Report from Dagstuhl Seminar 15251

Sparse Modelling and Multi-exponential Analysis

Edited by

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Abstract

The research fields of harmonic analysis, approximation theory and computer algebra are seemingly different domains and are studied by seemingly separated research communities. However, all of these are connected to each other in many ways.

The connection between harmonic analysis and approximation theory is not accidental: several constructions among which wavelets and Fourier series, provide major insights into central problems in approximation theory. And the intimate connection between approximation theory and computer algebra exists even longer: polynomial interpolation is a long-studied and important problem in both symbolic and numeric computing, in the former to counter expression swell and in the latter to construct a simple data model.

A common underlying problem statement in many applications is that of determining the number of components, and for each component the value of the frequency, damping factor, amplitude and phase in a multi-exponential model. It occurs, for instance, in magnetic resonance and infrared spectroscopy, vibration analysis, seismic data analysis, electronic odour recognition, keystroke recognition, nuclear science, music signal processing, transient detection, motor fault diagnosis, electrophysiology, drug clearance monitoring and glucose tolerance testing, to name just a few.

The general technique of multi-exponential modeling is closely related to what is commonly known as the Pad/'e-Laplace method in approximation theory, and the technique of sparse interpolation in the field of computer algebra. The problem statement is also solved using a stochastic perturbation method in harmonic analysis. The problem of multi-exponential modeling is an inverse problem and therefore may be severely ill-posed, depending on the relative location of the frequencies and phases. Besides the reliability of the estimated parameters, the sparsity of the multi-exponential representation has become important. A representation is called sparse if it is a combination of only a few elements instead of all available generating elements. In sparse interpolation, the aim is to determine all the parameters from only a small amount of data samples, and with a complexity proportional to the number of terms in the representation.

Despite the close connections between these fields, there is a clear lack of communication in the scientific literature. The aim of this seminar is to bring researchers together from the three mentioned fields, with scientists from the varied application domains.

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1 Executive Summary

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The seminar brought together a number of researchers from polynomial interpolation, rational approximation and exponential analysis. The five day seminar centered around talks on Exponential Analysis (Day 2), Rational Approximation (Day 3) and Sparse Interpolation (Day 4). Applications were grouped on Day 1 in order to challenge the participants to discuss them further while related topics, mainly from Numerical Linear Algebra, were scheduled on Day 5.

The seminar itself started with a talk by Cuyt and Lee pointing out the considerable intersection of the three main themes, particularly as they all strongly overlap. In order to reach out to industry and connect the scientific research to the industrial needs, several participants working at industrial or real-life applications were invited for a presentation on the first day of the seminar. Then interaction about these topics would occur naturally throughout the week. We mention talks on Mobile sampling and sensor networks (Karlheinz Gröchenig), High-speed fluorescence lifetime imaging (David Li), The estimation of variable star periods (Daniel Lichtblau) and Imaging of structured arrays (Adhemar Bultheel).

In the past the three communities have mostly been following distinct paths of research and methods for computation. One of the highlights of the seminar was the realization of significant commonalities between the communities, something nicely pointed out in the talk of Roche. Prony's method takes center stage in this case, with its origin in 1795 being used to solve problems in exponential analysis. Prony's method appeared much later in the case of sparse polynomial interpolation with it's use by Blahout, Ben-or/Tiwari, and Giesbrecht/Labahn/Lee. Prony's method takes samples at multiples of a common point to determine the support and then makes use of separate Hankel methods for determining the individual coefficients or weights of the expression.

Numerical conditioning was a significant issue in many talks at the seminar. Beckermann and later Matos looked at numerical conditioning of Padé and rational approximation problems. In the former case Beckermann used the close relationship of Padé approximation to Prony's method to point out that the latter is, for the most part, a provably ill-conditioned problem. Still there were a number of approaches in both areas which attempted to address this conditioning issue. In the case of numerical computation of sparse polynomial interpolants, use is made of randomization to produce a better conditioning of the problem, primarily by separating the roots appearing in Prony's problem. A similar idea also appears in exponential analysis making use of the notion of stride length. In both cases the object is to spread out the roots which arise in Prony's method.

Rather than spreading out the roots one can instead spread out the coefficients of a sparse polynomial/exponential expression for improving numerical performance. Sparse interpolation does this by making use of the concept of diversification where the coefficients are spread out multiplying evaluation points using a random multiplier. A corresponding concept in exponential analysis is the use of shifted samples which is useful to address the problem of anti-aliasing.

Sparse interpolation also makes use of the concept of small primes sparse interpolation where exponents are reduced modulo a small prime. This recovers the exponents modulo the small prime. Doing this for a number of small primes (which can be done in parallel) allows one to reconstruct the true exponents. Of course one encounters the problem of collisions and inadvertant combinations of exponents. It was noticed at the seminar that exponential analysis has a corresponding technique which made use of sub-sampling. Collisions in this case correspond to aliasing. Again the different communities reported on their methods for overcoming such collisions/aliasing problems.

Researchers at the seminar also showed interest in multivariate Prony methods. In the case of sparse interpolation one encounters Zippel's method while in exponential analysis there are projection methods. In these cases one attempts recursive methods for estimating the support of the underlying multivariable expression. In the case of multivariate polynomial interpolation a second approach is to convert the multivariate problem into a univariate problem by making use of randomized Kronecker substitution. Exponential sums takes a similar approach using random lattice projection.

