of
the game as a mapping of the configurations of the PTG to the least cost that Min can
guarantee while reaching the target, whatever the choices of Max. Similarly, the lower value \Val returns the greatest cost that Max can ensure (letting the cost being +\infty in case the
target locations are not reached).

An example of PTG is given in Figure 1, where the locations of Min (respectively, Max)
are represented by circles (respectively, rectangles), and the integers next to the locations
are their price-rates, i.e., the cost of spending one time unit in the location. Moreover,
there is only one clock \ell in the game, which is never reset and all guards on transitions
are \ell \in [0, 1] (hence this guard is not displayed and transitions are only labeled by their
respective discrete cost): this is an example of simple priced timed game (we will define them
properly later). It is easy to check that Min can force reaching the target location \ell_f from all
configurations \( (\ell, \nu) \) of the game, where \ell is a location and \nu is a real valuation of the clock
in [0, 1]. Let us comment on the optimal strategies for both players. From a configuration
\( (\ell_4, \nu) \), with \nu \in [0, 1], Max better waits until the clock takes value 1, before taking the
transition to \ell_f (he is forced to move, by the rules of the game). Hence, Max’s optimal value
is 3(1 - \nu) - 7 = -3\nu - 4 from all configurations \( (\ell_4, \nu) \). Symmetrically, it is easy to check
that Min better waits as long as possible in \ell_7, hence his optimal value is -16(1 - \nu) from all
configurations \( (\ell_7, \nu) \). However, optimal value functions are not always that simple, see for
instance the lower value function of \ell_1 on the right of Figure 1, which is a piecewise affine
function. To understand why value functions can be piecewise affine, consider the sub-game
enclosed in the dotted rectangle in Figure 1, and consider the value that Min can guarantee
from a configuration of the form \( (\ell_3, \nu) \) in this sub-game. Clearly, Min must decide how long
he will spend in \ell_3 and whether he will go to \ell_4 or \ell_7. His optimal value from all \( (\ell_3, \nu) \) is thus
\inf_{0 \leq \ell_1 \leq 1} \min (4\ell + (-3(\nu + t) - 4), 4\ell + 6 - 16(1 - (\nu + t))) = \min(-3\nu - 4, 16\nu - 10). Since
16\nu - 10 \geq -3\nu - 4 \text{ if and only if } \nu \leq 6/19, the best choice of Min is to move instantaneously
to \ell_f if \nu \in [0, 6/19] and to move instantaneously to \ell_4 if \nu \in (6/19, 1], hence the value
function of \ell_3 (in the subgame) is a piecewise affine function with two pieces.
Related work. PTGs were independently investigated in [8] and [1]. For (non-necessarily turn-based) PTGs with non-negative prices, semi-algorithms are given to decide the value problem that is to say, whether the upper value of a location (the best cost that Min can guarantee in valuation 0), is below a given threshold. They have also shown that, under the strongly non-Zeno assumption on prices (asking the existence of $\kappa > 0$ such that every cycle in the underlying region graph has a cost at least $\kappa$), the proposed semi-algorithms always terminate. This assumption was justified in [11, 7] by showing that, without it, the existence problem, that is to decide whether Min has a strategy guaranteeing to reach a target location with a cost below a given threshold, is indeed undecidable for PTGs with non-negative prices and three or more clocks. This result was recently extended in [9] to show that the value problem is also undecidable for PTGs with non-negative prices and four or more clocks. In [5], the undecidability of the existence problem has also been shown for PTGs with arbitrary price-rates (without prices on transitions), and two or more clocks. On a positive side, the value problem was shown decidable by [10] for PTGs with one clock when the prices are non-negative: a 3-exponential time algorithm was first proposed, further refined in [18, 16] into an exponential time algorithm. The key point of those algorithms is to reduce the problem to the computation of optimal values in a restricted family of PTGs called Simple Priced Timed Games (SPTGs for short), where the underlying automata contain no guard, no reset, and the play is forced to stop after one time unit. More precisely, the PTG is decomposed into a sequence of SPTGs whose value functions are computed and re-assembled to yield the value function of the original PTG. Alternatively, and with radically different techniques, a pseudo-polynomial time algorithm to solve one-clock PTGs with arbitrary prices on transitions, and price-rates restricted to two values amongst $\{-d, 0, +d\}$ (with $d \in \mathbb{N}$) was given in [14].

Contributions. Following the decidability results sketched above, we consider PTGs with one clock. We extend those results by considering arbitrary (positive and negative) prices. Indeed, all previous works on PTGs with only one clock (except [14]) have considered non-negative weights only, and the status of the more general case with arbitrary weights has so far remained elusive. Yet, arbitrary weights are an important modeling feature. Consider, for instance, a system which can consume but also produce energy at different rates. In this case, energy consumption could be modeled as a positive price-rate, and production by a negative price-rate. We propose an exponential time algorithm to compute the value of one-clock SPTGs with arbitrary weights. While this result might sound limited due to the restricted class of simple PTGs we can handle, we recall that the previous works mentioned above [10, 18, 16] have demonstrated that solving SPTGs is a key result towards solving more general PTGs. Moreover, this algorithm is, as far as we know, the first to handle the full class of SPTGs with arbitrary weights, and we note that the solutions (either the algorithms or the proofs) known so far do not generalise to this case. Finally, as a side result, this algorithm allows us to solve the more general class of reset-acyclic one-clock PTGs that we introduce. Thus, although we can not (yet) solve the whole class of PTGs with arbitrary weights, our result may be seen as a potentially important milestone towards this goal. Due to lack of space most proofs and technical details may be found in a detailed version [12].

2 Priced timed games: syntax, semantics, and preliminary results

Notations and definitions. Let $x$ denote a positive real-valued variable called clock. A guard (or clock constraint) is an interval with endpoints in $\mathbb{N} \cup \{+\infty\}$. We often abbreviate
guards, for instance $x \leq 5$ instead of $[0, 5]$. Let $S \subseteq \text{Guard}(x)$ be a finite set of guards. We let $[S] = \bigcup_{I \in S} I$. Assuming $M_0 = 0 < M_1 < \cdots < M_k$ are all the endpoints of the intervals in $S$ (to which we add $0$), we let $\text{Reg}_S = \{(M_i, M_{i+1}) \mid 0 \leq i \leq k - 1\} \cup \{(M_i) \mid 0 \leq i \leq k\}$ be the set of regions of $S$. Observe that $\text{Reg}_S$ is also a set of guards.

