

# Grouping Time-Varying Data for Interactive Exploration

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## Abstract

We present algorithms and data structures that support the interactive analysis of the grouping structure of one-, two-, or higher-dimensional time-varying data while varying all defining parameters. Grouping structures characterise important patterns in the temporal evaluation of sets of time-varying data. We follow Buchin et al. [9] who define groups using three parameters: group-size, group-duration, and inter-entity distance. We give upper and lower bounds on the number of maximal groups over all parameter values, and show how to compute them efficiently. Furthermore, we describe data structures that can report changes in the set of maximal groups in an output-sensitive manner. Our results hold in  $\mathbb{R}^d$  for fixed  $d$ .

**1998 ACM Subject Classification** F.2.2 Analysis of Algorithms and Problem Complexity

**Keywords and phrases** trajectory, time series, moving entity, grouping, algorithm, data structure

**Digital Object Identifier** 10.4230/LIPIcs.SoCG.2016.61

## 1 Introduction

Time-varying phenomena are ubiquitous and hence the rapid increase in available tracking, recording, and storing technologies has led to an explosive growth in time-varying data. Such data comes in various forms: time-series (tracking a one-dimensional variable such as stock prices), two- or higher-dimensional trajectories (tracking moving objects such as animals, cars, or sport players), or ensembles (sets of model runs under varying initial conditions for one-dimensional variables such as temperature or rain fall), to name a few. Efficient tools to extract information from time-varying data are needed in a variety of applications, such as

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\* A. v. G. is supported by the Netherlands Organisation for Scientific Research (NWO) under project nr. 612.001.102.

† B. S. is supported by the Netherlands Organisation for Scientific Research (NWO) under project nr. 639.023.208.

‡ F. S. is supported by the Danish National Research Foundation under grant nr. DNRF84.



predicting traffic flow [18], understanding animal movement [7], coaching sports teams [13], or forecasting the weather [21]. Consequently, recent years have seen a flurry of algorithmic methods to analyse time-varying data which can, for example, identify important geographical locations from a set of trajectories [6, 15], determine good average representations [8], or find patterns, such as groups traveling together [9, 14, 17].

Most, if not all, of these algorithms use several parameters to model the applied problem at hand. The assumption is that the domain scientists, who are the users of the algorithm, know from years of experience which parameter values to use in their analysis. However, in many cases this assumption is not valid. Domain scientists do *not* always know the correct parameter settings and in fact need algorithmic support to interactively explore their data in, for example, a visual analytics system [3, 16].

We present algorithms and data structures that support the interactive analysis of the grouping structure of one-, two-, or higher-dimensional time-varying data while varying all defining parameters. Grouping structures (which track the formation and dissolution of groups) characterise important patterns in the temporal evaluation of sets of time-varying data. Classic examples are herds of animals or groups of people. But also for one-dimensional ensembles grouping is meaningful, for example, when detecting trends in weather models [20].

Buchin et al. [9] proposed a grouping structure for sets of moving entities. Their definition was later extended by Kostitsyna et al. [17] to geodesic distances. In this paper we use the same trajectory grouping structure. Our contributions are data structures and query algorithms that allow the parameters of the grouping structure to vary interactively and hence make it suitable for explorative analysis of sets of time-varying data. Below we first briefly review the definitions of Buchin et al. [9] and then state our contributions in detail.

**Trajectory grouping structure [9].** Let  $\mathcal{X}$  be a set of  $n$  entities moving in  $\mathbb{R}^d$  and let  $\mathbb{T}$  denote time. The entities trace trajectories in  $\mathbb{T} \times \mathbb{R}^d$ . We assume that each individual trajectory is piecewise linear and consists of at most  $\tau$  vertices. Two entities  $a$  and  $b$  are  $\varepsilon$ -connected if there is a chain of entities  $a = c_1, \dots, c_k = b$  such that for any pair of consecutive entities  $c_i$  and  $c_{i+1}$  the distance is at most  $\varepsilon$ . A set  $G$  is  $\varepsilon$ -connected, if for any pair  $a, b \in G$ , the entities are  $\varepsilon$ -connected. Given parameters  $m$ ,  $\varepsilon$ , and  $\delta$ , a set of entities  $G$  is an  $(m, \varepsilon, \delta)$ -group during time interval  $I$  if (and only if) (i)  $G$  has size at least  $m$ , (ii)  $\text{duration}(I) \geq \delta$ , and (iii)  $G$  is  $\varepsilon$ -connected at any time  $t \in I$ . An  $(m, \varepsilon, \delta)$ -group  $(G, I)$  is *maximal* if  $G$  is maximal in size or  $I$  is maximal in duration, that is, if there is no group  $H \supset G$  that is also  $\varepsilon$ -connected during  $I$ , and no interval  $J \supset I$  such that  $G$  is  $\varepsilon$ -connected during  $J$ .

**Results and Organization.** We want to create a data structure  $\mathcal{D}$  that represents the grouping structure, that is, its maximal groups, while allowing us to efficiently change the parameters. As we show below, the complexity of the problem is already fully apparent for one-dimensional time-varying data. Hence we restrict our description to  $\mathbb{R}^1$  in Sections 2–4 and then explain in Section 5 how to extend our results to higher dimensions.

If all three parameters  $m$ ,  $\varepsilon$ , and  $\delta$  can vary independently the question arises what constitutes a meaningful maximal group. Consider a maximal  $(m, \varepsilon, \delta)$ -group  $(G, I)$ . If we slightly increase  $\varepsilon$  to  $\varepsilon'$ , and consider a slightly longer time interval  $I' \supseteq I$  then  $(G, I')$  is a maximal  $(m, \varepsilon', \delta)$ -group. Intuitively, these groups  $(G, I)$  and  $(G, I')$  are the same. Thus, we are interested only in (maximal) groups that are “combinatorially different”. Note that the set of entities  $G$  may also be a maximal  $(m, \varepsilon, \delta)$ -group during a time interval  $J$  completely different from  $I$ , we also wish to consider  $(G, I)$  and  $(G, J)$  to be combinatorially different groups. In Section 2 we formally define when two (maximal)  $(m, \varepsilon, \delta)$ -groups are

(combinatorially) different. We prove that there are at most  $O(|\mathcal{A}|n^2)$  such groups, where  $\mathcal{A}$  is the arrangement of the trajectories in  $\mathbb{T} \times \mathbb{R}^1$ , and  $|\mathcal{A}|$  is its complexity. We also argue that the number of maximal groups may be as large as  $\Omega(\tau n^3)$ , even for fixed parameters  $m$ ,  $\varepsilon$ , and  $\delta$  and in  $\mathbb{R}^1$ . This significantly strengthens the lower bound of Buchin et al. [9].

In Section 3 we present an  $O(|\mathcal{A}|n^2 \log^2 n)$  time algorithm to compute all combinatorially different maximal groups. In Section 4 we describe a data structure that allows us to efficiently obtain all groups for a given set of parameter values. Furthermore we also describe data structures for the interactive exploration of the data. Specifically, given the set of maximal  $(m, \varepsilon, \delta)$ -groups we want to change one or more of the parameters and efficiently report only those maximal groups which either ceased to be a maximal group or became a maximal group. That is, our data structures can answer so-called *symmetric-difference queries* which are gaining in importance as part of interactive analysis systems [12]. As mentioned above, in Section 5 we extend our data structures and algorithms to  $\mathbb{R}^d$ , for fixed  $d$ .