While there were strong commonalities between the main research areas, there were also some strong differences between the topics noted at the seminar. The most telling of these differences was the analysis of exponential sums which have polynomial, rather than constant coefficients. Such expressions appear naturally when modeling solutions of linear differential equations where the associated polynomial has repeated roots. Of course such problems have considerable numerical issues when the roots of the associated polynomial are close but not numerically equal. Sidi and Batenkov both pointed out the importance and difficulties when dealing with such problems.

The seminar was also important for illustrating the applications of the three research areas. In many cases the applications involved the need to only work with sums having a small sparse support rather than with the complete set of possible nonzero elements. Methods from the multivariate Prony problem were exploited by Collowald and Hubert to determine new cubature formulas invariant to some specific finite groups action. Markovsky showed the similarities to the exponential sum problems with the notion of low rank approximation of structured matrices. Software was also discussed. Numerical analysis of errors on experimental runs also brought up the issue of the type of random distributions used when simulating errors for the experiments.

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3 Overview of Talks

3.1 Multidimensional approximation of functions sampled at unequally spaced points by sums of exponentials

Fredrik Andersson (Lund University, SE)

Joint work of Andersson, Fredrik; Carlsson, Marcus; Wendt, Herwig

Let f be sampled at unequally spaced points $x_m \in \mathbb{R}^d$. We consider the problem of finding

$$g(x) = \sum_{k=1}^{K} c_k e^{2\pi i x \cdot \xi_k}, \tag{1}$$

so that $f(x_m) \approx g(x_m)$.

Let Ξ and Υ be subsets of equally spaced grids in $\mathbb{R}^d$ and let $\Omega = \Xi + \Upsilon = \{x + y : x \in \Xi, y \in \Upsilon\}$. Given a function g on Ω, consider the generalized multidimensional Hankel operator

$$\Gamma_g h(x) = \sum_{y \in \Upsilon} g(x+y)h(y), \quad x \in \Xi \tag{2}$$

By Kronecker;s theorem Γ_g has rank K if g is of the form (1). It also turns out that range of Γ_g is the space of all linear combinations of the functions $e^{2\pi i x \cdot \xi_k}$ on Ξ (See Lemma 4.2 of [1]). Let us represent the operator with the matrix $\mathbf{\Gamma}_g$.

Let $\mathbf{J}$ be an interpolation matrix that interpolates the values at the equally spaced point in Ω to unequally spaced points $\Psi = \{x_m\}_{m=1}^{M}$ in $\mathbb{R}^d$. In order to approximate the function f sampled at Ψ using K exponentials, we consider the optimization problem

$$\begin{aligned} &\underset{g}{\text{minimize}} \quad \sum_{m=1}^{M} |(\mathbf{J}g)_m - f(x_m)|^2 \\ &\text{subject to} \quad \operatorname{rank} \mathbf{\Gamma}_g = K \end{aligned} \tag{3}$$

where $(\mathbf{J}g)_m$ is the interpolated value of g at x_m.

We follow the setup in [4], and formulate (3) using the alternating direction method of multipliers [5]. The problem formulations is not convex, and there is hence no guarantee that the procedure will converge. However, it will typically give a matrix values of g such that the singular values σ_k of $\mathbf{\Gamma}_g$ are small if $k > K$. To estimate the (multidimensional) frequencies ξ_k associated with g we can then follow the approach gives in [2, 3] by solving systems of polynomial equations with coefficients taken from the singular vectors of $\mathbf{\Gamma}_g$ for $k > K$.

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3.2 Fourier-Sparsity Testing of Boolean Functions

Andrew Arnold (University of Waterloo, CA)

Joint work of Arnold, Andrew; Blais, Eric

We consider the problem of testing whether a function f has at most s nonzero Fourier coefficients, in which case we say f is s-sparse, given black-box access to f. We restrict our attention to perhaps the simplest case when f is a Boolean function acting on n bits. The analogous problem of learning the Fourier transform of an s-sparse Boolean function f was studied in previous work by Kushilevitz and Mansour [SIAM J. Computing, vol 22. (1992)], and Levin [J. Symb. Logic, vol 58. (1993)], the latter resulting in an $O(ns)$ Monte Carlo Sparse Fourier Transform (SFT) algorithm. Their work was the foundation for subsequent Sparse Fourier Transform algorithms in more general settings.

We say an algorithm is an ϵ-tester for sparse Boolean functions if it accepts if f is s-sparse and rejects if f is ϵ-far from s-sparse in terms of ℓ_2 norm, each with probability at least 2/3. Gopolan et al. [SIAM J. Computing, vol 40. (2011)], gave the first such tester with query-complexity polynomial in s and ϵ^{-1}.

We improve upon this result, present a sparsity tester with query-complexity $O(s \log s\epsilon^{-2} + \epsilon^{-4})$. Our tester relies on dimensionality-reduction techniques developed in the aforementioned previous work. Using these techniques, we reduce sparsity testing to the problem of homomorphism testing, which in turn may be solved via the Blum- Luby-Rubinfeld (BLR) linearity test [J. Comput. Syst. Sci. Int., vol 47. (1993)].