We rely on the notion of cost function to formalise the notion of optimal value function sketched in the introduction. Formally, for a set of guards $S \subseteq \text{Guard}(x)$, a cost function over $S$ is a function $f : [\text{Reg}_S] \rightarrow \mathbb{R} \cup \{+\infty, -\infty\}$ such that over all regions $r \in \text{Reg}_S$, $f$ is either infinite or a continuous piecewise affine function, with a finite set of cutpoints (points where the first derivative is not defined) $\{(\kappa_1, \ldots, \kappa_p) \subseteq \mathbb{Q}\}$, and with $f(\kappa_i) \in \mathbb{Q}$ for all $1 \leq i \leq p$. In particular, if $f(r) = \{f(\nu) \mid \nu \in r\}$ contains $+\infty$ (respectively, $-\infty$) for some region $r$, then $f(r) = \{+\infty\}$ ($\{+\infty\}$). We denote by $\text{CF}_S$ the set of all cost functions over $S$. In our algorithm to solve $\text{SPTG}$s, we will need to combine cost functions thanks to the $\triangleright$ operator. Let $f \in \text{CF}_S$ and $f' \in \text{CF}_{S'}$ be two costs functions on set of guards $S, S' \subseteq \text{Guard}(x)$, such that $[S] \cap [S']$ is a singleton. We let $f \triangleright f'$ be the cost function in $\text{CF}_{S \cup S'}$ such that $(f \triangleright f')(\nu) = f(\nu)$ for all $\nu \in [\text{Reg}_S]$, and $(f \triangleright f')(\nu) = f'(\nu)$ for all $\nu \in [\text{Reg}_{S'}] \setminus [\text{Reg}_S]$. We consider an extended notion of one-clock priced timed games ($\text{PTG}$s for short) allowing for the use of urgent locations, where only a zero delay can be spent, and final cost functions which are associated with each final location and incur an extra cost to be paid when ending the game in this location. Formally, a $\text{PTG}$ $\mathcal{G}$ is a tuple $(L_{\text{Min}}, L_{\text{Max}}, L_f, L_u, \varphi, \Delta, \pi)$ where $L_{\text{Min}}$ (respectively, $L_{\text{Max}}$) is a finite set of locations for player $\text{Min}$ (respectively, $\text{Max}$), with $L_{\text{Min}} \cap L_{\text{Max}} = \emptyset$; $L_f$ is a finite set of final locations, and we let $L = L_{\text{Min}} \cup L_{\text{Max}} \cup L_f$ be the whole location space; $L_u \subseteq L \setminus L_f$ indicates urgent locations\(^1\); $\Delta \subseteq (L \setminus L_f) \times \text{Guard}(x) \times \{\top, \bot\} \times L$ is a finite set of transitions; $\varphi = (\varphi_\ell)_{\ell \in L_f}$ associates to each $\ell \in L_f$ its final cost function, that is an affine\(^2\) cost function $\varphi_\ell$ over $S_\ell = \{I \mid \exists \ell', R, \ell' : (\ell, I, R, \ell') \in \Delta\}$; $\pi : L \cup \Delta \rightarrow \mathbb{Z}$ mapping an integer price to each location – its price-rate – and transition.

Intuitively, a transition $(\ell, I, R, \ell')$ changes the current location from $\ell$ to $\ell'$ if the clock has value in $I$ and the clock is reset according to the Boolean $R$. We assume that, in all $\text{PTG}$s, the clock $x$ is bounded, i.e., there is $M \in \mathbb{N}$ such that for all guards $I \in S_\ell$, $I \subseteq [0, M]$\(^3\). We denote by $\text{Reg}_G$ the set of locations of $\mathcal{G}$. We further denote\(^4\) by $\Pi^0_G$, $\Pi^\infty_G$ and $\Pi^\text{fin}_G$ respectively the values $\max_{\ell \in \Delta} |\pi(\ell)|$, $\max_{\ell \in L} |\pi(\ell)|$ and $\max_{\nu \in [0, M]} \max_{\ell \in L} |\varphi_\ell(\nu)| = \max_{\ell \in L} \max(\{\varphi_\ell(0)\}, \varphi_\ell(M))$. That is, $\Pi^0_G$, $\Pi^\infty_G$ and $\Pi^\text{fin}_G$ are the largest absolute values of the location prices, transition prices and final cost functions.

Let $\mathcal{G} = (L_{\text{Min}}, L_{\text{Max}}, L_f, L_u, \varphi, \Delta, \pi)$ be a $\text{PTG}$. A configuration of $\mathcal{G}$ is a pair $s = (\ell, \nu) \in L \times \mathbb{R}^+$. We denote by $\text{Conf}_G$ the set of configurations of $\mathcal{G}$. Let $(\ell, \nu)$ and $(\ell', \nu')$ be two configurations. Let $\delta = (\ell, I, R, \ell') \in \Delta$ be a transition of $\mathcal{G}$ and $\ell \in \mathbb{R}^+$ be a delay. Then, there is a $(t, \delta)$-transition from $(\ell, \nu)$ to $(\ell', \nu')$ with cost $c$, denoted by $(\ell, \nu) \xrightarrow{t,\delta \in \mathcal{G}} (\ell', \nu')$, if

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\(^1\) Here we differ from [10] where $L_u \subseteq L_{\text{Max}}$.

\(^2\) The affine restriction on final cost function is to simplify our further arguments, though we do believe that all of our results could be adapted to cope with general cost functions.

\(^3\) Observe that this last restriction is not without loss of generality in the case of $\text{PTG}$s. While all timed automata $\mathcal{A}$ can be turned into an equivalent (with respect to reachability properties) $\mathcal{A}'$ whose clocks are bounded [4], this technique cannot be applied to $\text{PTG}$s, in particular with arbitrary prices.

\(^4\) Throughout the paper, we often drop the $\mathcal{G}$ in the subscript of several notations when the game is clear from the context.
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(i) \( \ell \in L_u \) implies \( t = 0 \);
(ii) \( \nu + t \in I \);
(iii) \( R = \top \) implies \( \nu' = 0 \);
(iv) \( R = \bot \) implies \( \nu' = \nu + t \);
(v) \( c = \pi(\delta) + t \times \pi(\ell) \).

Observe that the cost of \((t, \delta)\) takes into account the price-rate of \(\ell\), the delay spent in \(\ell\), and the price of \(\delta\). We assume that the game has no deadlock: for all \(s \in \text{Conf}_G\), there are \((t, \delta, c)\) and \(s' \in \text{Conf}_G\) such that \(s \xrightarrow{t,\delta,c} s'\). Finally, we write \(s \xrightarrow{t,\delta,c} s'\) whenever there are \(t, \delta\) such that \(s \xrightarrow{t,\delta,c} s'\).

A strategy for player \text{Min} is a function \(\sigma_{\text{Min}}\) mapping every finite play ending in location of \text{Min} to a pair \((\ell, \delta) \in \mathbb{R}^+ \times \Delta\), indicating what \text{Min} should play. We also request that the strategy proposes only valid pairs \((\ell, \delta)\), i.e., that for all runs \(\rho\) ending in \((\ell, \nu)\), \(\sigma_{\text{Min}}(\rho) = (t, (\ell, I, R, \ell'))\) implies that \(\nu + t \in I\). Strategies \(\sigma_{\text{Max}}\) of player \text{Max} are defined accordingly. We let \(\text{Strat}_{\text{Min}}(G)\) and \(\text{Strat}_{\text{Max}}(G)\) be the sets of strategies of \text{Min} and \text{Max}, respectively. A pair of strategies \((\sigma_{\text{Min}}, \sigma_{\text{Max}})\) in \(\text{Strat}_{\text{Min}}(G) \times \text{Strat}_{\text{Max}}(G)\) is called a profile of strategies. Together with an initial configuration \(s_0 = (\ell_0, \nu_0)\), it defines a unique play \(\text{Play}(s_0, \sigma_{\text{Min}}, \sigma_{\text{Max}}) = s_0 \xrightarrow{c_0} s_1 \xrightarrow{c_1} s_2 ... s_k \xrightarrow{c_k} ...\) where for all \(j \geq 0\), \(s_{j+1}\) is the unique configuration such that \(s_j \xrightarrow{c_{j+1}} s_{j+1}\) with \(c_{j+1} = \sigma_{\text{Min}}(s_j) = \sigma_{\text{Max}}(s_j)\) if \(t_j \in L_{\text{Min}}\); and \((t_j, \delta_j) = (t_j, \delta_j) = \sigma_{\text{Max}}(s_j) = \sigma_{\text{Min}}(s_j)\) if \(t_j \in L_{\text{Max}}\). We let \(\text{Play}(\sigma_{\text{Min}})\) (respectively, \(\text{Play}(s_0, \sigma_{\text{Min}})\)) be the set of plays that conform with \(\sigma_{\text{Min}}\) (and start in \(s_0\)).