## 2 Combinatorially Different Maximal Groups

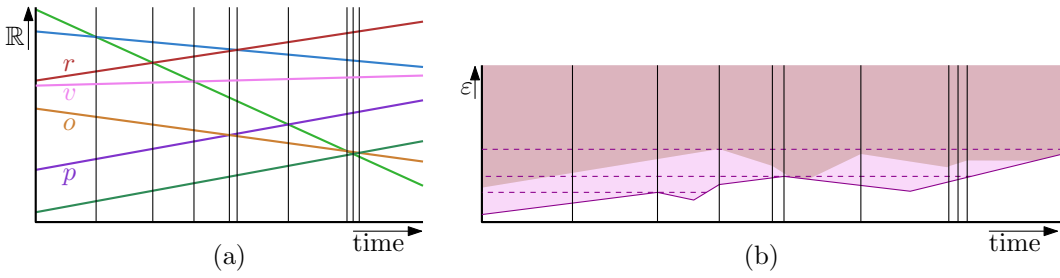
We consider entities moving in  $\mathbb{R}^1$ , hence the trajectories form an arrangement  $\mathcal{A}$  in  $\mathbb{T} \times \mathbb{R}^1$ . We assume that no three pairs of entities have equal distance at the same time. Consider the four-dimensional *parameter space*  $\mathbb{P}$  with axes time, size, distance, and duration. A set of entities  $G$  defines a region  $A_G$  in this space in which it is *alive*: a point  $p = (p_t, p_m, p_\varepsilon, p_\delta) = (t, m, \varepsilon, \delta)$  lies in  $A_G$  if and only if  $G$  is a  $(m, \varepsilon, \delta)$ -group at time  $t$ . We use these regions to define when groups are combinatorially different. First (Section 2.1) we fix  $m = 1$  and  $\delta = 0$  and define and count the number of combinatorially different maximal  $(1, \varepsilon, 0)$ -groups, over all choices of parameter  $\varepsilon$ . We then extend our results to include other values of  $\delta$  and  $m$  in Section 2.2.

### 2.1 The Number of Distinct Maximal $(1, \varepsilon, 0)$ -Groups, over all $\varepsilon$

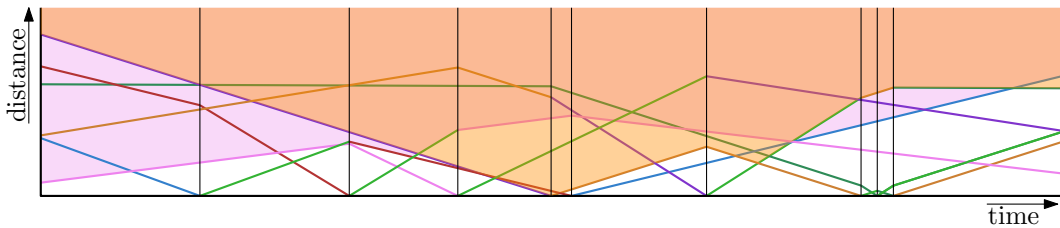
Consider the  $(t, \varepsilon)$ -plane in  $\mathbb{P}$  through  $\delta = 0$  and  $m = 1$ . The intersection of all regions  $A_G$  with this plane give us the points  $(t, \varepsilon)$  for which  $G$  is a  $(1, \varepsilon, 0)$ -group. Note that  $G$  is a  $(1, \varepsilon, 0)$ -group at time  $t$  if and only if the set  $G$  is  $\varepsilon$ -connected at time  $t$ . Hence the region  $A_G$ , restricted to this plane, corresponds to the set of points  $(t, \varepsilon)$  for which  $G$  is  $\varepsilon$ -connected.  $A_G$ , restricted to this plane, is simply connected. Furthermore, as the distance between any pair of entities moving in  $\mathbb{R}^1$  varies linearly,  $A_G$  is bounded from below by a  $t$ -monotone polyline  $f_G$ . The region is unbounded from above: if  $G$  is  $\varepsilon$ -connected (at time  $t$ ) for some value  $\varepsilon$ , then it is also  $\varepsilon'$ -connected for any  $\varepsilon' \geq \varepsilon$  (see Figure 1). Every maximal length segment in the intersection between (the restricted)  $A_G$  and the horizontal line  $\ell_\varepsilon$  at height  $\varepsilon$  corresponds to a (maximal) time interval  $I$  during which  $(G, I)$  is a  $(1, \varepsilon, 0)$ -group, or an  $\varepsilon$ -group for short. Every such a segment corresponds to an *instance* of  $\varepsilon$ -group  $G$ .

► **Observation 1.** *Set  $G$  is a maximal  $\varepsilon$ -group on  $I$ , iff the line segment  $s_{\varepsilon, I} = \{(t, \varepsilon) \mid t \in I\}$  is a maximal length segment in  $A_G$ , and is not contained in  $A_H$ , for a supergroup  $H \supset G$ .*

Two instances of  $\varepsilon$ -group  $G$  may *merge*. Let  $v$  be a local maximum of  $f_G$  and  $I_1 = [t_1, v_t]$  and  $I_2 = [v_t, t_2]$  be two instances of group  $G$  meeting at  $v$ . At  $v_\varepsilon$ , the two instances  $G$  that are alive during  $[t_1, v_t]$  and  $[v_t, t_2]$  merge and we now have a single time interval  $I = [t_1, t_2]$  on which  $G$  is a group. We say that  $I$  is a new instance of  $G$ , different from  $I_1$  and  $I_2$ . We can thus decompose  $A_G$  into maximally-connected regions, each corresponding to a distinct instance of group  $G$ , using horizontal segments through the local maxima of  $f_G$ . We further split each region at the values  $\varepsilon$  where  $G$  changes between being maximal and being dominated. Let  $\mathcal{P}_G$  denote the obtained set of regions in which  $G$  is maximal. Each such a



■ **Figure 1** (a) A set of trajectories for a set of entities moving in  $\mathbb{R}^1$  (b) The region  $A_{\{r,v\}}$  during which  $\{r, v\}$  is alive, and its decomposition into polygons, each corresponding to a distinct instance. In all such regions, except the top one  $\{r, v\}$  is a maximal group: in the top region  $\{r, v\}$  is dominated by  $\{r, v, o\}$  (darker region).



■ **Figure 2** The arrangement  $\mathcal{H}$  and the regions  $A_{\{r,v\}}$  (purple) and  $A_{\{p,o\}}$  (orange) for the trajectories shown in Figure 1(a). The arrangement  $\mathcal{H}$  corresponds to the arrangement of functions  $h_a(t)$  that represent the distance from  $a$  to the entity directly above  $a$  at time  $t$ .

region  $P$  corresponds to a *combinatorially distinct* instance on which  $G$  is a maximal group (with at least one member and duration at least zero). The region  $P$  is bounded by at most two horizontal line segments and two  $\varepsilon$ -monotone chains (see Figure 1(b)).

**Counting maximal  $\varepsilon$ -groups.** To bound the number of distinct maximal  $\varepsilon$ -groups, over all values of  $\varepsilon$ , we have to count the number of polygons in  $\mathcal{P}_G$  over all sets  $G$ . While there are possibly exponentially many sets, there is structure in the regions  $A_G$  which we can exploit.

Consider a set of entities  $G$  and a region  $P \in \mathcal{P}_G$  corresponding to a distinct instance of the maximal  $\varepsilon$ -group  $G$ . We observe that all vertices of  $P$  lie on the polyline  $f_G$ : they are either directly vertices of  $f_G$ , or they are points  $(t, \varepsilon)$  on the edges of  $f_G$  where  $G$  starts or stops being maximal. For the latter case there must be a polyline  $f_H$ , for some subgroup or superset of  $G$ , that intersects  $f_G$  at such a point. Furthermore, observe that any vertex (of either type) is used by at most a constant number of regions from  $\mathcal{P}_G$ .

Below we show that the complexity of the arrangement  $\mathcal{H}$ , of all polylines  $f_G$  over all  $G$ , is bounded by  $O(|\mathcal{A}|n)$ . Furthermore, we show that each vertex of  $\mathcal{H}$  can be incident to at most  $O(n)$  regions. It follows that the complexity of all polygons  $P \in \mathcal{P}_G$ , over all groups (sets)  $G$ , and thus also the number of such sets, is at most  $O(|\mathcal{A}|n^2)$ .