3.3 Numerical stability of the parameter estimation problem in sparse generalized exponential sums

Dmitry Batenkov (Technion – Haifa, IL)

Joint work of Batenkov, Dmitry; Yomdin, Yosef

We consider the parameter estimation problem in sparse generalized exponential sums of the form $m(k) = \sum_{j=1}^{s} e^{\imath x_j k} \sum_{\ell=0}^{d_j-1} a_{\ell,j} k^{\ell}$, when $m(k)$ are known only approximately.

We provide estimates on the component-wise condition numbers of the parameters x_j and $a_{\ell,j}$ above, and show that they can be accurately recovered by sampling at arithmetic progressions and polynomial homotopy methods.

We also discuss the application of these ideas to the problem of recovering a piecewise-smooth function (including the positions of the discontinuities) from its Fourier coefficients.

3.4 On the conditioning of the Padé map and related questions

Bernhard Beckermann (University of Lille, FR)

Joint work of Beckermann, Bernhard; Matos, Ana C.

Padé approximants play an important role in signal processing, sparse interpolation and exponential analysis. In this talk we will report about recent results from [1] concerning the forward and backward conditioning of the (real) Padé map, which sends a vector of Taylor coefficients onto the normalized vector of coefficients of the Padé numerator and denominator. In particular, we show that this map is not necessarily well conditioned for robust Padé approximants in the sense of Trefethen et al. [2].

We will also discuss the condition number of related non-linear maps.

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3.5 Sub-Nyquist spectral analysis

Matteo Briani (University of Antwerp, BE)

Joint work of Briani, Matteo; Cuyt, Annie; Lee, Wen-shin

In the field of sparse interpolation, parametric methods aim to retrieve the values of parameters of a linear combination of exponential functions from samples in a uniform time grid. These samples are collected following the Shannon-Nyquist theorem that dictates the minimum sampling rate that prevents the aliasing effect. In this paper we explain how it is possible, by means of undersampling, to use a coarser time grid and still be able to solve the aliasing effect. This reflects into a better conditioning of the problem and this behavior is explained by means of the ill-disposedness and a link to Padé approximation theory. Avoiding the aliasing effect, and using a coarser time grid, it is possible to perform several smaller independent analysis from the original set of samples. Joining these analysis together we obtain a method that brings higher accuracy to the existing parametric methods and introduces an extra parameter that can be use as validator. This is especially useful when the parametric method has to deal with signals consisting of close frequencies in a broad spectrum.

3.6 Order parameter for images of structured arrays

Adhemar Bultheel (KU Leuven, BE)

Joint work of Bultheel, Adhemar; Kaatz, Forrest

In nature (e.g. a bee honeycomb, muscle structure, crystals) or in engineering (e.g. micro lens arrays, nano pore/pillar arrays, solar cells) two-dimensional highly regular arrays are produced. Hexagonal, square or triangular grids are most common. Perfect symmetry of the grids does not exist in practical situations. Given the image of some array, one may analyse the properties of each of the individual nodes of the grid and compute parameters like their size, the location of their centers, perhaps their orientation, etc. These parameters could be combined to define some number indicating the deviation from the ideal grid.

We have tried to compute some order parameter from the Fourier transform of the image. For example a perfect hexagonal array has a Fourier spectrum that consists of a central peak, surrounded by six smaller peaks and their harmonics. This is a sparse exponential representation. The more the nodes in the image are dislocated from the perfect grid, the more noise will show up in the spectrum. Thus the amount of noise in the Fourier domain can be used as a measure for the disorder of the original grid.

Unfortunately, images may depend on many parameters (number of nodes, size of the nodes, shape of the nodes, orientation of the image,. . .) so that the Fourier technique only works in a rather restrictive number of situations and it is probably not useful in practical situations.

3.7 A moment matrix approach to symmetric cubatures

Mathieu Collowald (INRIA Sophia Antipolis – Méditerranée, FR)

Joint work of Collowald, Mathieu; Hubert, Evelyne

Quadrature and sparse interpolation are closely linked. The common key issue is the construction of a linear form

$$\Lambda : \mathbb{R}[x] \to \mathbb{R}, p \mapsto \sum_{j=1}^{r} a_j p(\xi_j)$$

from the knowledge of its restriction to $\mathbb{R}[x]_{\leq d}$. The unknowns are the weights a_j and the nodes ξ_j.

Cubature is a generalization of quadrature in higher dimension. An approach based on moment matrices was proposed in [2, 4]. We give a basis-free version in terms of the Hankel operator $\mathcal{H}$ associated to Λ. The existence of a cubature of degree d with r nodes boils down to conditions of ranks and positive semidefiniteness on $\mathcal{H}$. The nodes are then the solutions of a generalized eigenvalue problem.

Standard domains of integration are symmetric under the action of a finite group. It is natural to look for cubatures that respect this symmetry [1, 3]. Introducing adapted bases obtained from representation theory, the symmetry constraint allows to block diagonalize the Hankel operator $\mathcal{H}$. The size of the blocks is explicitly related to the orbit types of the nodes. From the computational point of view, we then deal with smaller-sized matrices both for securing the existence of the cubature and computing the nodes.

