The Independence of Markov’s Principle in Type Theory

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Abstract

In this paper, we show that Markov’s principle is not derivable in dependent type theory with natural numbers and one universe. One tentative way to prove this would be to remark that Markov’s principle does not hold in a sheaf model of type theory over Cantor space, since Markov’s principle does not hold for the generic point of this model. It is however not clear how to interpret the universe in a sheaf model [9, 17, 21]. Instead we design an extension of type theory, which intuitively extends type theory by the addition of a generic point of Cantor space. We then show the consistency of this extension by a normalization argument. Markov’s principle does not hold in this extension, and it follows that it cannot be proved in type theory.

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1 Introduction

Markov’s principle has a special status in constructive mathematics. One way to formulate this principle is that if it is impossible that a given algorithm does not terminate, then it does terminate. It is equivalent to the fact that if a set of natural number and its complement are both computably enumerable, then this set is decidable. This form is often used in recursivity theory. This principle was first formulated by Markov, who called it “Leningrad’s principle”, and founded a branch of constructive mathematics around this principle [14].

This principle is also equivalent to the fact that if a given real number is not equal to 0 then this number is apart from 0 (that is this number is $< -r$ or $> r$ for some rational number $r > 0$). On this form, it was explicitly refuted by Brouwer in intuitionistic mathematics, who gave an example of a real number (well defined intuitionistically) which is not equal to 0, but also not apart from 0. (The motivation of Brouwer for this example was to show the necessity of using negation in intuitionistic mathematics [4].) The idea of Brouwer can be represented formally using topological models [19].

In a neutral approach to mathematics, such as Bishop’s [3], Markov’s principle is simply left undecided. We also expect to be able to prove that Markov’s principle is not provable in formal system in which we can express Bishop’s mathematics. For instance, Kreisel [12] introduced modified realizability to show that Markov’s principle is not derivable in the formal system $HA^ω$. Similarly, one would expect that Markov’s principle is not derivable in...
We say that a statement $A$ is independent of some formal system if $A$ cannot be derived in that system. A statement in the formal system of Martin-Löf type theory (MLTT) is represented by a closed type. A statement/type $A$ is derivable if it is inhabited by some term $t$ (written $\text{MLTT} \vdash t : A$). This is the so-called propositions-as-types principle. Correspondingly we say that a statement $A$ (represented as a type) is independent of MLTT if there is no term $t$ such that $\text{MLTT} \vdash t : A$.

The main result of this paper is to show that Markov’s principle is independent of Martin-Löf type theory.

The main idea for proving this independence is to follow Brouwer’s argument. We want to extend type theory with a “generic” infinite sequence of 0 and 1 and establish that it is both absurd that this generic sequence is never 0, but also that we cannot show that it has to take the value 0. To add such a generic sequence is exactly like adding a Cohen real [5] in forcing extension of set theory. A natural attempt for doing this will be to consider a topological model of type theory (sheaf model over Cantor space), extending the work [19] to type theory. However, while it is well understood how to represent universes in presheaf model [9], it has turned out to be surprisingly difficult to represent universes in sheaf models, as we learnt from works of Chuangjie Xu and Martin Escardo [21] and works of Thomas Streicher [17]. Our approach is here instead a purely syntactical description of a forcing extension of type theory (refining previous work of [7]), which contains a formal symbol for the generic sequence and a proof that it is absurd that this generic sequence is never 0, together with a normalization theorem, from which we can deduce that we cannot prove that this generic sequence has to take the value 0. Since this formal system is an extension of type theory, the independence of Markov’s principle follows.

As stated in [11], which describes an elegant generalization of this principle in type theory, Markov’s principle is an important technical tool for proving termination of computations, and thus can play a crucial role if type theory is extended with general recursion as in [6].

This paper is organized as follows. We first describe the rules of the version of type theory we are considering. This version can be seen as a simplified version of type theory as represented in the system Agda [16], and in particular, contrary to the work [7], we allow $\eta$-conversion, and we express conversion as judgment. Markov’s principle can be formulated in a natural way in this formal system. We describe then the forcing extension of type theory, where we add a Cohen real. For proving normalization, we follow Tait’s computability method [18, 15], but we have to consider an extension of this with a computability relation in order to interpret the conversion judgment. This can be seen as a forcing extension of the technique used in [1]. Using this computability argument, it is then possible to show that we cannot show that the generic sequence has to take the value 0. We end by a refinement of this method, giving a consistent extension of type theory where the negation of Markov’s principle is provable.

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1 The paper [10] presents a model of the calculus of constructions using the idea of modified realizability, and it seems possible to use also this technique to interpret the type theory we consider and prove in this way the independence of Markov’s principle.