**The complexity of  $\mathcal{H}$ .** The *span*  $S_G(t) = \{a \mid a \in \mathcal{X} \wedge a(t) \in [\min_{b \in G} b(t), \max_{b \in G} b(t)]\}$  of a set of entities  $G$  at time  $t$  is the set of entities between the lowest and highest entity of  $G$  at time  $t$  (for technical reasons, we include the lowest entity of  $G$  in the span, but not the highest). Let  $h_a(t)$  denote the distance from entity  $a$  to the entity directly above  $a$  at time  $t$ , that is,  $h_a(t)$  is the height of the face in  $\mathcal{A}$  that has  $a$  on its lower boundary at time  $t$ .

► **Observation 2.** A set  $G$  is  $\varepsilon$ -connected at time  $t$ , if and only if the largest distance among consecutive entities in  $S_G(t)$  is at most  $\varepsilon$ . That is,

$$f_G(t) = \max_{a \in S_G(t)} h_a(t)$$

It follows that  $\mathcal{H}$  is a subset of the arrangement of the  $n$  functions  $h_a$ , for  $a \in \mathcal{X}$  (see Figure 2). We use this fact to show that  $\mathcal{H}$  has complexity at most  $O(|\mathcal{A}|n)$ :

► **Lemma 3.** Let  $\mathcal{A}$  be an arrangement of  $n$  line segments, and let  $k$  be the maximum number of line segments intersected by a vertical line. The number of triplets  $(F, F', x)$  such that the faces  $F \in \mathcal{A}$  and  $F' \in \mathcal{A}$  have equal height  $h$  at  $x$ -coordinate  $x$  is at most  $O(|\mathcal{A}|k) \subseteq O(|\mathcal{A}|n) \subseteq O(n^3)$ .

► **Remark.** Interestingly, this bound is tight in the worst case. In the full version of this paper we give a construction where there are  $\Omega(n^3)$  triplets  $(F, F', x)$  such that  $F$  and  $F'$  have equal height at  $x$ , even if we use lines instead of line segments [22].

► **Lemma 4.** The arrangement  $\mathcal{H}$  has complexity  $O(|\mathcal{A}|n)$ .

What remains to show is that each vertex  $v$  of  $\mathcal{H}$  can be incident to at most  $O(n)$  polygons from different sets. We use Lemma 5, which follows from Buchin et al. [9]:

► **Lemma 5.** Let  $\mathcal{R}$  be the Reeb graph for a fixed value  $\varepsilon$  capturing the movement of a set of  $n$  entities moving along piecewise-linear trajectories in  $\mathbb{R}^d$  (for some constant  $d$ ), and let  $v$  be a vertex of  $\mathcal{R}$ . There are at most  $O(n)$  maximal groups that start or end at  $v$ .

► **Lemma 6.** Let  $v$  be a vertex of  $\mathcal{H}$ . Vertex  $v$  is incident to at most  $O(n)$  polygons from  $\mathcal{P} = \bigcup_{G \subseteq \mathcal{X}} \mathcal{P}_G$ .

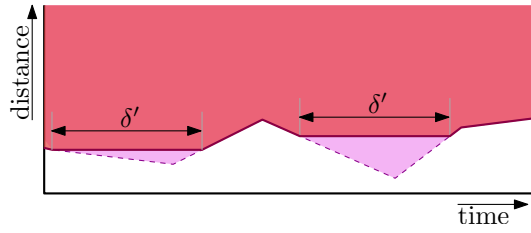
**Proof.** Let  $P \in \mathcal{P}_G$  be a region that uses  $v$ . Thus,  $G$  either starts or ends as a maximal  $v_\varepsilon$ -group at time  $v_t$ . This means,  $v$  correspond to a single vertex  $u$  in the Reeb graph, built with parameter  $v_\varepsilon$ . By Lemma 5, there are at most  $O(n)$  maximal  $v_\varepsilon$ -groups that start or end at  $u$ . Hence,  $v$  can occur in regions of at most  $O(n)$  different sets  $G$ . For a fixed set  $G$ , the regions in  $\mathcal{P}_G$  are disjoint, so there are only  $O(1)$  regions from  $\mathcal{P}_G$ , that contain  $v$ . ◀

► **Lemma 7.** The number of distinct  $\varepsilon$ -groups, over all values  $\varepsilon$ , and the total complexity of all regions  $\mathcal{P} = \bigcup_{G \subseteq \mathcal{X}} \mathcal{P}_G$ , are both at most  $O(|\mathcal{H}|n) = O(|\mathcal{A}|n^2)$ .

## 2.2 The Number of Distinct Maximal Groups, over all Parameters

Maximal groups are monotonic in  $m$  and  $\delta$  (see Buchin et al. [9]); hence a maximal  $(m, \varepsilon, \delta)$ -group is also a maximal  $(m', \varepsilon, \delta')$ -group for any parameters  $m' \leq m$  and  $\delta' \leq \delta$ . It follows that the number of combinatorially different maximal groups is still at most  $O(|\mathcal{A}|n^2)$ .

For the complexity of the regions in  $\bigcup \mathcal{P}_G$ : fix  $m = 0$ , and consider the remaining subspace of  $\mathbb{P}$  with axes time, distance, and duration, and the restriction of  $A_G$ , for any set  $G$ , into this space. In the  $\delta = 0$  plane we simply have the regions  $A_G$ , that are bounded from below by a  $t$ -monotone polyline  $f_G$ , as described in Section 2.1. As we increase  $\delta$  we observe that the local minima in the boundary  $f_G$  get replaced by a horizontal line segment of width  $\delta$  (see Figure 3). For arbitrarily small values of  $\delta > 0$ , the total complexity of this boundary is still  $O(|\mathcal{A}|n^2)$ . Further increasing  $\delta$ , monotonically decreases the number of vertices on the functions  $f_G$ . It follows that the regions  $A_G$ , restricted to the time, distance, duration space also have total complexity  $O(|\mathcal{A}|n^2)$ . Finally, consider the regions  $A_G$  in the full four dimensional space. Clearly,  $A_G \cap \{p \mid p \in \mathbb{P} \wedge p_m < |G|\} = \emptyset$ . For values  $m \geq |G|$ , the boundary of  $A_G$  is constant in  $m$ . We conclude:



■ **Figure 3** A cross section of the region  $A_{\{r,v\}}$  with the plane through  $\delta = \delta'$ . The boundary of the original region (i.e. the cross section with the plane through  $\delta = 0$ ) is dashed.

► **Theorem 8.** *Let  $\mathcal{X}$  be a set of  $n$  entities, in which each entity travels along a piecewise-linear trajectory of  $\tau$  edges in  $\mathbb{R}^1$ , and let  $\mathcal{A}$  be the resulting trajectory arrangement. The number of distinct maximal groups is at most  $O(|\mathcal{A}|n^2) = O(\tau n^4)$ , and the total complexity of all regions in the parameter space corresponding to these groups is also  $O(|\mathcal{A}|n^2) = O(\tau n^4)$ .*

In the full version [22] we prove Lemma 9: even for fixed parameters  $\varepsilon$ ,  $m$ , and  $\delta$ , the number of maximal  $(m, \varepsilon, \delta)$ -groups, for entities moving in  $\mathbb{R}^1$ , may be as large as  $\Omega(\tau n^3)$ . This strengthens the result of Buchin et al. [9], who established this bound for entities in  $\mathbb{R}^2$ .