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3.8 Exponential analysis, Sparse interpolation and Padé approximation

Annie Cuyt (University of Antwerp, BE)

Joint work of Cuyt, Annie; Lee, Wen-shin

A common underlying problem statement in many applications is that of determining the number of components, and for each component the value of the frequency, damping factor, amplitude and phase in a multi-exponential model. It occurs, for instance, in magnetic resonance and infrared spectroscopy, vibration analysis, seismic data analysis, electronic odour recognition, keystroke recognition, nuclear science, music signal processing, transient detection, motor fault diagnosis, electrophysiology, drug clearance monitoring and glucose tolerance testing, to name just a few.

The general technique of multi-exponential modeling is closely related to what is commonly known as the Padé-Laplace method in approximation theory, and the technique of sparse interpolation in the field of computer algebra. The problem of multi-exponential modeling is an inverse problem and therefore may be severely ill-posed, depending on the relative location of the frequencies and phases. Besides the reliability of the estimated parameters, the sparsity of the multi-exponential representation has become important. A representation is called sparse if it is a combination of only a few elements instead of all available generating elements.

Despite the close connections between these fields, there is a clear lack of communication in the scientific literature. The aim of this seminar is to bring researchers together from the three mentioned fields, with scientists from the varied application domains.

3.9 Inverse Problems regularised by Sparsity

Pier Luigi Dragotti (Imperial College London, GB)

Modelling signals as sparse in a proper domain has proved useful in many signal processing tasks and, here, we show how sparsity can be used to solve inverse problems. We first recall that many inverse problems involve the reconstruction of continuous-time or continuous-space signals from discrete measurements and show how to relate the discrete measurements to some properties of the original signal (e.g., its Fourier transform at specific frequencies). Given this partial knowledge of the original signal, we then solve the inverse problem using sparsity. We focus on two specific problems which have important practical implications: localisation of diffusion sources from sensor measurements and reconstruction of planar domains from samples. First, we show how to reconstruct specific planar domains whose contours are determined using implicit functions, then we localise diffusion sources using a variation of the 'reciprocity gap' method which involves analytic test functions.

In both cases, the problem is solved by building a Prony's type system and by building structured matrices which, in the ideal settings, are simultaneously Toeplitz and rank deficient.

3.10 Mobile Sampling

Karlheinz Groechenig (Universität Wien, AT)

Joint work of Groechenig, Karlheinz; Romero, Jose Luis; Unnikrishnan, Jayakrishnan; Vetterli, Martin

We study the design of sampling trajectories for stable sampling and the reconstruction of bandlimited spatial fields using mobile sensors. The spectrum is assumed to be a symmetric convex set. As a performance metric we use the path density of the set of sampling trajectories that is defined as the total distance traveled by the moving sensors per unit spatial volume of the spatial region being monitored. Focussing first on parallel lines, we identify the set of parallel lines with minimal path density that contains a set of stable sampling for fields bandlimited to a known set. We then show that the problem becomes ill-posed when the optimization is performed over all trajectories by demonstrating a feasible trajectory set with arbitrarily low path density. However, the problem becomes well-posed if we explicitly specify the stability margins. We demonstrate this by obtaining a non-trivial lower bound on the path density of an arbitrary set of trajectories that contain a sampling set with explicitly specified stability bounds.

This is joint work with Jose Luis Romero, Univ. of Vienna, Jayakrishnan Unnikrishnan and Martin Vetterli from Ecole Polytechnique Fédérale de Lausanne (EPFL), Switzerland.

3.11 Error-Correcting Sparse Interpolation in Chebyshev Basis

Erich Kaltofen (North Carolina State University – Raleigh, US)

Joint work of Arnold, Andrew; Kaltofen, Erich L.

We present an error-correcting interpolation algorithm for a univariate black-box polynomial that has a sparse representation using Chebyshev polynomials as a term basis. Our algorithm assumes that an upper bound on the number of erroneous evaluations is given as input, and is a generalization of the algorithm by Lakshman and Saunders [SIAM J. Comput., vol. 24 (1995)] for interpolating sparse Chebyshev polynomials and the techniques in error-correcting sparse interpolation in the usual basis of consecutive powers of the variable due to Comer, Kaltofen, and Pernet [Proc. ISSAC 2012 and 2014]. We prove the correctness of our list-decoder-based algorithm with a Descartes-rule-of-signs-like property for sparse polynomials in Chebyshev basis. We also give a new algorithm that reduces the sparse interpolation in Chebyshev basis to that in power basis, thus making the many techniques for the sparse interpolation in power basis, for instance, supersparse (lacunary) interpolation over large finite fields, available to interpolation in Chebyshev basis. Furthermore, we can customize the randomized early termination algorithms from Kaltofen and Lee [J. Symb. Comput., vol. 36 (2003)] to our new approach.

3.12 A multivariate generalization of Prony's method

Stefan Kunis (Universität Osnabrück, DE), Ulrich von der Ohe (Universität Osnabrück, DE)

Joint work of Kunis, Stefan; Peter, Thomas; Römer, Tim; von der Ohe, Ulrich

A classical solution to the problem of parameter reconstruction for an exponential sum from a finite number of samples is given by Prony's method and the parameters are recovered as roots of a single univariate polynomial. We present a generalization of this method for exponential sums in an arbitrary finite number of variables and realize the parameters as common roots of several multivariate polynomials. Finally, the coefficients of the exponential sum arise as solutions to a linear system of equations.