As sketched in the introduction, we consider optimal reachability-cost games on \(PTGs\), where the aim of player \text{Min} is to reach a location of \(L_f\) while minimising the cost. To formalise this objective, we let the value of a strategy \(\sigma_{\text{Min}}\) for \text{Min} be the function \(\text{Val}_{G}^{\text{Min}}: \text{Conf}_G \rightarrow \mathbb{R}^+\) such that for all \(s \in \text{Conf}_G\): \(\text{Val}_{G}^{\text{Min}}(s) = \sup_{\sigma_{\text{Max}} \in \text{Strat}_{\text{Max}}} \text{Cost}(\text{Play}(s, \sigma_{\text{Min}}, \sigma_{\text{Max}}))\). Intuitively, \(\text{Val}_{G}^{\text{Min}}(s)\) is the largest value that \text{Max} can achieve when playing against strategy \(\sigma_{\text{Min}}\) of \text{Min} (it is thus a worst case from the point of view of \text{Min}). Symmetrically, for \(\sigma_{\text{Max}} \in \text{Strat}_{\text{Max}}, \text{Val}_{G}^{\text{Max}}(s) = \inf_{\sigma_{\text{Min}} \in \text{Strat}_{\text{Min}}} \text{Cost}(\text{Play}(s, \sigma_{\text{Min}}, \sigma_{\text{Max}}))\), for all \(s \in \text{Conf}_G\). Then, the upper and lower values of \(G\) are respectively the functions \(\overline{\text{Val}}_G: \text{Conf}_G \rightarrow \mathbb{R}\) and \(\underline{\text{Val}}_G: \text{Conf}_G \rightarrow \mathbb{R}\) where, for all \(s \in \text{Conf}_G\): \(\overline{\text{Val}}_G(s) = \inf_{\sigma_{\text{Min}} \in \text{Strat}_{\text{Min}}} \text{Val}_{G}^{\text{Min}}(s)\) and \(\underline{\text{Val}}_G(s) = \sup_{\sigma_{\text{Max}} \in \text{Strat}_{\text{Max}}} \text{Val}_{G}^{\text{Max}}(s)\). We say that a game is determined if the lower and the upper values match for every configuration \(s\), and in this case, we say that the optimal value \(\overline{\text{Val}}_G\) of the game \(G\) exists, defined by \(\overline{\text{Val}}_G = \overline{\text{Val}}_G = \underline{\text{Val}}_G\). A strategy \(\sigma_{\text{Min}}\) of \text{Min} is optimal (respectively, \(\varepsilon\)-optimal) if \(\overline{\text{Val}}_G^{\text{Min}} = \overline{\text{Val}}_G, (\overline{\text{Val}}_G^{\text{Min}} \leq \overline{\text{Val}}_G + \varepsilon)\), i.e., \(\sigma_{\text{Min}}\) ensures that the cost of the plays will be at most \(\overline{\text{Val}}_G\) (\(\overline{\text{Val}}_G + \varepsilon\)). Symmetrically, a strategy \(\sigma_{\text{Max}}\) of \text{Max} is optimal (respectively, \(\varepsilon\)-optimal) if \(\overline{\text{Val}}_G^{\text{Max}} = \overline{\text{Val}}_G, (\overline{\text{Val}}_G^{\text{Max}} \geq \overline{\text{Val}}_G - \varepsilon)\).

Properties of the value. Let us now prove useful preliminary properties of the value function of \(PTGs\), that – as far as we know – had hitherto never been established. Using a general determinacy result by Gale and Stewart [15], we can show that \(PTGs\) (with one clock) are determined. Hence, the value function \(\text{Val}_G\) exists for all \(PTG\ G\). We can further show that, for all locations \(\ell\), \(\text{Val}_G(\ell)\) is a piecewise continuous function that might exhibit discontinuities only on the borders of the regions of \(\text{Reg}_G\) (where \(\text{Val}_G(\ell)\) is the function such that \(\text{Val}_G(\ell)(\nu) = \text{Val}_G(\ell, \nu)\) for all \(\nu \in \mathbb{R}^+\)).
The value functions for all locations, as well as a pair of optimal strategies (always exist if no values are infinite) can be computed in exponential time. Infinite, or continuous and piecewise-affine with at most an exponential number of cutpoints.

Switching

This paper is devoted mainly to proving the following theorem on SPTGs:

**Theorem 1.** For all (one-clock) PTGs $G$:

(i) $\text{Val}_G = \text{Val}_G$, i.e., PTGs are determined; and

(ii) for all $r \in \text{Reg}_G$, for all $\ell \in L$, $\text{Val}_G(\ell)$ is either infinite or continuous over $r$.

Simple priced timed games. As sketched in the introduction, our main contribution is to solve the special case of simple one-clock priced timed games with arbitrary costs. Formally, an $r$-SPTG, with $r \in \mathbb{Q}^+ \cap [0, 1]$, is a PTG $G = (L_{\text{Min}}, L_{\text{Max}}, L_I, L_u, \varphi, \Delta, \pi)$ such that for all transitions $(\ell, I, R, \ell') \in \Delta$, $I = [0, r]$ and $R = \perp$. Hence, transitions of $r$-SPTGs are henceforth denoted by $(\ell, \ell')$, dropping the guard and the reset. Then, an SPTG is a 1-SPTG.

This paper is devoted mainly to proving the following theorem on SPTGs:

**Theorem 2.** Let $G$ be an SPTG. Then, for all locations $\ell \in L$, the function $\text{Val}_G(\ell)$ is either infinite, or continuous and piecewise-affine with at most an exponential number of cutpoints. The value functions for all locations, as well as a pair of optimal strategies $(\sigma_{\text{Min}}, \sigma_{\text{Max}})$ (that always exist if no values are infinite) can be computed in exponential time.

Before sketching the proof of this theorem, we discuss a class of (simple) strategies that are sufficient to play optimally. Roughly speaking, Max has always a memoryless optimal strategy, while Min might need (finite) memory to play optimally – it is already the case in untimed quantitative reachability games with arbitrary weights [13]. Moreover, these strategies are finitely representable (recall that even a memoryless strategy depends on the current configuration and that there are infinitely many in our time setting).