► **Lemma 9.** *For a set  $\mathcal{X}$  of  $n$  entities, in which each entity travels along a piecewise-linear trajectory of  $\tau$  edges in  $\mathbb{R}^1$ , there can be  $\Omega(\tau n^3)$  maximal  $\varepsilon$ -groups.*

### 3 Algorithm

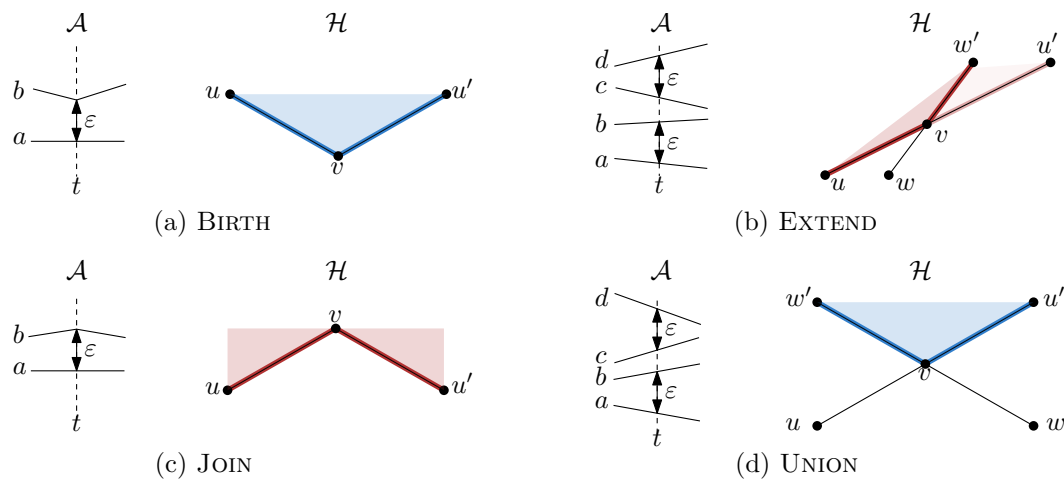
In the following we refer to combinatorially different maximal groups simply as groups. Our algorithm computes a representation (of size  $O(|\mathcal{A}|n^2)$ ) of all groups, which we can use to list all groups and, given a pointer to a group  $G$ , list all its members and the polygon  $Q_G \in \mathcal{P}_G$ . We assume  $\delta = 0$  and  $m = 1$ , since the sets of maximal groups for  $\delta > 0$  and  $m > 1$  are a subset of the set for  $\delta = 0$  and  $m = 1$ .

#### 3.1 Overview

Our algorithm uses the arrangement  $\mathcal{H}$  located in the  $(t, \varepsilon)$ -plane. Line segments in  $\mathcal{H}$  correspond to the height function of the faces in  $\mathcal{A}$ . Let  $a, b \in S_G(t)$  be the pair of consecutive entities in the span of a group  $G$  with maximum vertical distance at time  $t$ . We refer to  $(a, b)$  as the *critical pair* of  $G$  at time  $t$ . The pair  $(a, b)$  determines the minimal value of  $\varepsilon$  that is required for the group  $G$  to be  $\varepsilon$ -connected at time  $t$ . The distance between a critical pair  $(a, b)$  defines an edge of the polygon bounding  $G$  in  $\mathcal{H}$ .

Our representation will consist of the arrangement  $\mathcal{H}$  in which each edge  $e$  is annotated with a data structure  $\mathcal{T}_e$ , a list  $\mathcal{L}$  (or array) with the top edge in each *group polygon*  $Q_G \in \mathcal{P}_G$ , and an additional data structure  $\mathcal{S}$  to support reconstructing the grouping polygons. We start by computing the arrangement  $\mathcal{H}$ . This takes  $O(|\mathcal{H}|) = O(\tau n^3)$  time [2]. The arrangement is built from the set of height-functions of the faces of  $\mathcal{A}$ . With each edge we store the pair of edges in  $\mathcal{A}$  responsible for it.

Given arrangement  $\mathcal{H}$  we use a sweep line algorithm to construct the rest of the representation. A horizontal line  $\ell(\varepsilon)$  is swept at height  $\varepsilon$  upwards, and all groups  $G$  whose group polygon  $Q_G$  currently intersects  $\ell$  are maintained. To achieve this we maintain a two-part status structure. First, a set  $\mathcal{S}$  with for each group  $G$  the time interval  $I(G, \varepsilon) = Q_G \cap \ell(\varepsilon)$ . Second, for each edge  $e \in \mathcal{H}$  intersected by  $\ell(\varepsilon)$  a data structure  $\mathcal{T}_e$  with the sets of entities



**Figure 4** The different types of vertex events shown both in the arrangement  $\mathcal{A}$  and in  $\mathcal{H}$ . The EXTEND event has a horizontally symmetric case.

whose time interval starts or ends at  $e$ , that is,  $G \in \mathcal{T}_e$  if and only if  $I(G, \varepsilon) = [s, t]$  with  $s = e \cap \ell(\varepsilon)$  or  $t = e \cap \ell(\varepsilon)$ . We postpone the implementation of  $\mathcal{T}$  to Section 3.3. The data structures support the following operations:

Operation	Input	Action
$\text{FILTER}(\mathcal{T}_e, X)$	A data structure $\mathcal{T}_e$ A set of entities $X$	Create a data structure $\mathcal{T}' = \{G \cap X \mid G \in \mathcal{T}_e\}$
$\text{INSERT}(\mathcal{T}_e, G)$	A data structure $\mathcal{T}_e$ A pointer to a representation of $G$	Create a data structure $\mathcal{T}' = \mathcal{T}_e \cup \{G\}$ .
$\text{DELETE}(\mathcal{T}_e, G)$	A data structure $\mathcal{T}_e$ A pointer to a representation of $G$	Create a data structure $\mathcal{T}' = \mathcal{T}_e \setminus \{G\}$ .
$\text{MERGE}(\mathcal{T}_e, \mathcal{T}_f)$	Two data structures $\mathcal{T}_e, \mathcal{T}_f$ , belonging to two edges $e, f$ having the same starting or ending vertex	Create a data structure $\mathcal{T}' = \mathcal{T}_e \cup \mathcal{T}_f$ .
$\text{CONTAINS}(\mathcal{T}_e, G)$	A data structure $\mathcal{T}_e$ A pointer to a representation of $G$ ending or starting on edge $e$	Test if $\mathcal{T}_e$ contains set $G$ .
$\text{HASSUPERSET}(\mathcal{T}_e, G)$	A data structure $\mathcal{T}_e$ A pointer to a representation of $G$ ending or starting on edge $e$	Test if $\mathcal{T}_e$ contains a set $H \supseteq G$ , and return the smallest such set if so.

The end points of the time interval  $I(G, \varepsilon) = [\text{start}(G, \varepsilon), \text{end}(G, \varepsilon)]$  vary non-stop along the sweep. For each group  $G$ , the set  $\mathcal{S}$  instead stores the edges  $e$  and  $f$  of  $\mathcal{H}$  that contain the starting time  $\text{start}(G, \varepsilon)$  and ending time  $\text{end}(G, \varepsilon)$ , respectively, and pointers to the representation of  $G$  in  $\mathcal{T}_e$  and  $\mathcal{T}_f$ . We refer to  $e$  and  $f$  as the *starting edge* and *ending edge* of  $G$ . In addition, we store with each interval  $I(G, \varepsilon)$  a pointer to the previous version of the interval  $I(G, \varepsilon')$  if (and only if) the starting time (ending time) of  $G$  changed to edge  $e$  (edge  $f$ ) at  $\varepsilon'$ . Note that updates for both  $\mathcal{S}$  and  $\mathcal{T}$  occur only when a vertex is hit by the sweep line  $\ell(\varepsilon)$ . For all unbounded groups we add  $I(G, \infty)$  to  $\mathcal{L}$  after the sweep line algorithm.

### 3.2 Sweepline Events

The sweep line algorithm results in four different vertex events (see Figure 4). The EXTEND-event has a symmetrical version in which  $\overline{uu'}$  and  $\overline{ww'}$  both have a negative incline. We describe how to update our hypothetical data structures in all cases.

**Case I – Birth.** Vertex  $v$  is a local minimum of one of the functions  $h_a$ , with  $a \in \mathcal{X}$  (see Figure 4(a)). When the sweep line intersects  $v$  a new maximal group  $G$  is born. We can find the maximal group spawned in  $O(|G|)$  time by checking which trajectories are  $\varepsilon$ -connected for this value of  $t$  and  $\varepsilon$ . To this end we traverse the (vertical decomposition of)  $\mathcal{A}$  starting at the entities defining  $v$ .