2 Some authors define independence in the stronger sense “A statement is independent of a formal system if neither the statement nor its negation is provable in the system”, e.g. [13]. We will establish the independence of Markov’s principle in this stronger sense with the help of known results from the literature.
2 Type theory and forcing extension

A dependent type theory is given by: A syntax describing the set objects of discourse, forms of judgments, and rules of inference for deriving valid judgments.

The syntax of our type theory is given by the grammar:

\[
\begin{align*}
t, u, A, B & := x \mid \bot \mid \text{rec} (\lambda x. A) \mid \text{unitrec} (\lambda x. A) t \mid \text{boolrec} (\lambda x. A) t u \mid \text{natrec} (\lambda x. A) t u \\
& \mid U \mid N \mid N_0 \mid N_1 \mid N_2 \mid 0 \mid 1 \mid S \mid t \mid \Pi(x : A) B \mid \lambda x. t \mid t u \mid \Sigma(x : A) B \mid (t, u) \mid t.1 \mid t.2
\end{align*}
\]

The terms \( N_0, N_1, N_2, \) and \( N \) will denote, respectively, the empty type, the unit type, the type of booleans, and the type of natural numbers. The term \( U \) will denote the universe, i.e. the type of small types. We use the notation \( \mathbb{N} \) as a short hand for the term \( S^0 0 \), where \( S \) is the successor constructor of natural numbers.

2.1 Type system

We describe a type theory with one universe à la Russell, natural numbers, functional extensionality and surjective pairing, hereafter referred to as MLTT. The type theory has the following judgment forms: 1. \( \Gamma \vdash A \). 2. \( \Gamma \vdash A \). 3. \( \Gamma \vdash t : A \). 4. \( \Gamma \vdash A = B \). 5. \( \Gamma \vdash t = u : A \).

The first expresses that \( \Gamma \) is a well-formed context, the second that \( A \) is a type in the context \( \Gamma \), and the third that \( t \) is a term of type \( A \) in the context \( \Gamma \). The fourth and fifth express type and term equality respectively. Below we outline the inference rules of this type theory.

We use the notation \( F \rightarrow G \) for \( \Pi(x : F)G \) when \( G \) doesn’t depend on \( F \) and \( \neg A \) for \( A \rightarrow N_0 \).

### Natural numbers:

\[
\begin{align*}
\Gamma & \vdash N \\
\Gamma & \vdash 0 : N \\
\text{nat-rec-I} & : \Gamma, x : N \vdash F \quad \Gamma, x : N \vdash a : F[0] \\
& \quad \quad \Gamma \vdash g : \Pi(x : N)(F[x] \rightarrow F[S x]) \\
\Gamma & \vdash \text{nat-rec} (\lambda x. F) a g : \Pi(x : N) F \\
\text{nat-rec-0} & : \Gamma, x : N \vdash F \\
& \quad \quad \Gamma, x : N \vdash a : F[0] \\
& \quad \quad \Gamma \vdash g : \Pi(x : N)(F[x] \rightarrow F[S x]) \\
\Gamma & \vdash \text{nat-rec} (\lambda x. F) a g 0 \equiv a_0 : F[0] \\
\text{nat-rec-suc} & : \Gamma, x : N \vdash F \\
& \quad \quad \Gamma, x : N \vdash a : F[0] \\
& \quad \quad \Gamma \vdash n : N \\
& \quad \quad \Gamma \vdash g : \Pi(x : N)(F[x] \rightarrow F[S x]) \\
\Gamma & \vdash \text{nat-rec} (\lambda x. F) a g n \equiv g n (\text{nat-rec} (\lambda x. F) a g n) : F[S n] \\
\text{nat-rec-suc} & : \Gamma, x : N \vdash F \\
& \quad \quad \Gamma, x : N \vdash a : F[0] \\
& \quad \quad \Gamma \vdash g : \Pi(x : N)(F[x] \rightarrow F[S x]) \\
\Gamma & \vdash \text{nat-rec} (\lambda x. F) a g n \equiv \text{nat-rec} (\lambda x. G) a 0 g : \Pi(x : N) F
\end{align*}
\]

### Booleans:

\[
\begin{align*}
\Gamma & \vdash N_2 \\
\Gamma & \vdash 0 : N_2 \\
\Gamma & \vdash 1 : N_2 \\
\text{bool-rec-1} & : \Gamma, x : N_2 \vdash F \\
& \quad \quad \Gamma, x : N_2 \vdash a : F[0] \\
& \quad \quad \Gamma \vdash a_1 : F[1] \\
\Gamma & \vdash \text{bool-rec} (\lambda x. F) a 0 a_1 : \Pi(x : N_2) F \\
\text{bool-rec-0} & : \Gamma, x : N_2 \vdash F \\
& \quad \quad \Gamma, x : N_2 \vdash a : F[0] \\
& \quad \quad \Gamma \vdash a_1 : F[1] \\
\Gamma & \vdash \text{bool-rec} (\lambda x. F) a 0 a_1 0 \equiv a_0 : F[0]
\end{align*}
\]