Strategies of Max are formalised with the notion of finite positional strategies (FP-strategies): they are memoryless strategies $\sigma$ (i.e., for all finite plays $\rho_1 = \rho'_1 \xrightarrow{\ell_1} s$ and $\rho_2 = \rho'_2 \xrightarrow{\ell_2} s$ ending in the same configuration, we have $\sigma(\rho_1) = \sigma(\rho_2)$), such that for all locations $\ell$, there exists a finite sequence of rationals $0 \leq \nu_{1,\ell} < \nu_{2,\ell} < \cdots < \nu_{k,\ell} = 1$ and a finite sequence of transitions $\delta_1, \ldots, \delta_k \in \Delta$ such that

(i) for all $1 \leq i \leq k$, for all $\nu \in (\nu_{i-1,\ell}, \nu_i)$, either $\sigma(\ell, \nu) = (0, \delta_1)$, or $\sigma(q, \nu) = (\nu_j - \nu, \delta_i)$ (assuming $\nu_0 = \min(0, \nu_1)$); and

(ii) if $\nu_i > 0$, then $\sigma(\ell, 0) = (\nu_i, \delta_i)$.

We let $\text{pts}(\sigma)$ be the set of $\nu_i$ for all $\ell$ and $i$, and $\text{int}(\sigma)$ be the set of all successive intervals generated by $\text{pts}(\sigma)$. Finally, we let $|\sigma| = |\text{int}(\sigma)|$ be the size of $\sigma$. Intuitively, in an interval $(\nu_{i-1}, \nu_i)$, $\sigma$ always returns the same move: either to take immediately $\delta_i$ or to wait until the clock reaches the endpoint $\nu_i$ and then take $\delta_i$.

Min, however may require memory to play optimally. Informally, we will compute optimal switching strategies, as introduced in [13] (in the untimed setting). A switching strategy is described by a pair $(\sigma_{\text{Min}}^1, \sigma_{\text{Min}}^2)$ of FP-strategies and a switch threshold $K$, and consists in playing $\sigma_{\text{Min}}^1$ until the total accumulated cost of the discrete transitions is below $K$; and then to switch to strategy $\sigma_{\text{Min}}^2$. The role of $\sigma_{\text{Min}}^2$ is to ensure reaching a final location: it is thus a (classical) attractor strategy. The role of $\sigma_{\text{Min}}^1$, on the other hand, is to allow Min to decrease the cost low enough (possibly by forcing negative cycles) to secure a cost below $K$, and the computation of $\sigma_{\text{Min}}^1$ is thus the critical point in the computation of an optimal switching strategy. To characterise $\sigma_{\text{Min}}^1$, we introduce the notion of negative cycle strategy (NC-strategy). Formally, an NC-strategy $\sigma_{\text{Min}}$ of Min is an FP-strategy such that for all runs $\rho = (t_1, \nu) \xrightarrow{c_1} \cdots \xrightarrow{c_{k-1}} (t_k, \nu') \in \text{Play}(\sigma_{\text{Min}})$ with $t_1 = t_k$, and $\nu, \nu'$ in the same interval of $\text{int}(\sigma_{\text{Min}})$, the sum of prices of discrete transitions is at most $-1$, i.e., $\pi(t_1, \ell_2 + \cdots + \pi(t_{k-1}, t_k) \leq -1$. To characterise the fact that $\sigma_{\text{Min}}^1$ must allow Min to reach a cost which is small enough, without necessarily reaching a target state, we define the fake value of an NC-strategy $\sigma_{\text{Min}}$ from a configuration $s$ as $f_{\text{Min}}^s = \sup |\text{Cost}(\rho)|$.
\(\rho \in \text{Play}(s, \sigma_{\text{Min}}), \rho \) reaches a target, i.e., the value obtained when ignoring the \(\sigma_{\text{Min}}\)-induced plays that do not reach the target. Thus, clearly, \(\text{fake}_{G}^{\min}(s) \leq \text{Val}_{G}^{\min}(s)\). We say that an NC-strategy is \textit{fake-optimal} if its fake value, in every configuration, is equal to the optimal value of the configuration in the game. This is justified by the following result whose proof relies on the switching strategies described before:

\[\text{Lemma 3. If } \text{Val}_{G}(\ell, \nu) \neq +\infty, \text{ for all } \ell \text{ and } \nu, \text{ then for all NC-strategies } \sigma_{\text{Min}}, \text{ there is a strategy } \sigma'_{\text{Min}}, \text{ such that } \text{Val}_{G}^{\min}(s) \leq \text{fake}_{G}^{\min}(s) \text{ for all configurations } s. \text{ In particular, if } \sigma_{\text{Min}} \text{ is a fake-optimal NC-strategy, then } \sigma'_{\text{Min}} \text{ is an optimal (switching) strategy of the SPTG.} \]

Then, an SPTG is called \textit{finitely optimal if}:

\(\text{(i) } \text{Min has a fake-optimal NC-strategy; } \)
\(\text{(ii) Max has an optimal FP-strategy; and } \)
\(\text{(iii) Val}_{G}(\ell) \text{ is a cost function, for all locations } \ell. \)

The central point in establishing Theorem 2 will thus be to prove that \textit{all SPTGs are finitely optimal}, as this guarantees the existence of well-behaved optimal strategies and value functions. We will also show that they can be computed in exponential time. The proof is by induction on the number of urgent locations of the SPTG. In Section 3, we address the base case of SPTGs with urgent locations only (where no time can elapse). Since these SPTGs are very close to the \textit{untimed} min-cost reachability games of [13], we adapt the algorithm in this work and obtain the \texttt{solveInstant} function (Algorithm 1). This function can also compute \(\text{Val}_{G}(\ell, 1)\) for all \(\ell\) and all games \(G\) (even with non-urgent locations) since time can not elapse anymore when the clock has valuation 1. Next, using the continuity result of Theorem 1, we can detect locations \(\ell\) where \(\text{Val}_{G}(\ell, \nu) \in (+\infty, -\infty)\), for all \(\nu \in [0, 1]\), and remove them from the game. Finally, in Section 4 we handle SPTGs with non-urgent locations by refining the technique of [10, 18] (that work only on SPTGs with non-negative costs). Compared to [10, 18], our algorithm is simpler, being iterative, instead of recursive.

### 3 SPTGs with only urgent locations

Throughout this section, we consider an \(r\)-SPTG \(G = (L_{\text{Min}}, L_{\text{Max}}, L_{f}, L_{u}, \varphi, \Delta, \pi)\) where all locations are urgent, i.e., \(L = L_{\text{Min}} \cup L_{\text{Max}}\). We first explain briefly how we can compute the value function of the game for a \textit{fixed} clock valuation \(\nu \in [0, r]\) (more precisely, we can compute the vector \((\text{Val}_{G}(\ell, \nu))_{\ell \in L}\). Since no time can elapse, we can adapt the techniques developed in [13] to solve (untimed) \textit{min-cost reachability games}. The adaptation consists in taking into account the final cost functions. This yields the function \texttt{solveInstant} (Algorithm 1), that computes the vector \((\text{Val}_{G}(\ell, \nu))_{\ell \in L}\) for a fixed \(\nu\). The results of [13] also allow us to compute associated optimal strategies: when \(\text{Val}(\ell, \nu) \notin \{-\infty, +\infty\}\) the optimal strategy for \text{Max} is memoryless, and the optimal strategy for \text{Min} is a switching strategy \((\sigma_{1}^{\text{Min}}, \sigma_{2}^{\text{Min}})\) with a threshold \(K\) (as described in the previous section).