**Case II – Extend.** Vertex  $v$  is the intersection of two line segments  $s_{ab} = \overline{uu'}$  and  $s_{cd} = \overline{ww'}$ , both with a positive incline (see Figure 4(b)). The case in which  $s_{ab}$  and  $s_{cd}$  have negative incline can be handled symmetrically. Assume without loss of generality that  $s_{cd}$  is steeper than  $s_{ab}$ . We start with the following observation:

► **Observation 10.** *None of the groups arriving on edge  $(w, v)$  continue on edge  $(v, u')$ .*

**Proof.** Let  $G$  be a group that arrives at  $v$  using edge  $(w, v)$ . As  $G$  uses  $(w, v)$ , it must contain entities both above and below the face  $F$  defined by critical pair  $(c, d)$ . We know that  $u_\varepsilon$  and  $w_\varepsilon$  are strictly smaller than  $v_\varepsilon$  and  $\varepsilon$  is never smaller than zero. Thus,  $v_\varepsilon$  is strictly positive and  $F$  has a strictly positive height at  $t$ . Therefore,  $G$  still contains entities above and below  $F$  after time  $t$ . But then the critical pair  $(c, d)$  is still part of  $G$  and  $s_{cd}$  is a lower bound for the group. It follows that  $G$  must use edge  $(v, w')$ . ◀

We first compute the groups on outgoing edge  $(v, u')$ . By Observation 10 all these groups arrive on edge  $(u, v)$ . In particular, they are the maximal size subsets from  $\mathcal{T}_{(u,v)}$  for which all entities lie below entity  $d$  at time  $t$ , that is,  $\mathcal{T}_{(v,u')} = \text{FILTER}(\mathcal{T}_{(u,v)}, \text{below}(d, t))$ , where  $\text{below}(y, t) = \{x \mid x \in \mathcal{X} \wedge x(t) < y(t)\}$ . For each group  $G$  in  $\mathcal{T}_{(v,u')}$  we update the time-interval in  $\mathcal{S}$ . If  $G$  was dominated by a maximal group  $H \supset G$  on incoming edge  $(u, v)$ , we insert a new time interval with starting edge  $f = \text{start}(H, \varepsilon)$  and ending edge  $(v, u')$  into  $\mathcal{S}$ , and insert  $G$  into  $\mathcal{T}_f$ . Note that  $G$  and  $H$  indeed have the same starting time:  $G$  is a subset of  $H$ , and is thus  $\varepsilon$ -connected at any time where  $H$  is  $\varepsilon$ -connected. Since  $G$  was not maximal before, it did not start earlier than  $H$  either.

The groups from  $\mathcal{T}_{(u,v)}$  that contain entities on both sides of critical pair  $(c, d)$ , continue onto edge  $(v, w')$ . Let  $\mathcal{T}'$  denote these groups. We update the interval  $I(G)$  in  $\mathcal{S}$  for each group  $G \in \mathcal{T}'$  by setting the ending edge to  $(v, w')$ .

Next, we determine which groups from  $\mathcal{T}_{(w,v)}$  die at  $v$ . A maximal group  $G \in \mathcal{T}_{(w,v)}$  dies at  $v$  if there is a group  $H$  on  $(v, w')$  that dominates  $G$ . Any such group  $H$  must arrive at  $v$  by edge  $(u, v)$ . Hence, for each group  $G \in \mathcal{T}_{(w,v)}$  we check if there is a group  $H \in \mathcal{T}'$  with  $H \supset G$  and  $I(H) \supseteq I(G)$ . For each of these groups we remove the interval  $I(G, \varepsilon)$  from  $\mathcal{S}$ , add  $I(G, \varepsilon)$  to  $\mathcal{L}$ , and delete the set  $G$  from the data structure  $\mathcal{T}_f$ , where  $f$  is the starting edge of  $G$  (at height  $\varepsilon$ ).

The remaining (not dominated) groups from  $\mathcal{T}_{(w,v)}$  continue onto edge  $(v, w')$ . Let  $\mathcal{T}''$  denote this set. We obtain  $\mathcal{T}_{(v,w')}$  by merging  $\mathcal{T}'$  and  $\mathcal{T}''$ , that is,  $\mathcal{T}_{(v,w')} = \text{MERGE}(\mathcal{T}', \mathcal{T}'')$ . Since we now have the data structures  $\mathcal{T}_{(v,u')}$  and  $\mathcal{T}_{(v,w')}$ , and we updated  $\mathcal{S}$  accordingly, our status structure again reflects the maximal groups currently intersected by the sweep line.



**Case III – Join.** Vertex  $v$  is a local maximum of one of the functions  $h_a$ , with  $a \in \mathcal{X}$  (see Figure 4(c)). Two combinatorially different maximal groups  $G_u$  and  $G_w$  with the same set of entities die at  $v$  (and get replaced by a new maximal group  $G^*$ ) if and only if  $G_u$  is a maximal group in  $\mathcal{T}_{(u,v)}$  and  $G_w$  is a maximal group in  $\mathcal{T}_{(w,v)}$ . We test this with a call to  $\text{CONTAINS}(\mathcal{T}_{(w,v)}, G_u)$  for each group  $G_u \in \mathcal{T}_{(u,v)}$ . Let  $G$  be a group in  $\mathcal{T}_{(u,v)}$ , and let  $H \in \mathcal{T}_{(w,v)}$  be the smallest supergroup of  $G$ , if such a group exists. At  $v$  the group  $G$  will immediately extend to the ending edge of  $H$ . We can find  $H$  by using a  $\text{HASUPERSET}(\mathcal{T}_{(w,v)}, G)$  call. If  $H$  exists we insert  $G$  into  $\mathcal{T}_e$ , and update  $I(G, \varepsilon)$  in  $\mathcal{S}$  accordingly. We process the groups  $G$  in  $\mathcal{T}_{(w,v)}$  that have a group  $H \in \mathcal{T}_{(u,v)}$  whose starting time jumps at  $v$  analogously.

**Case IV – Union.** Vertex  $v$  is the intersection of a line segment  $s_{ab} = \overline{uw'}$  with positive incline and a line segment  $s_{cd} = \overline{ww'}$ , with negative incline (see Figure 4(d)). The UNION event is a special case of the BIRTH event. Incoming groups on edge  $(u, v)$  are below the line segment  $s_{cd}$  and, hence, can not contain any elements that are above  $c$ . As a consequence the line segment  $s_{cd}$  does not limit these groups and for a group  $G \in \mathcal{T}_{(u,v)}$  we can safely add it to  $\mathcal{T}_{(v,u')}$ . We also update the interval  $I(G)$  in  $\mathcal{S}$  by setting the ending edge to  $(v, u')$ . An analogous argument can be made for groups arriving on edge  $(w, v)$ .

Furthermore a new maximal group is formed. Let  $H$  be the set of all entities  $\varepsilon$ -connected to entity  $a$  at time  $t$ . We insert  $H$  into  $\mathcal{T}_{(v,u')}$  and  $\mathcal{T}_{(v,w')}$  and we insert a time interval  $I(H)$  into  $\mathcal{S}$  with starting edge  $(v, w')$  and ending edge  $(v, u')$ .

### 3.3 Data Structure

We can implement  $\mathcal{S}$  using any standard balanced binary search tree, the only requirement is that, given a (representation of) set  $G$  in a data structure  $\mathcal{T}_e$ , we can efficiently find its corresponding interval in  $\mathcal{S}$ .

**The data structure  $\mathcal{T}_e$ .** We need a data structure  $\mathcal{T} = \mathcal{T}_e$  that supports FILTER, INSERT, DELETE, MERGE, CONTAINS, and HASUPERSET efficiently. We describe a structure of size  $O(n)$ , that supports CONTAINS and HASUPERSET in  $O(\log n)$  time, FILTER in  $O(n)$  time, and INSERT and DELETE in amortized  $O(\log^2 n)$  time. In general, answering CONTAINS and HASUPERSET queries in a dynamic setting is hard and may require  $O(n^2)$  space [24].