In the first part of the talk we explain this approach and its algebraic properties. Provided we sample the exponential sum on an equidistant grid with a number of grid points in each coordinate direction bounded from below by the number parameters, unique reconstruction is guaranteed and this bound is shown to be sharp. In its simplest form, the reconstruction method consists of setting up a certain multilevel Toeplitz matrix of the samples, compute a basis of its kernel, and compute by some method of choice the set of common roots of the multivariate polynomials whose coefficients are given in the second step.

The second part of the talk is dedicated to numerical properties of our approach. Provided the number of grid points in each coordinate direction is bounded from below by some small constant divided by the separation distance of the parameters, the kernel of the above Toeplitz matrix can be stably computed. Moreover, we relate our approach to a recent semidefinite optimization formulation and show a couple of numerical experiments.

3.13 Behavior preserving extension of univariate and bivariate functions

David Levin (Tel Aviv University, IL)

Given function values on a domain D_0, possibly with noise, we examine the possibility of extending the function to a larger domain D, $D_0 \subset D$. In addition to smoothness at the boundary of D_0, the extension on $D \setminus D_0$ should also inherit behavioral trends of the function on D_0, such as growth and decay or even oscillations. The approach chosen here is based upon the framework of linear models, univariate or bivariate, with constant or varying coefficients.

3.14 Estimating Variable Star Periods from Unevenly Sampled Light Curve Data

Daniel Lichtblau (Wolfram Research – Champaign, US)

Joint work of Lichtblau, Daniel; Bryant, Jeffrey

A problem of interest in astronomy is determining the period of variable stars. Data collection is of necessity irregular (can only sample on clear nights) and noisy (from light pollution, atmospheric differences, etc.) We describe several ways in which period estimation can be performed on such data. Some are by now classical (from 60's–80's). One newer method will use Diophantine approximation.

3.15 High-speed fluorescence lifetime imaging (FLIM) instruments with fast hardware-friendly exponential analysis

David Li (The University of Strathclyde – Glasgow, GB)

Fast fluorescence lifetime imaging (FLIM) techniques are powerful tools for visualising protein interaction networks in living cells. FLIM has been used for cancer diagnosis, assessing drug efficacy in cancer therapy, understanding brain functions, etc. It can also sense physiological parameters such as Ca^{2+}, pH, O_2, temperature, viscosity, etc [1, 2]. For real- time applications, such as visualising neuronal activities or fast biophysical phenomena, it is desirable to apply innovative solid-state single-photon sensors [3, 4] and fast hardware embedded exponential analysis processors that can boost FLIM imaging [5, 6, 7]. But is it easy to have a hardware-friendly and high-efficient exponential analysis method for such applications?

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3.16 Structured low-rank approximation: Theory, algorithms, and applications

Ivan Markovsky (Free University of Brussels, BE)

URL http://slra.github.io/

Mathematical engineering continuously addresses new applications and solves new problems. The expansion of existing methods and applications makes it difficult to maintain a common theoretical framework. This talk shows the potential of the structured low-rank approximation setting to unify problems of data modeling from diverse application areas. An example treated in more details in the presentation is identification of a linear time-invariant system from observed trajectories of the system. We present an optimization method based on the variable projection principle. The method can deal with data with exact and missing (unknown) values. Problems with exact and missing values occur in data driven simulation and control – a new trend of model-free methods for system dynamical analysis and control.

3.17 Well conditioned rational functions approximants versus numerically co-prime polynomials

Ana C. Matos (Lille I University, FR)

Joint work of Matos, Ana C.; Beckermann, Bernd; Labahn, George

Rational functions like for instance Padé approximants play an important role in signal processing, sparse interpolation and exponential analysis. However, for a successful modeling with help of rational functions we want to make sure that there is no "similar" rational function being degenerate, i.e., having strictly smaller degree of both degrees of numerator and denominator. In particular, we prefer having rational functions without Froissart doublets (i.e., roots close to a pole), and without spurious poles (i.e., simple poles having small residuals).

In a recent paper [1] we showed that, provided that the Sylvester matrix built with the coefficients of the numerator and denominator is well-conditioned, the corresponding rational function has neither Froissart doublets nor spurious poles, and this is also true to sufficiently "close" rational functions. Here closeness is measured with two different metrics, in terms of the chordal distance of the values on the unit disk, or in terms of the distance of normalized coefficient vectors. The paper [1] also contained a comparison of these two metrics.

In [2] the authors introduced a measure for numerical coprimeness representing the minimal distance in the coefficient vector metric to a couple of degenerate polynomials (with a joint root allowing for canceling the fraction). They also showed that if the underlying Sylvester matrix is well-conditioned then a couple of polynomials is numerically coprime, the reciprocal being wrong.

The aim of this talk is to provide precise inequalities implying that also the larger class of rational functions with numerator and denominator being numerically coprime do not have neither Froissart doublets nor spurious poles.

This is a joint work with Bernd Beckermann and George Labahn.