Now let us explain how we can reduce the computation of \(\text{Val}_{G}(\ell): \nu \in [0, r] \mapsto \text{Val}(\ell, \nu)\) (for all \(\ell\)) to a \textit{finite number of calls} to \texttt{solveInstant}. Let \(F_{G}\) be the set of affine functions over \([0, r]\) such that \(F_{G} = \{k + \varphi_{\ell} | \ell \in L_{f} \land k \in \mathbb{Z}\} \) where \(\mathcal{I} = [-(|L| - 1)\Pi^{r}, |L|\Pi^{r}] \cap \mathbb{Z}\). Observe that \(F_{G}\) has cardinality \(2|L|^{2}\Pi^{r}\), i.e., pseudo-polynomial in the size of \(G\). From [13], we conclude that the functions in \(F_{G}\) are sufficient to characterise \(\text{Val}_{G}\), in the following sense: for all \(\ell \in L\) and \(\nu \in [0, r]\) such that \(\text{Val}(\ell, \nu) \notin \{-\infty, +\infty\}\), there is \(f \in F_{G}\) with \(\text{Val}(\ell, \nu) = f(\nu)\). Using the continuity of \(\text{Val}_{G}\) (Theorem 1), we show that all the cutpoints of \(\text{Val}_{G}\) are intersections of functions from \(F_{G}\), i.e., belong to the set of \textit{possible cutpoints} \(\text{PossCP}_{G} = \{\nu \in [0, r] | \exists f_{1}, f_{2} \in F_{G} | f_{1} \neq f_{2} \land f_{1}(\nu) = f_{2}(\nu)\}\). Observe that \(\text{PossCP}_{G}\)
contains at most $|F_G|^2 = 4|L_f|^4|\Pi_r|^2$ points (also a pseudo-polynomial in the size of $G$) since all functions in $F_G$ are affine, and can thus intersect at most once with every other function. Moreover, $\text{PossCP}_G \subseteq \mathbb{Q}$, since all functions of $F_G$ take rational values in $0$ and $r \in \mathbb{Q}$. Thus, for all $\ell$, $\text{Val}_G(\ell)$ is a cost function (with cutpoints in $\text{PossCP}_G$ and pieces from $F_G$). Since $\text{Val}_G(\ell)$ is a piecewise affine function, we can characterise it completely by computing only its value on its cutpoints. Hence, we can reconstruct $\text{Val}_G(\ell)$ by calling $\text{solveInstant}$ on each rational valuation $r \in \text{PossCP}_G$. From the optimal strategies computed along $\text{solveInstant}$ [13], we can also reconstruct a fake-optimal NC-strategy for $\text{Min}$ and an optimal FP-strategy for $\text{Max}$, hence:

\begin{itemize}
  \item \textbf{Proposition 4.} Every $r$-SPTG $G$ with only urgent locations is finitely optimal. Moreover, for all locations $\ell$, the piecewise affine function $\text{Val}_G(\ell)$ has cutpoints in $\text{PossCP}_G$ of cardinality $4|L_f|^4|\Pi_r|^2$, pseudo-polynomial in the size of $G$.
\end{itemize}

\section{Solving general SPTGs}

In this section, we consider SPTGs with possibly non-urgent locations. We first prove that all such SPTGs are finitely optimal. Then, we introduce Algorithm 2 to compute optimal values and strategies of SPTGs. To the best of our knowledge, this is the first algorithm to solve SPTGs with arbitrary weights. Throughout the section, we fix an SPTG $G = (L_{\text{Min}}, L_{\text{Max}}, L_f, L_u, \varphi, \Delta, \pi)$ with possibly non-urgent locations. Before presenting our core contributions, let us explain how we can detect locations with infinite values. As already argued, we can compute $\text{Val}(\ell, 1)$ for all $\ell$ assuming all locations are urgent, since time can not elapse anymore when the clock has valuation $1$. This can be done with $\text{solveInstant}$. Then, by continuity, $\text{Val}(\ell, 1) = +\infty$ (respectively, $\text{Val}(\ell, 1) = -\infty$), for some $\ell$ if and only if $\text{Val}(\ell, \nu) = +\infty$ (respectively, $\text{Val}(\ell, \nu) = -\infty$) for all $\nu \in [0, 1]$. We remove from the game all locations with infinite value without changing the values of other locations (as justified in [13]). Thus, we henceforth assume that $\text{Val}(\ell, \nu) \in \mathbb{R}$ for all $(\ell, \nu)$.

\textbf{The $G_{L',r}$ construction.} To prove finite optimality of SPTGs and to establish correctness of our algorithm, we rely in both cases on a construction that consists in decomposing $G$ into a sequence of SPTGs with more urgent locations. Intuitively, a game with more urgent locations is easier to solve since it is closer to an untimed game (in particular, when all locations are urgent, we can apply the techniques of Section 3). More precisely, given a set $L'$ of non-urgent locations, and a valuation $r_0 \in [0, 1]$, we will define a (possibly infinite)
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sequence of valuations $1 = r_0 > r_1 > \cdots$ and a sequence $G_{L',r_0}, G_{L',r_1}, \ldots$ of SPTGs such that

(i) all locations of $G$ are also present in each $G_{L',r_i}$, except that the locations of $L'$ are now urgent; and

(ii) for all $i \geq 0$, the value function of $G_{L',r_i}$ is equal to $Val_G$ on the interval $[r_{i+1}, r_i]$. Hence, we can re-construct $Val_G$ by assembling well-chosen parts of the values functions of the $G_{L',r_i}$ (assuming $\inf_i r_i = 0$).

This basic result will be exploited in two directions. First, we prove by induction on the number of urgent locations that all SPTGs are finitely optimal, by re-constructing $Val_G$ (as well as optimal strategies) as a $\sqcup$-concatenation of the value functions of a finite sequence of SPTGs with one more urgent locations. The base case, with only urgent locations, is solved by Proposition 4. This construction suggests a recursive algorithm in the spirit of [10, 18] (for non-negative prices). Second, we show that this recursion can be avoided (see Algorithm 2). Instead of turning locations urgent one at a time, this algorithm makes them all urgent and computes directly the sequence of SPTGs with only urgent locations. Its proof of correctness relies on the finite optimality of SPTGs and, again, on our basic result linking the values functions of $G$ and games $G_{L',r_i}$.

Let us formalise these constructions. Let $G$ be an SPTG, let $r \in [0,1]$ be an endpoint, and let $x = (x_\ell)_{\ell \in L}$ be a vector of rational values. Then, $wait(G,r,x)$ is an $r$-SPTG in which both players may now decide, in all non-urgent locations $\ell$, to $wait$ until the clock takes value $r$, and then to stop the game, adding the cost $x_\ell$ to the current cost of the play. Formally, $wait(G,r,x) = (L_{\text{Min}}, L_{\text{Max}}, L_f', L_u, \varphi', T', \pi')$ is such that $L_f' = L_f \cup \{\ell | \ell \in L \setminus L_u\};$ for all $\ell' \in L_f$ and $\nu \in [0,r]$, $\varphi'_\nu(\ell') = \varphi_\nu(\ell)$, for all $\ell \in L \setminus L_u$, $\varphi'_\ell(\nu) = (r - \nu) \cdot \pi(\ell) + x_\ell; T' = T \cup \{(\ell,0,[r],\ldots,\ell') | \ell \in L \setminus L_u\};$ for all $\delta \in T'$, $\pi'(\delta) = \pi(\delta)$ if $\delta \in T$, and $\pi'(\delta) = 0$ otherwise. Then, we let $G_r = wait(G,r,\{(Val_G(\ell,\nu))_{\ell \in L}\}$, i.e., the game obtained thanks to $wait$ by letting $x$ be the value of $G$ in $r$. One can check that this first transformation does not alter the value of the game, for valuations before $r$: $Val_G(\ell,\nu) = Val_{G_r}(\ell,\nu)$ for all $\nu \leq r$.