► **Lemma 11.** *Let  $G$  and  $H$  be two non-empty  $\varepsilon$ -groups that both end at time  $t$ . We have:*

$$(G \cap H \neq \emptyset \wedge |G| \leq |H|) \iff G \subseteq H \wedge G \neq \emptyset.$$

We implement  $\mathcal{T}$  with a tree similar to the *grouping-tree* used by Buchin et al. [9]. Let  $\{G_1, \dots, G_k\}$  denote the groups stored in  $\mathcal{T}$ , and let  $\mathcal{X}' = \bigcup_{i \in [1, \dots, k]} G_i$  denote the entities in these groups. Our tree  $\mathcal{T}$  has a leaf for every entity in  $\mathcal{X}'$ . Each group  $G_i$  is represented by an internal node  $v_i$ . For each internal node  $v_i$  the set of leaves in the subtree rooted at  $v_i$  corresponds exactly to the entities in  $G_i$ . By Lemma 11 these sets indeed form a tree. With each node  $v_i$ , we store the size of the group  $G_i$ , and (a pointer to) an arbitrary entity in  $G_i$ . Next to the tree we store an array containing for each entity a pointer to the leaf in the tree that represents it (or NIL if the entity does not occur in any group). We preprocess  $\mathcal{T}$  in  $O(n)$  time to support level-ancestor (LA) queries as well as lowest common ancestor (LCA) queries, using the methods of Bender and Farach-Colton [4, 5]. Both methods work only for *static* trees, whereas we need to allow updates to  $\mathcal{T}$  as well. However, as we need to query  $\mathcal{T}_e$  only when processing the upper end vertex of  $e$ , we can be lazy in updating  $\mathcal{T}_e$ . More specifically, we delay all updates, and simply rebuild  $\mathcal{T}_e$  when we handle its upper end vertex.



has constructed the complete tree. Updating the array with pointers to the leaves takes  $O(n)$  time, as does rebuilding the tree for future LA and LCA queries. ◀

The final function FILTER can easily be implemented in linear time by pruning the tree from the bottom up. We thus conclude:

► **Lemma 15.** *We can handle each event in  $O(n \log^2 n)$  time.*

### 3.4 Maximal Groups

**Reconstructing the grouping polygons.** Given a group  $G$ , represented by a pointer to the top edge of  $Q_G$  in  $\mathcal{L}$ , we can construct the complete group polygon  $Q_G$  in  $O(|Q_G|)$  time, and list all group members of  $G$  in  $O(|G|)$  time. We have access to the top edge of  $Q_G$ . This is an interval  $I(G, \hat{\varepsilon})$  in  $\mathcal{S}$ , specifically, the version corresponding to  $\hat{\varepsilon}$ , where  $\hat{\varepsilon}$  is the value at which  $G$  dies as a maximal group. We then follow the pointers to the previous versions of  $I(G, \cdot)$  to construct the left and right chains of  $Q_G$ . When we encounter the value  $\check{\varepsilon}$  at which  $G$  is born, these chains either meet at the same vertex, or we add the final bottom edge of  $Q_G$  connecting them. To report the group members of  $G$ , we follow the pointer to  $I(G, \hat{\varepsilon})$  in  $\mathcal{S}$ . This interval stores a pointer to its starting edge  $e$ , and to a subtree in  $\mathcal{T}_e$  of which the leaves represent the entities in  $G$ .

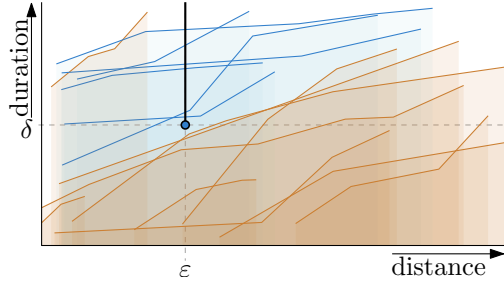
**Analysis.** The list  $\mathcal{L}$  contains  $O(g) = O(|\mathcal{A}|n^2)$  entries (Theorem 8), each of constant size. The total size of all  $\mathcal{S}$ 's is  $O(|\mathcal{H}|n)$ : at each vertex of  $\mathcal{H}$ , there are only a linear number of changes in the intervals in  $\mathcal{S}$ . Each edge  $e$  of  $\mathcal{H}$  stores a data structure  $\mathcal{T}_e$  of size  $O(n)$ . It follows that our representation uses a total of  $O(|\mathcal{H}|n) = O(|\mathcal{A}|n^2)$  space. Handling each of the  $O(|\mathcal{H}|)$  nodes requires  $O(n \log^2 n)$  time, so the total running time is  $O(|\mathcal{A}|n^2 \log^2 n)$ .

► **Theorem 16.** *Given a set  $\mathcal{X}$  of  $n$  entities, in which each entity travels along a trajectory of  $\tau$  edges, we can compute a representation of all  $g = O(|\mathcal{A}|n^2)$  combinatorial maximal groups  $\mathcal{G}$  such that for each group in  $\mathcal{G}$  we can report its grouping polygon and its members in time linear in its complexity and size, respectively. The representation has size  $O(|\mathcal{A}|n^2)$  and takes  $O(|\mathcal{A}|n^2 \log^2 n)$  time to compute, where  $|\mathcal{A}| = O(\tau n^2)$  is the complexity of the trajectory arrangement.*

## 4 Data Structures for Maximal Group Queries

In this section we present data structures that allow us to efficiently obtain all groups for a given set of parameter values. Throughout this section,  $n$  denotes the number of entities considered,  $\tau$  the number of vertices in any trajectory,  $k$  the output complexity, i.e. the number of groups reported,  $g$  the number of maximal groups,  $g'$  the maximum number of maximal groups for a given (fixed) value of  $\varepsilon$ , and  $\Pi$  the total complexity of the regions corresponding to the  $g$  combinatorially different maximal groups. So we have  $g' = O(\tau n^3)$  and  $g \leq \Pi = O(\tau n^4)$ . When  $g'$ ,  $g$ , or  $\Pi$  appear as the argument of a logarithm, we write  $O(\log n\tau)$ . We show that we can store all groups in a data structure of size  $O(\Pi \log n\tau \log n)$  that can be built in  $O(\Pi \log^2 n\tau \log n)$  time, and allows reporting all  $(m, \varepsilon, \delta)$ -groups in  $O(\log^2 n\tau \log n + k)$  time. We use the following three-level tree to achieve this.

On the first level we have a balanced binary tree with in the leaves the group sizes  $1..n$ . Each internal node  $v$  corresponds to a range  $R_v$  of group sizes and stores all groups whose size lies in the range  $R_v$ . Let  $\mathcal{G}_v$  denote this set of groups, and for each such group let  $D_G$  denote the duration of group  $G$  as a function of  $\varepsilon$ . The functions  $D_G$  are piecewise-linear,



■ **Figure 6** The functions  $D_G$  expressing the duration of group  $G$  as a function of  $\varepsilon$ . Assuming all groups have size at least  $m$ , all  $(m, \varepsilon, \delta)$ -groups intersect the upward vertical half-ray starting in point  $(\varepsilon, \delta)$ .

$\delta$ -monotone, and may intersect (see Figure 6). By Theorem 8 the total complexity of these functions is  $O(\Pi)$ . We store all functions  $D_G$ , with  $G \in \mathcal{G}_v$ , in a data structure that can answer the following *polyline stabbing* queries in  $O(\log^2 n\tau + k)$  time: Given a query point  $q = (\varepsilon, \delta)$ , report all polylines that pass above point  $q$ , that is, for which  $D_G(\varepsilon) \geq \delta$ . Thus, given parameters  $m$ ,  $\varepsilon$ , and  $\delta$ , finding all  $(m, \varepsilon, \delta)$ -groups takes  $O(\log^2 n\tau \log n + k)$  time.