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3.18 Using noise to detect faint signals: tricks with Padé approximants to Z-transforms

Luca Perotti (Texas Southern Unversity – Houston, US)

Joint work of Perotti, Luca; Bessis, Daniel; Regimbau, Tania

Adding small amounts of noise is a recognized method to stabilize Padé approximants; due to the nonlinearity of the Padé approximant, larger amounts of noise can be added to generate different time series and thus increase the statistics when detection probabilities are low. Recently we proposed a new technique based on the observation that the presence of even a weak signal significantly perturbs the universal properties of noise poles and zeros of the Padé approximants to the Z-transform of a data series. For data from two channels, combined in a single complex sequence, the different behavior of poles corresponding to complex noise and poles corresponding to coherent signal can also be used as a signature of the presence of a signal in heavy noise.

3.19 The generalized Prony method and its application I and II

Thomas Peter (Universität Osnabrück, DE), Gerlind Plonka (Universität Göttingen, DE)

Joint work of Peter, Thomas; Plonka, Gerlind

In this paper, we want to present a new very general approach for the reconstruction of sparse expansions of eigenfunctions of suitable linear operators. This approach provides us with a tool to unify all Prony-like methods on the one hand and to essentially generalize the Prony approach on the other hand. Thus it will establish a much broader field of applications of the method. In particular, we will show that all well-known Prony-like reconstruction methods for exponentials and polynomials known so far, can be seen as special cases of this approach. For example, the new insight into Prony-like methods enables us to derive new reconstruction algorithms for orthogonal polynomial expansions including Jacobi, Laguerre, and Hermite polynomials. The approach also applies to finite dimensional vector spaces, and we derive a deterministic reconstruction method for M-sparse vectors from only $2M$ measurements.

The talk will be split into two parts given by the two authors. In the **first part** we concentrate on deriving the new general approach to apply Prony's method to sparse expansions of eigenfunctions of linear operators and present the close connection to the well-known Prony-method.

The **second part** of the talk is especially dedicated to the advantages that the new more general insights give us for applications, as e.g. the use of different linear operators, the influence of the choice of functionals in case of noisy data and further numerical issues.

3.20 High dimensional approximation with trigonometric polynomials

Daniel Potts (TU Chemnitz, DE)

Joint work of Lutz Kämmerer, Lutz; Potts, Daniel; Volkmer, Toni

In this talk, we present algorithms for the approximation of multivariate functions by trigonometric polynomials. The approximation is based on sampling of multivariate functions on rank-1 lattices. To this end, we study the approximation of functions in periodic Sobolev spaces of dominating mixed smoothness. The proposed algorithm based mainly on a one-dimensional fast Fourier transform, and the arithmetic complexity of the algorithm depends only on the cardinality of the support of the trigonometric polynomial in the frequency domain. Therefore, we investigate trigonometric polynomials with frequencies supported on hyperbolic crosses and energy based hyperbolic crosses in more detail. Furthermore, we present algorithms where the support of the trigonometric polynomial is unknown.

References

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3.21 Using univariate algorithms to solve multivariate problems

Daniel Roche (U.S. Naval Academy – Annapolis, US)

A key feature of sparse interpolation algorithms is that their complexity should scale nicely (often linearly) with the number of variables in the unknown function. In fact, such algorithms can usually be decomposed into two parts: a "base case" univariate interpolation algorithm, and a method to reduce a given multivariate problem to one or more instances of a univariate one.

We will look at both historical and very recent approaches to the second part, the multivariate-to-univariate reduction. As has been frequently observed, many of these reductions are essentially orthogonal to the choice of underlying univariate algorithm, allowing for a wide range of hybrid approaches – not all of which are equally effective. We will examine the strengths and weaknesses of the various variable reduction strategies, and aim to give some insights into how they may be most effectively chosen and applied to new problems.

3.22 New approaches to Vector-Valued Rational Interpolation

Avraham Sidi (Technion – Haifa, IL)

We discuss some recent vector-valued rational interpolation procedures for vector-valued functions $F(z)$, $F : \mathbb{C} \to \mathbb{C}^N$. The interpolants produced by these procedures are all of the simple form

$$R_{p,k}(z) = \frac{U_{p,k}(z)}{V_{p,k}(z)} = \frac{\sum_{j=0}^{k} c_j\, \psi_{1,j}(z)\, G_{j+1,p}(z)}{\sum_{j=0}^{k} c_j\, \psi_{1,j}(z)}.$$

Here

$$\psi_{m,n}(z) = \prod_{r=m}^{n}(z - \xi_r), \quad n \geq m \geq 1; \quad \psi_{m,m-1}(z) = 1, \quad m \geq 1,$$

and $G_{m,n}(z)$ is the vector-valued polynomial of interpolation to $F(z)$ at the points ξ_i, $m \leq i \leq n$. The c_j are scalars, and they are determined in different ways by the different methods. As such, $R_{p,k}(z)$ interpolates $F(z)$ at ξ_i, $1 \leq i \leq p$, in the generalized Hermite sense.

We first discuss the algebraic properties of these interpolants, namely, their uniqueness, symmetry, and reproducing properties. We next discuss their use in approximating vector-valued meromorphic functions $F(z)$ in the complex plane.