Next, we make locations urgent. For a set $L' \subseteq L \setminus L_u$ of non-urgent locations, we let $G_{L',r}$ be the SPTG obtained from $G_r$ by making urgent every location $\ell$ of $L'$. Observe that, although all locations $\ell \in L'$ are now urgent in $G_{L',r}$, their clones $\ell'$ allow the players to $wait$ until $r$. When $L'$ is a singleton $\{\ell\}$, we write $G_{\ell,r}$ instead of $G_{\{\ell\},r}$. While the construction of $G_r$ does not change the value of the game, introducing urgent locations does. We can characterise an interval $[a, r]$ on which the value functions of $H = G_{L',r}$ and $H^+ = G_{L',r}L_\cup(\ell),r$ coincide, as stated by the next proposition. The interval $[a, r]$ depends on the slopes of the pieces of $Val_{H^+}$ as depicted in Figure 2: for each location $\ell$ of Min, the slopes of the pieces of $Val_{H^+}$ contained in $[a, r]$ should be $\leq -\pi(\ell)$ (and $\geq -\pi(\ell)$ when $\ell$ belongs to Max). It is proved by lifting optimal strategies of $H^+$ into $H$, and strongly relies on the determinacy result of Theorem 1:

\textbf{Proposition 5.} Let $0 < a < r < 1$, $L' \subseteq L \setminus L_u$ and $\ell \notin L' \cup L_u$ a non-urgent location of $\text{Min}$ (respectively, $\text{Max}$). Assume that $G_{L' \cup(\ell),r}$ is finitely optimal, and for all $a \leq \nu_1 < \nu_2 \leq r$

\[ \frac{Val_{G_{L' \cup(\ell),r}}(\ell, \nu_2) - Val_{G_{L' \cup(\ell),r}}(\ell, \nu_1)}{\nu_2 - \nu_1} \geq -\pi(\ell) \quad \text{(respectively,} \leq -\pi(\ell)) \quad . \]

Then, for all $\nu \in [a, r]$ and $\ell' \in L$, $Val_{G_{L' \cup(\ell),r}}(\ell', \nu) = Val_{G_{\ell',r}}(\ell', \nu)$. Furthermore, fake-optimal NC-strategies and optimal FP-strategies in $G_{L' \cup(\ell),r}$ are also fake-optimal and optimal over $[a, r]$ in $G_{\ell',r}$.
valprove that $\pi$ on the number of non-urgent locations and instantiate the previous results to the case where $SPTG \star \{ = \min L \}$ to obtain an algorithm to solve them.

different ways of choosing such that the (potentially infinite) sequence of games $\ell \in L' \leq r$ that all slopes on its right are smaller than or equal to $-\pi(\ell)$ in the graph of $Val_{L',r}(\ell, r)$. Dashed lines have slope $-\pi(\ell^*)$.

Given an SPTG $G$ and some finitely optimal $G_{L',r}$, we now characterise precisely the left endpoint of the maximal interval ending in $r$ where the value functions of $G$ and $G_{L',r}$ coincide, with the operator $\text{left}_{L'}: (0, 1] \rightarrow [0, 1]$ (or simply left, if $L'$ is clear) defined as:

$$\text{left}_{L'}(r) = \sup \{ r' \leq r \mid \forall \ell \in L \forall r \in [r', r] \text{ } Val_{L',r}(\ell, r) = Val_G(\ell, r) \}.$$ 

By continuity of the value (Theorem 1), this supremum exists and $Val_G(\ell, \text{left}_{L'}(r)) = Val_{G_{L',r}}(\ell, \text{left}_{L'}(r))$. Moreover, $Val_G(\ell)$ is a cost function on $[\text{left}_{L'}(r), r]$ since $G_{L',r}$ is finitely optimal. However, this definition of $\text{left}_{L'}(r)$ is semantical. Yet, building on the ideas of Proposition 5, we can effectively compute $\text{left}_{L'}(r)$, given $Val_{G_{L',r}}$. We claim that $\text{left}_{L'}(r)$ is the minimal valuation such that for all locations $\ell \in L' \cap L_{\text{Min}}$ (respectively, $\ell \in L' \cap L_{\text{Max}}$), the slopes of the affine sections of the cost function $Val_{G_{L',r}}(\ell)$ on $[\text{left}_{L'}(r), r]$ are at least (at most) $-\pi(\ell)$. Hence, $\text{left}_{L'}(r)$ can be obtained (see Figure 3), by inspecting iteratively, for all $\ell$ of $\text{Min}$ (respectively, $\text{Max}$), the slopes of $Val_{G_{L',r}}(\ell)$, by decreasing valuations, until we find a piece with a slope $> -\pi(\ell)$ (respectively, $< -\pi(\ell)$). This enumeration of the slopes is effective as $Val_{G_{L',r}}$ has finitely many pieces, by hypothesis. Moreover, this guarantees that $\text{left}_{L'}(r) < r$. Thus, one can reconstruct $Val_G$ on $[\inf r_i, r_0]$ from the value functions of the (potentially infinite) sequence of games $G_{L',r_0}, G_{L',r_1}, \ldots$ where $r_{i+1} = \text{left}_{L'}(r_i)$ for all $i$ such that $r_i > 0$, for all possible choices of non-urgent locations $L'$. Next, we will define two different ways of choosing $L'$: the former to prove finite optimality of all SPTGs, the latter to obtain an algorithm to solve them.

SPTGs are finitely optimal. To prove finite optimality of all SPTGs we reason by induction on the number of non-urgent locations and instantiate the previous results to the case where $L' = \{ \ell^* \}$ where $\ell^*$ is a non-urgent location of minimum price-rate (i.e., for all $\ell \in L$, $\pi(\ell^*) \leq \pi(\ell)$). Given $r_0 \in [0, 1]$, we let $r_0 > r_1 > \cdots$ be the decreasing sequence of valuations such that $r_i = \text{left}_{L'}(r_{i-1})$ for all $i > 0$. As explained before, we will build $Val_G$ on $[\inf r_i, r_i, r_0]$ from the value functions of games $G_{L',r_i}$. Assuming finite optimality of those games, this will prove that $G$ is finitely optimal under the condition that $r_0 > r_1 > \cdots$ eventually stops, i.e., $r_i = 0$ for some $i$. This property is given by the next lemma, which ensures that, for all $i$, the owner of $\ell^*$ has a strictly better strategy in configuration $(\ell^*, r_{i+1})$ than waiting until $r_i$ in location $\ell^*$. 

---

**Figure 2** The condition (1) (in the case $L' = \emptyset$ and $\ell \in L_{\text{Min}}$): graphically, it means that the slope between every two points of the plot in $[a, r]$ (represented with a thick line) is greater than or equal to $-\pi(\ell)$ (represented with dashed line).