We build a segment tree storing the ( $\varepsilon$ -extent of the) individual edges of all polylines stored at  $v$ . An internal node  $u$  of the segment tree corresponds to an interval  $I(u)$ , and stores the set of edges  $\text{Ints}(u)$  that completely span  $I(u)$ . Hence, with respect to  $u$ , we can consider these segments as lines. For a query with a point  $q$ , we have to be able to report all (possibly intersecting) lines from  $\text{Ints}(u)$  that pass above  $q$ . We use a duality transform to map each line  $\ell$  to a point  $\ell^*$  and query point  $q$  to a line  $q^*$ . The problem is then to report all points  $\ell^*$  in the half-plane below  $q^*$ . Such queries can be answered in  $O(\log h + k)$  time, using  $O(h)$  space and  $O(h \log h)$  preprocessing time, where  $h$  is the number of points stored [11]. It follows that we can find all  $k$  polylines that pass above  $q$  in  $O(\log^2 n\tau + k)$  time, using  $O(\Pi \log n\tau)$  space, and  $O(\Pi \log^2 n\tau)$  preprocessing time. We conclude:

► **Theorem 17.** *Given parameters  $m$ ,  $\varepsilon$ , and  $\delta$ , we can build a data structure of size  $O(\Pi \log n\tau \log n)$ , using  $O(\Pi \log^2 n\tau \log n)$  preprocessing time, which can report all  $(m, \varepsilon, \delta)$ -groups in  $O(\log^2 n\tau \log n + k)$  time, where  $k$  is the output complexity.*

In the full version [22], we also describe data structures for the interactive exploration of the data. Here, we have all  $(m, \varepsilon, \delta)$ -groups, for some parameters  $m$ ,  $\varepsilon$ , and  $\delta$ , and we want to change (some of) the parameters. Say to  $m'$ ,  $\varepsilon'$ , and  $\delta'$ , respectively. This requires us to solve *symmetric difference queries*, in which we want to efficiently report all maximal  $(m, \varepsilon, \delta)$ -groups that are no longer maximal for parameters  $m'$ ,  $\varepsilon'$ , and  $\delta'$ , and all maximal  $(m', \varepsilon', \delta')$ -groups that were not maximal for parameters  $m$ ,  $\varepsilon$ , and  $\delta$ . That is, we wish to report  $\mathcal{G}(m, \varepsilon, \delta) \Delta \mathcal{G}(m', \varepsilon', \delta')$ . The following theorem summarizes our results.

► **Theorem 18.** *Let  $(m, \varepsilon, \delta)$  and  $(m', \varepsilon', \delta')$  be two configurations of parameters. In  $O(P(\Pi, g', n))$  time we can build a data structure of size  $O(S(\Pi, g', n))$  for symmetric difference queries, that is, we can report all groups in  $\mathcal{G}(\varepsilon, m, \delta) \Delta \mathcal{G}(\varepsilon', m', \delta')$ , in  $O(Q(\Pi, g', n, k))$  time. In these results  $\Pi$  denotes the total complexity of all combinatorially different maximal groups (over all values  $\varepsilon$ ),  $g'$  the number of maximal groups for a fixed value  $\varepsilon$ ,  $n$  the number of entities,  $\tau$  the number of vertices in a trajectory, and  $k$  the output complexity. The functions  $P$ ,  $S$ , and  $Q$  depend on which of the parameters are allowed to change (other parameters are assumed to be fixed and known at preprocessing time). We have*

Variable Param.	Query time $Q(\Pi, g', n, k)$	Space $S(\Pi, g', n)$	Preproc. $P(\Pi, g', n)$
<i>Changing one parameter at a time</i>			
$\delta$	$\log n\tau + k$	$g' \log n\tau$	$g' \log n\tau$
$m$	$\log n\tau + k$	$g' \log n\tau$	$g' \log n\tau$
$\varepsilon$	$\log n\tau + k$	$\Pi \log n\tau$	$\Pi \log n\tau$
$\delta, m$	$\log n\tau + k$	$g' \log n\tau$	$g' \log n\tau$
$\varepsilon, m$	$\log^2 n\tau + k$	$\Pi \log^2 n\tau$	$\Pi \log^2 n\tau$
$\varepsilon, \delta$	$\sqrt{g'} 2^{\log^* n\tau} \log^2 n\tau + k$	$\Pi \log^2 n\tau$	$\Pi \log^3 n\tau$
$\varepsilon, \delta, m$	$\sqrt{g'} 2^{\log^* n\tau} \log^2 n\tau \log n + k$	$\Pi \log^2 n\tau$	$\Pi \log^3 n\tau$
<i>Changing multiple parameters at the same time</i>			
$\delta, m$	$\log n\tau + k$	$g' \log n\tau$	$g' \log n\tau$
$\varepsilon, m$	$\log^2 n\tau + k$	$\Pi \log^2 n\tau$	$\Pi \log^2 n\tau$
$\varepsilon, \delta$	$\sqrt{g'} 2^{\log^* n\tau} \log^3 n\tau + k$	$\Pi \log^2 n\tau$	$\Pi \log^3 n\tau$
$\varepsilon, \delta, m$	$\sqrt{g'} 2^{\log^* n\tau} \log^3 n\tau \log^2 n + k$	$\Pi \log^2 n\tau \log^2 n$	$\Pi \log^3 n\tau \log^2 n$

## 5 Entities Moving in $\mathbb{R}^d$

We now describe how our results can be extended to entities moving in  $\mathbb{R}^d$ , for any constant dimension  $d$ .

### 5.1 Bounding the Complexity

Recall that  $\mathcal{X}(t)$  denotes the locations of the entities at time  $t$ . We still consider the  $(t, \varepsilon)$ -plane in  $\mathbb{P}$ , and the regions  $A_G$ , for subsets  $G \subseteq \mathcal{X}$ , in which set  $G$  is alive. Such a region is still bounded from below by a function  $f_G$  that expresses the minimum distance for which  $G$  is  $\varepsilon$ -connected. We again consider the arrangement  $\mathcal{H}$  of these functions  $f_G$ , over all sets  $G$ .

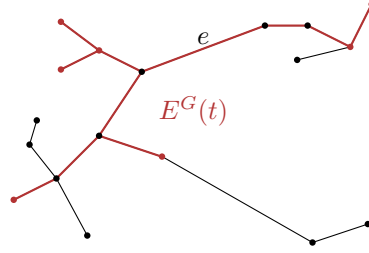
Let  $\mathcal{H}''$  be the arrangement of all pairwise distance functions  $h_{ab}(t) = \|a(t)b(t)\|$ . For any subset of entities  $G$  and any time  $t$ ,  $f_G(t) = \|a(t)b(t)\|$  for some pair of entities  $a$  and  $b$ . Thus,  $\mathcal{H}$  is a sub-arrangement of  $\mathcal{H}''$ . This immediately gives an  $O(\tau n^4)$  bound on the complexity of  $\mathcal{H}$ . We instead show that  $\mathcal{H}$  has complexity at most  $O(\tau n^3 \beta_4(n))$ , where  $\beta_s(n) = \lambda_s(n)/n$ , and  $\lambda_s(n)$  is the maximum length of a Davenport-Schinzel sequence of order  $s$  on  $n$  symbols. Using exactly the same argument as in Lemma 6 we then get a bound of  $O(\tau n^4 \beta_4(n))$  on the number of combinatorially different groups.

Let  $\mathcal{E}(t)$  be the Euclidean minimum spanning tree (EMST) of the points in  $\mathcal{X}(t)$ .