Next, choosing the interpolation points appropriately, for $p \to \infty$ and k fixed, we derive de Montessus type convergence results for the interpolants and Koenig type convergence results for their poles and residues, which show that these interpolants, despite their simple appearance, are effective approximation tools. Especially interesting Koenig type results are obtained when the residues of $F(z)$ form a mutually orthogonal set. (Note that, for any type of rational interpolation problem, whether scalar or vector, the crucial test for deciding whether these are useful approximation tools is the existence of de Montessus and Koenig type theories.)

Finally, we consider the fully confluent case in which all interpolation points ξ_i coincide, and show the connection of the resulting interpolants with Krylov subspace methods.

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3.23 Efficient spectral estimations by MUSIC and related algorithms

Manfred Tasche (Universität Rostock, DE)

Joint work of Tasche, Manfred; Potts, Daniel; Volkmer, Toni

In the spectral estimation, one has to determine all parameters of a univariate resp. multivariate exponential sum h, if only finitely many (noisy) sampled data of h are given. A frequently used method for spectral estimation is the known MUSIC algorithm. Another popular methods are ESPRIT and the approximate Prony method (APM). We show that both MUSIC and APM are based on an orthogonal projection onto a so-called noise space, whereas ESPRIT uses an orthogonal projection onto the orthogonal complement of the noise space, the so-called signal space. These orthogonal projections can be constructed by (partial) singular value decomposition or QR decomposition of a rectangular Hankel matrix formed by the given sampled data of h.

In this talk, we describe that MUSIC and the related algorithms can be efficiently realized by sampling of h on special grids and using sparse fast Fourier transforms. Numerical experiments illustrate the procedure.

3.24 Towards simplified construction of subresultant matrix of multiple univariate polynomials

Akira Terui (University of Tsukuba, JP)

For three or more inputs of univariate polynomials with the real coefficients, we discuss a new construction of subresultant-like matrix which enable us to estimate the degree of the greatest common divisor (GCD) of the input polynomials from its rank. Such matrix is used in approximate GCD algorithms using optimization techniques with its degree is given in advance, especially for constructing constraints. Therefore, in these algorithms, it is important to construct the matrix in a more simplified form to make the overall algorithm more efficient. In this talk, for those purposes, we discuss towards a proposal of a new simplified construction of the matrix.

3.25 Hankel and Quasi-Hankel low-rank matrix completion: a convex relaxation

Konstantin Usevich (GRIPSA Lab – Saint Martin d'Hères, FR)

Joint work of Comon, Pierre; Usevich, Konstantin

The completion of matrices with missing values under the rank constraint is a non-convex optimization problem. A popular convex relaxation is based on minimization of the nuclear norm (sum of singular values) of the matrix. For this relaxation, an important question is when the two optimization problems lead to the same solution. This question was addressed in the literature mostly in the case of random positions of missing elements and random known elements. In this contribution, we analyze the case of structured matrices with fixed pattern of missing values, in particular, the case of Hankel and quasi-Hankel matrix completion, which appears as a subproblem in the computation of symmetric tensor canonical polyadic decomposition. Similar matrix completion problems appear in other applications, where a function can be approximated as a sum of complex exponentials (time series analysis, medical imaging). We extend existing results on completion of rank-one real Hankel matrices to completion of rank-r complex Hankel and quasi-Hankel matrices.

3.26 A deterministic sparse FFT algorithm for vectors with short support

Katrin Wannenwetsch (Universität Göttingen, DE)

Joint work of Wannenwetsch, Katrin; Plonka, Gerlind

It is well known that usual FFT algorithms for the discrete Fourier transform of a vector of length N require $\mathcal{O}(N \log N)$ arithmetical operations. Within the last years, there has been a great interest in sublinear time Fourier algorithms for sparse vectors.

In this talk we consider the special case where a signal $\mathbf{x} \in \mathbb{C}^N$ is known to vanish outside a support interval of length $m < N$. If the support length m of $\mathbf{x}$ or a good bound of it is a-priori known we derive a sublinear algorithm to compute $\mathbf{x}$ from its discrete Fourier transform $\widehat{\mathbf{x}} \in \mathbb{C}^N$. The proposed algorithm is deterministic and numerically stable.

In case of exact Fourier measurements we require only $\mathcal{O}(m \log m)$ arithmetical operations. For noisy measurements, we propose a stable $\mathcal{O}(m \log N)$ algorithm.

This is joint work with Gerlind Plonka.

3.27 Sparsity with Symbolic Polynomials

Stephen M. Watt (University of Western Ontario – London, CA)

We are interested in algorithms for "symbolic polynomials", that is multivariate polynomials generalized so the exponents are themeselves integer-valued multivariate polynomials, for example $x^{n^2/2-n/2} - y^m$. These objects may be used to model parameterized families of Laurent polynomials, with integer evaluations of the exponent variables giving specific Laurent polynomials. We have shown elsewhere that when polynomials with coefficents in a particular ring form a unique factorization domain, then so do the corresponding symbolic polynomials. We have given algorithms to compute their GCDs and factorizations in this case. Some of these algorithms rely on reduction to algorithms on sparse polynomials with many more variables, as will be explored in this talk. We additionally describe some new directions on Groebner bases for symbolic polynomials.