**Figure 3** In this example $L' = \{ \ell^* \}$ and $\ell^* \in L_{\text{Min}}$, $\text{left}_{L'}(r)$ is the leftmost point such that all slopes on its right are smaller than or equal to $-\pi(\ell^*)$. Dashed lines have slope $-\pi(\ell^*)$. 
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Lemma 6. If $G_{r', r_i}$ is finitely optimal for all $i \geq 0$, then

1. if $\ell^{*} \in L_{\text{Min}}$ (respectively, $L_{\text{Max}}$), $\text{Val}_G(\ell^{*}, r_{i+1}) < \text{Val}_G(\ell^{*}, r_i) + (r_{i} - r_{i+1})\pi(\ell^{*})$ (respectively, $\text{Val}_G(\ell^{*}, r_{i+1}) > \text{Val}_G(\ell^{*}, r_i) + (r_{i} - r_{i+1})\pi(\ell^{*})$), for all $i$; and
2. there is $i \leq |F_G|^2 + 2$ such that $r_i = 0$.

By iterating this construction, we make all locations urgent iteratively, and obtain:

Proposition 7. Every SPTG $G$ is finitely optimal and for all locations $\ell$, $\text{Val}_G(\ell)$ has at most $O((|F_G|^2L^2)2^{L+2})$ cutpoints.

Proof. As announced, we show by induction on $n \geq 0$ that every $r$-SPTG $G$ with $n$ non-urgent locations is finitely optimal, and that the number of cutpoints of $\text{Val}_G(\ell)$ is at most $O((|F_G|^2(L_f + n^2))2^{n+2})$, which suffices to show the above bound, since $|L_f| + n^2 \leq |L|^2$.

The base case $n = 0$ is given by Proposition 4. Now, assume that $G$ has at least one non-urgent location, and consider $\ell^{*}$ one with minimum price. By induction hypothesis, all $r'$-SPTGs $G_{r', r}$ are finitely optimal for all $r' \in [0, r]$. Let $r_0 > r_1 > \cdots$ be the decreasing sequence defined by $r_0 = r$ and $r_i = \text{left}_{r_i}(r_{i-1})$ for all $i \geq 1$. By Lemma 6, there exists $j \leq |F_G|^2 + 2$ such that $r_j = 0$. Moreover, for all $0 < i \leq j$, $\text{Val}_{G_{r_i}} = \text{Val}_{G_{r, r_{i-1}}}$ on $[r_i, r_{i-1}]$ by definition of $r_i = \text{left}_{r}(r_{i-1})$, so that $\text{Val}_{G_{r_{i-1}}}(\ell)$ is a cost function on this interval, for all $\ell$, and the number of cutpoints on this interval is bounded by $O((|F_G|^2(L_f + n - 1)^2 + n^2))2^{n-1}2^{n-2}) = O((|F_G|^2(L_f + n^2))2^{n-1}2^{n-2})$ by induction hypothesis (notice that maximal transition prices are the same in $G$ and $G_{r_{i-1}}$, but that we add $n$ more final locations in $G_{r, r_{i-1}}$). Adding the cutpoint 1, summing over $i$ from 0 to $j \leq |F_G|^2 + 2$, and observing that $|F_G| \leq 2|F_G|^2|L_f|$, we bound the number of cutpoints of $\text{Val}_{G_{r_{i-1}}}(\ell)$ by $O((|F_G|^2(L_f + n^2))2^{n+2})$. Finally, we can reconstruct fake-optimal and optimal strategies in $G$ from the fake-optimal and optimal strategies of $G_{r, r_{i-1}}$.

Computing the value functions. The finite optimality of SPTGs allows us to compute the value functions. The proof of Proposition 7 suggests a recursive algorithm to do so: from an SPTG $G$ with minimal non-urgent location $\ell^{*}$, solve recursively $G_{\ell^{*}, 1}$, $G_{\ell^{*}, \text{left}(1)}$, $G_{\ell^{*}, \text{left}(\text{left}(1))}$, etc. handling the base case where all locations are urgent with Algorithm 1. While our results above show that this is correct and terminates, we propose instead to solve – without the need for recursion – the sequence of games $G_{L_n, 1}$, $G_{L_n, \text{left}(1)}$, etc., i.e., making all locations urgent at once. Again, the arguments given above prove that this scheme is correct, but the key argument of Lemma 6 that ensures termination cannot be applied in this case. Instead, we rely on the following lemma, stating that there will be at least one cutpoint of $\text{Val}_{G_{\ell}}$ in each interval $[\text{left}(r), r]$. Observe that this lemma relies on the fact that $G$ is finitely optimal, hence the need to first prove this fact independently with the sequence $G_{r, 1}$, $G_{r, \text{left}(1)}$, $G_{r, \text{left}(\text{left}(1))}$, etc. Termination then follows from the fact that $\text{Val}_{G_{\ell}}$ has finitely many cutpoints by finite optimality.

Lemma 8. Let $r_0 \in [0, 1]$ such that $G_{L', r_0}$ is finitely optimal. Suppose that $r_1 = \text{left}_{L'}(r_0)$, and let $r_2 = \text{left}_{L'}(r_1)$. There exists $r' \in [r_2, r_1]$ and $\ell \in L'$ such that

1. $\text{Val}_{G_{\ell}}(r')$ is affine on $[r', r_1]$, of slope equal to $-\pi(\ell)$, and
2. $\text{Val}_{G_{\ell}}(r_1) \neq \text{Val}_{G_{\ell}}(r_0) + \pi(\ell)(r_0 - r_1)$.

As a consequence, $\text{Val}_{G_{\ell}}(\ell)$ has a cutpoint in $[r_1, r_0]$.

Algorithm 2 implements these ideas. Each iteration of the while loop computes a new game in the sequence $G_{L_n, 1}$, $G_{L_n, \text{left}(1)}$,... described above; solves it thanks to solveinstant; and thus computes a new portion of $\text{Val}_{G_{\ell}}$ on an interval on the left of the current point $r \in [0, 1]$. More precisely, the vector $(\text{Val}_{G_{\ell}}(\ell, 1))_{\ell \in L}$ is first computed in line 1.
Then, the algorithm enters the while loop, and the game $G'$ obtained when reaching line 6 is $G'_{L_1 \setminus L_u, 1}$. Then, the algorithm enters the repeat loop to analyse this game. Instead of building the whole value function of $G'$, Algorithm 2 builds only the parts of $\text{Val} _{G'}$ that coincide with $\text{Val} _{G}$. It proceeds by enumerating the possible cutpoints $a$ of $\text{Val} _{G'}$, starting in $r$, by decreasing valuations (line 8), and computes the value of $\text{Val} _{G'}$ in each cutpoint thanks to $\text{solveInstant}$ (line 9), which yields a new piece of $\text{Val} _{G'}$. Then, the if in line 10 checks whether this new piece coincides with $\text{Val} _{G}$, using the condition given by Proposition 5. If it is the case, the piece of $\text{Val} _{G'}$ is added to $f_r$ (line 11); repeat is stopped otherwise. When exiting the repeat loop, variable $b$ has value left(1). Hence, at the next iteration of the while loop, $G' = G_{L_1 \setminus L_u, \text{left}(1)}$ when reaching line 6. By continuing this reasoning inductively, one concludes that the successive iterations of the while loop compute the sequence $G_{L_1 \setminus L_u, 1}, G_{L_1 \setminus L_u, \text{left}(1)}, \ldots$ as announced, and rebuilds $\text{Val} _{G}$ from them. Termination in exponential time is ensured by Lemma 8: each iteration of the while loop discovers at least one new cutpoint of $\text{Val} _{G}$, and there are at most exponentially many (note that a tighter bound on this number of cutpoints would entail a better complexity of our algorithm).