► **Observation 19.** *A subset of entities  $G \subseteq \mathcal{X}$  is  $\varepsilon$ -connected at time  $t$  if and only if for any two entities  $p, q \in G$  the longest edge in the Euclidean minimum spanning tree  $\mathcal{E}(t)$  on the path between  $p$  and  $q$  has length at most  $\varepsilon$ .*

Specifically, let  $\mathcal{E}^G(t)$  be the minimum (connected) subtree of  $\mathcal{E}(t)$  containing all points in  $G(t)$ , and let  $\check{\varepsilon}_G(t)$  be the length of the longest edge  $e$  in  $\mathcal{E}^G(t)$  (see Figure 7). We have that  $G$  is an  $\varepsilon$ -group for all  $\varepsilon \geq \check{\varepsilon}_G(t)$ , and that  $f_G(t) = \max_{(a,b) \in \mathcal{E}^G(t)} \|a(t)b(t)\|$ .

It follows from Observation 19 that we are interested in the distance function  $h_{ab}$  on the time intervals during which  $(a, b)$  is part of the EMST. Hence, we need to consider only the arrangement of such partial functions. It is, however, difficult to bound the complexity of the resulting arrangement directly. Instead, we consider  $h_{ab}$  only during those time intervals in which  $(a, b)$  is an edge in the Yao-graph [23]. Let  $\mathcal{H}'$  be the resulting arrangement. Since the EMST is a subgraph of the Yao-graph it follows that  $\mathcal{H}$  is a sub-arrangement of  $\mathcal{H}'$  [23].



■ **Figure 7**  $\mathcal{E}(t)$  and its minimum subtree  $E^G(t)$  (red edges) for a subset of entities  $G$  (red vertices). Edge  $e$  determines the minimum  $\varepsilon$  for which  $G$  is  $\varepsilon$ -connected at  $t$ .

► **Lemma 20.**  $\mathcal{H}'$  has complexity  $O(\tau n^3 \beta_4(n))$ .

**Proof.** Fix an entity  $a$ , and consider the movement of the other entities with respect to  $a$ . This gives us a set of (piecewise linear) trajectories. Entity  $a$  is fixed at the origin. Partition this space into  $k = O(2^d) = O(1)$  equal size polyhedral cones  $C_1, \dots, C_k$  that have their common apex at the origin. For each such cone  $C_i$ , let  $\eta_a^i(t)$  denote the distance from  $a$  to the nearest entity in the cone. It is easy to show that  $\eta_a^i$  is piecewise hyperbolic, and consists of  $O(\tau \lambda_4(n))$  pieces [17].

Let  $\mathcal{H}^*$  be the arrangement of all functions  $\eta_a^i$ , over all entities  $a \in \mathcal{X}$  and all cones  $C_i$ . The total number of pieces (hyperbolic arcs), over all entities and all cones, is  $O(\tau n \lambda_4(n))$ . Partition time into  $O(\tau \lambda_4(n))$  time intervals, with  $O(n)$  pieces each. This may require splitting some of the pieces, but the total number of pieces remains  $O(\tau n \lambda_4(n))$ . In each time interval we now have  $O(n)$  hyperbolic arc pieces, that intersect at most  $O(n^2)$  times in total. It follows that  $\mathcal{H}^*$  has complexity  $O(\tau \lambda_4(n) n^2) = O(\tau n^3 \beta_4(n))$ .

Fix a time  $t$ , and consider the graph  $Y(t)$  that has an edge  $(a, b)$  if and only if  $b$  is the nearest neighbor of  $a$  in one of the cones  $C_i$  at time  $t$ , that is,  $\|a(t)b(t)\| = \eta_a^i(t)$ . Indeed,  $Y(t)$  is the Yao-graph of the entities at time  $t$  [23]. It follows that  $\mathcal{H}^* = \mathcal{H}'$ . ◀

Since  $\mathcal{H}$  is a sub arrangement of  $\mathcal{H}'$ , it follows that  $\mathcal{H}$  also has complexity  $O(\tau n^3 \beta_4(n))$ . Using exactly the same argument as in Lemma 6 we then get that the number of combinatorially different maximal groups is  $O(\tau n^4 \beta_4(n))$ . We conclude:

► **Theorem 21.** Let  $\mathcal{X}$  be a set of  $n$  entities, in which each entity travels along a piecewise-linear trajectory of  $\tau$  edges in  $\mathbb{R}^d$ , for any constant  $d$ . The number of maximal combinatorial groups as well as the total complexity of all their group polygons is at most  $O(\tau n^4 \beta_4(n))$ .

## 5.2 Algorithm

We can almost directly apply our algorithm from Section 3 in higher dimensions as well. Instead of the arrangement  $\mathcal{H}$ , we now use  $\mathcal{H}'$ . The only differences involve discovering the set entities involved in a BIRTH-event, and splitting the set of entities in case of an EXTEND-event. Let  $v$  denote the vertex of  $\mathcal{H}'$  we are processing. We use that at time  $v_t$ ,  $\mathcal{H}'$  encodes the Yao-graph  $Y$ . Using a breadth first search in  $Y$  we can find the entities connected to  $v$ . If an edge has length larger than  $\varepsilon$  we stop the exploration along it. Since  $Y$  is planar, and has  $O(n)$  vertices this takes  $O(n)$  time. This does not affect the running time, hence we get the same result as in Theorem 16 for entities moving in  $\mathbb{R}^d$ , for any constant  $d$ .

### 5.3 Data Structures

**Finding all maximal  $(m, \varepsilon, \delta)$ -groups.** We use the same approach as in Section 4. However, the functions  $\text{duration}_G$  are no longer (piecewise) linear functions in  $\varepsilon$ . Let  $\text{start}_G(\varepsilon) = f^{-1}(\varepsilon)$  and  $\text{end}_G(\varepsilon) = h^{-1}(\varepsilon)$  be some hyperbolic functions  $f$  and  $h$  corresponding to curves in  $\mathcal{H}$ . We have that  $\text{duration}_G(\varepsilon) = \text{end}_G(\varepsilon) - \text{start}_G(\varepsilon)$ . The function  $\text{duration}_G$  corresponds to a piecewise curve with pieces defined by polynomials of degree at most four. Hence, we have to solve the following sub-problem: given a set of  $g'$  algebraic curves of degree at most four, and query point  $q$ , report all curves that pass above  $q$ .

We can solve such queries as follows. We transform the curves into hyperplanes in  $\mathbb{R}^\ell$ , where  $\ell$  is the linearization dimension. We then apply a duality transform, after which the problem can be solved using a half-space range query. Since we have curves of degree at most four in  $\mathbb{R}^2$ , the linearization dimension is seven: the set of points above a curve can be described using a seven-variate polynomial (the five coefficients of the degree four curve, and the two coordinates of the point) [1]. It follows that we can find all curves above query point  $q$  in  $O(g'^{1-1/\lceil 7/2 \rceil} \text{polylog } n\tau + k) = O(g'^{2/3} \text{polylog } n\tau + k)$  time, using linear space [19]. Reporting all maximal  $(m, \varepsilon, \delta)$ -groups thus takes  $O(g'^{2/3} \text{polylog } n\tau + k)$  time, using  $O(\Pi \log n\tau \log n)$  space and  $O(\Pi \log^2 n\tau \log n)$  preprocessing time.

Alternatively, we can maintain the upper envelope of the curves in a dynamic data structure. To solve a query, we repeatedly delete the curve realizing the upper envelope at  $q_\varepsilon$ . This allows us to find all  $(m, \varepsilon, \delta)$ -groups in  $O(k\beta_4(g')^2 \text{polylog } n\tau)$  time [10].

**Symmetric Difference Queries.** Only the versions of the problem that involve changing both  $\varepsilon$  and  $\delta$  are affected. Instead of piecewise linear functions  $\text{duration}_G$  we again have piecewise curves of degree at most four. We use a similar approach as above to find the curves that intersect a vertical or horizontal query segment in  $O(g'^{2/3} \text{polylog } n\tau + k)$  time. Thus, we essentially replace the  $\sqrt{g'}$  terms in Theorem 18 by a  $g'^{2/3}$  term.

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