3.28 Reconstruction of Structured Functions from Sparse Fourier Data

Marius Wischerhoff (Universität Göttingen, DE)

In several scientific areas, such as radio astronomy, computed tomography, and magnetic resonance imaging, the reconstruction of structured functions from the knowledge of samples of their Fourier transform is a common problem. For the analysis of the examined object, it is important to reconstruct the underlying original signal as exactly as possible. We aim to uniquely recover structured functions from only a small number of Fourier samples. For this purpose, the Prony method, which is a deterministic method for the recovery of sparse trigonometric functions, is used as key instrument to derive algorithms for unique recovery by means of a smallest possible set of Fourier data.

We will give an overview of reconstruction results for different function classes, and we will consider two classes in detail.

First, we will examine linear combinations of N non-uniform shifts of a given bivariate function. Here, the unknown shift parameters and corresponding coefficients in the linear combination are recovered from sparse Fourier data. Unique recovery of the parameters is possible by using only $3N + 1$ Fourier samples on three lines through the origin. For this purpose, two predetermined lines are considered, while the third sampling line is chosen dependently on the results obtained by employing the samples from the first two lines. The presented approach can be generalized to the case of d-variate functions with $d > 2$.

Secondly, we turn to the reconstruction of polygonal shapes in the real plane. Here, a convex or non-convex polygonal domain D with N vertices is considered. It is shown that the vertices and their order can be reconstructed by taking $3N$ samples of the Fourier transform of the characteristic function of the polygonal domain D. Again, two predetermined sampling lines and an appropriately chosen third line are considered.

3.29 Accuracy of Spike-Train Fourier Reconstruction for Near-Colliding Nodes

Yosef Yomdin (Weizmann Institute – Rehovot, IL)

Joint work of Akinshin, Andrey; Batenkov, Dmitry; Yomdin, Yosef

We study reconstruction of “spike-train” signals F of the form

$$F(x) = \sum_{j=1}^{d} a_j \delta(x - x_j),$$

from their Fourier transform $\hat{F}(s)$, known for $s \in [-N, N]$, with an absolute error not exceeding $\epsilon > 0$. We concentrate on “near-collision” situations where the nodes x_j are known to form an l elements cluster of a size $h \ll 1$.

We show that in such situations the geometry of error amplification in the reconstruction is governed by the “Prony foliations” S_q whose leaves are defined by the Prony equations $\sum_{j=1}^{d} a_j x_j^k = \gamma_k$, with $k = 0, \ldots, q \leq l$, and with the arbitrary right-hand sides γ_k. On this base we give an “absolute” (i.e. valid with any reconstruction method) lower bound for the “worst case” reconstruction error of F from $\hat{F}$. We show that for the measurement error

$\epsilon > C_1(hN)^{2l-1}$, the inside configuration of the cluster nodes (in the worst case scenario) cannot be reconstructed at all.

Combining a proper rescaling with the "Decimation method" we show that for $\epsilon < C_2(hN)^{2l-1},\ C_2 \ll C_1$, an accurate (up to an error $\alpha h,\ \alpha \ll 1$) reconstruction of the cluster nodes is possible. The same algorithm reconstructs the non-cluster nodes and amplitudes with the full accuracy (of order $\frac{\epsilon}{N}$).

3.30 Semidefinite Representations of Noncompact Convex Sets

Lihong Zhi (MMRC – Beijing, CN)

Joint work of Guo, Feng; Wang, Chu; Zhi, Lihong

We consider the problem of the semidefinite representation of a class of non-compact basic semialgebraic sets. We introduce the conditions of pointedness and closedness at infinity of a semialgebraic set and show that under these conditions our modified hierarchies of nested theta bodies and Lasserre's relaxations converge to the closure of the convex hull of S. Moreover, if the PP-BDR property is satisfied, our theta body and Lasserre's relaxation are exact when the order is large enough; if the PP-BDR property does not hold, our hierarchies converge uniformly to the closure of the convex hull of S restricted to every fixed ball centered at the origin. We illustrate through a set of examples that the conditions of pointedness and closedness are essential to ensure the convergence. Finally, we provide some strategies to deal with cases where the conditions of pointedness and closedness are violated.

3.31 Trivariate polynomial approximation on Lissajous curves

Stefano de Marchi (University of Padova, IT)

Joint work of Bos Len; Vianello Marco; de March, Stefano
Main reference L. Bos, S. De Marchi, M. Vianello, "Trivariate polynomial approximation on Lissajous curves," arXiv:1502.04114v1 [math.NA], 2015.
URL http://arxiv.org/abs/1502.04114v1

We study Lissajous curves in the 3-cube, that generate algebraic cubature formulas on a special family of rank-1 Chebyshev lattices. These formulas are used to construct trivariate hyperinterpolation polynomials via a single 1-d Fast Chebyshev Transform (by the Chebfun package), and to compute discrete extremal sets of Fekete and Leja type for trivariate polynomial interpolation. Applications could arise in the framework of Lissajous sampling for MPI (Magnetic Particle Imaging).

4 Panel Discussions

At the closing meeting the organizers presented some slides summarizing the connections and similarities between the techniques used by the different communities gathered at the seminar. These slides are being complemented with reference material, an effort which is being continued after the seminar, and made available at the seminar's webpage or

https://www.uantwerpen.be/en/rg/cma/

as a shared document.

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