Example 9. Let us briefly sketch the execution of Algorithm 2 on the SPTG in Figure 1.

During the first iteration of the while loop, the algorithm computes the correct value functions until the cutpoint $\frac{1}{2}$: in the repeat loop, at first $a = 9/10$ but the slope in $\ell_1$ is smaller than the slope that would be granted by waiting, as depicted in Figure 1. Then, $a = 3/4$ where the algorithm gives a slope of value $-16$ in $\ell_2$ while the cost of this location of Max is $-14$. During the first iteration of the while loop, the inner repeat loop thus ends with $r = 3/4$. The next iterations of the while loop end with $r = \frac{1}{2}$ (because $\ell_1$ does not pass the test in line 10); $r = \frac{1}{4}$ (because of $\ell_2$) and finally with $r = 0$, giving us the value functions on the entire interval $[0, 1]$. 

<table>
<thead>
<tr>
<th>Algorithm 2: solve($G$)</th>
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<tbody>
<tr>
<td><strong>Input:</strong> SPTG $G = (L_{\text{Min}}, L_{\text{Max}}, L_f, L_u, \varphi, \Delta, \pi)$</td>
</tr>
<tr>
<td>1 $f = (f_\ell)<em>{\ell \in L} := \text{solveInstant}(G, 1)$ /* $f</em>\ell$: $[1] \rightarrow \mathbb{R}$ */</td>
</tr>
<tr>
<td>2 $r := 1$</td>
</tr>
<tr>
<td>3 while $0 &lt; r$ do /* Invariant: $f_\ell$: $[r, 1] \rightarrow \mathbb{R}$ */</td>
</tr>
<tr>
<td>4 \quad $G' := \text{wait}(G, r, f(r))$ /* $\text{r-SPTG}$ $G' = (L_{\text{Min}}, L_{\text{Max}}, L'_f, L'_u, \varphi', T', \pi')$ */</td>
</tr>
<tr>
<td>5 \quad $L'_u := L'_u \cup L$ /* every location is made urgent */</td>
</tr>
<tr>
<td>6 \quad $b := r$</td>
</tr>
<tr>
<td>7 \quad repeat /* Invariant: $f_\ell$: $[b, 1] \rightarrow \mathbb{R}$ */</td>
</tr>
<tr>
<td>8 \quad \quad $a := \max(\text{PossCP} _{G'} \cap [0, b))$</td>
</tr>
<tr>
<td>9 \quad \quad $x = (x_\ell)<em>{\ell \in L} := \text{solveInstant}(G', a)$ /* $x</em>\ell = \text{Val} _{G'}(\ell, a)$ */</td>
</tr>
<tr>
<td>10 \quad \quad if $\forall \ell \in L_{\text{Min}} \frac{f_\ell(b) - x_\ell}{b - a} \leq -\pi(\ell) \land \forall \ell \in L_{\text{Max}} \frac{f_\ell(b) - x_\ell}{b - a} \geq -\pi(\ell)$ then</td>
</tr>
<tr>
<td>11 \quad \quad \quad foreach $\ell \in L$ do $f_\ell := (\nu \in [a, b] \mapsto f_\ell(b) + (\nu - b) \frac{f_\ell(b) - x_\ell}{b - a}) \triangleright f_\ell$</td>
</tr>
<tr>
<td>12 \quad \quad \quad $b := a$; stop := false</td>
</tr>
<tr>
<td>13 \quad \quad else stop := true</td>
</tr>
<tr>
<td>14 \quad until $b = 0$ or stop</td>
</tr>
<tr>
<td>15 \quad $r := b$</td>
</tr>
<tr>
<td>16 return $f$</td>
</tr>
</tbody>
</table>
Thus, all plays according to \( \sigma \) that are minimal with respect to the top, turning them into a reset-acyclic PTG, we can solve them in exponential time, and show that their value functions are cost functions with at most exponentially many cutpoints, using our techniques. Unfortunately, the arguments to bound the number of resets do not hold for arbitrary costs, as shown by the PTG in Figure 4. We claim that \( \text{Val}(\ell_0) = 0 \); that \( \text{Min} \) has no optimal strategy, but a family of \( \varepsilon \)-optimal strategies \( \sigma_{\text{Min}}^* \) each with value \( \varepsilon \); and that each \( \sigma_{\text{Min}}^* \) requires memory whose size depends on \( \varepsilon \) and might yield a play visiting at least \( 1/\varepsilon \) times the reset between \( \ell_0 \) and \( \ell_1 \) (hence the number of resets can not be bounded). For all \( \varepsilon > 0 \), \( \sigma_{\text{Min}}^* \) consists in: waiting \( 1 - \varepsilon \) time units in \( \ell_0 \), then going to \( \ell_1 \) during the \( 1/\varepsilon \) first visits to \( \ell_0 \); and to go directly to \( \ell_f \) afterwards. Against \( \sigma_{\text{Min}}^* \), \( \text{Max} \) has two possible choices:

(i) either wait \( 0 \) time unit in \( \ell_1 \), wait \( \varepsilon \) time units in \( \ell_2 \), then reach \( \ell_f \); or
(ii) wait \( \varepsilon \) time unit in \( \ell_1 \) then force the cycle by going back to \( \ell_0 \) and wait for \( \text{Min} \)'s next move.

Thus, all plays according to \( \sigma_{\text{Min}}^* \) will visit a sequence of locations which is either of the form \( \ell_0(\ell_1\ell_0)^k\ell_1\ell_2\ell_f \), with \( 0 \leq k < \lceil 1/\varepsilon \rceil \); or of the form \( \ell_0(\ell_1\ell_0)^{\lceil 1/\varepsilon \rceil}\ell_f \). In the former case, the cost of the play will be \(-k\varepsilon + 0 + \varepsilon = -(k - 1)\varepsilon \leq \varepsilon \); in the latter, \(-\varepsilon([1/\varepsilon]) + 1 \leq 0 \). This shows that \( \text{Val}(\ell_0) = 0 \), but there is no optimal strategy as none of these strategies allow one to guarantee a cost of \( 0 \) (neither does the strategy that waits \( 1 \) time unit in \( \ell_0 \)).

However, we may apply the result on reset-acyclic PTGs to obtain:

\[\begin{align*}
\text{Theorem 10.} & \quad \text{The value functions of all one-clock PTGs are cost functions with at most exponentially many cutpoints.}
\end{align*}\]

\[\begin{proof}
\text{Let } G \text{ be a one-clock PTG. Let us replace all transitions } (\ell, g, \top, \ell') \text{ resetting the clock by } (\ell, g, \bot, \ell''), \text{ where } \ell'' \text{ is a new final location with } \varphi_{\ell''} = \text{Val}_G(\ell, 0) - \text{observe that } \text{Val}_G(\ell, 0) \text{ exists even if we can not compute it, so this transformation is well-defined. This yields a reset-acyclic PTG } G' \text{ such that } \text{Val}_{G'} = \text{Val}_G. \quad \end{proof}\]
References


Simple Priced Timed Games are not That Simple