---

3 This is a type system similar to Martin-löf’s [15] except that we have \( \eta \)-conversion and surjective pairing.
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\[
\text{boolrec-1} \quad \Gamma, x: N_2 \vdash F : \Gamma \vdash a_0 : F[0] \quad \Gamma \vdash a_1 : F[1] \\
\Gamma \vdash \text{boolrec} (\lambda x.F) a_0 a_1 = a_1 : F[1]
\]

\[
\text{boolrec-eq} \quad \Gamma, x: N_2 \vdash F = G \quad \Gamma \vdash a_0 : F[0] \quad \Gamma \vdash a_1 : F[1] \\
\Gamma \vdash \text{natrecc} (\lambda x.F) a_0 a_1 = \text{natrecc} (\lambda x.G) a_0 a_1 : \Pi(x: N_2) F
\]

**Dependent functions:**

\[
\begin{align*}
\Pi \text{-I} & \quad \Gamma \vdash F \quad \Gamma, x:F \vdash G \\
\Gamma & \vdash \Pi(x:F) G \\
\Pi \text{-eq} & \quad \Gamma \vdash F = H \quad \Gamma, x:F \vdash G = E \\
\Gamma & \vdash \Pi(x:F) G = \Pi(x:H) E \\
\lambda \text{-I} & \quad \Gamma, x:F \vdash t:G \\
\Gamma & \vdash \lambda x.t : \Pi(x:F) G \\
\text{fun-ext} & \quad \Gamma \vdash g : \Pi(x:F) G \quad \Gamma \vdash a : F \\
\Gamma & \vdash a g : G[a] \\
\beta & \quad \Gamma, x:F \vdash t:G \\
\Gamma & \vdash (\lambda x.t) a = t[a] : G[a]
\end{align*}
\]

\[
\begin{align*}
\text{fun-ap} & \quad \Gamma \vdash \lambda x.t : \Pi(x:F) G \\
\Gamma & \vdash \lambda x.t \ u : \Sigma(x: H) F \\
\text{fun-eq} & \quad \Gamma \vdash h : \Pi(x:F) G \\
\Gamma & \vdash u = v : F \\
\Gamma & \vdash h u = g v : G[u] \\
\end{align*}
\]

**Dependent product:**

\[
\begin{align*}
\Sigma \text{-I} & \quad \Gamma \vdash F \quad \Gamma, x:F \vdash G \\
\Gamma & \vdash \Sigma(x:F) G \\
\Sigma \text{-eq} & \quad \Gamma \vdash F = H \quad \Gamma, x:F \vdash G = E \\
\Gamma & \vdash \Sigma(x:F) G = \Sigma(x:H) E \\
\text{pr-1} & \quad \Gamma, x:F \vdash G \\
\Gamma & \vdash a : F \\
\Gamma & \vdash b : G[a] \\
\text{pr-e-1} & \quad \Gamma \vdash t : \Sigma(x:F) G \\
\Gamma & \vdash t.1 : F \\
\text{pr-e-2} & \quad \Gamma \vdash t : \Sigma(x:F) G \\
\Gamma & \vdash t.2 : G[t.1] \\
\text{pr-eq-1} & \quad \Gamma \vdash (t, u) : \Sigma(x:F) G \\
\Gamma & \vdash (t, u).1 = t : F \\
\text{pr-eq-2} & \quad \Gamma \vdash (t, u) : \Sigma(x:F) G \\
\Gamma & \vdash (t, u).2 = u : G[t] \\
\end{align*}
\]

\[
\begin{align*}
\text{pr-1} & \quad \Gamma \vdash t : \Sigma(x:F) G \\
\Gamma & \vdash t.1 = u.1 : F \\
\text{pr-eq-1} & \quad \Gamma \vdash t : \Sigma(x:F) G \\
\Gamma & \vdash t.2 = u.2 : G[t.1] \\
\text{pr-ext} & \quad \Gamma \vdash t : \Sigma(x:F) G \\
\Gamma & \vdash u : \Sigma(x:F) G \\
\Gamma & \vdash t.1 = u.1 : F \\
\Gamma & \vdash t.2 = u.2 : G[t.1] \\
\end{align*}
\]

**Universe:**

\[
\begin{align*}
\Gamma & \vdash U \\
\Gamma & \vdash F : U \\
\Gamma & \vdash F = G : U \\
\Gamma & \vdash F = U \\
\Gamma & \vdash N : U \\
\Gamma & \vdash N_2 : U \\
\Gamma & \vdash F : U \\
\Gamma, x : F \vdash G : U \\
\Gamma & \vdash \Pi(x:F) G : U \\
\Gamma & \vdash F : H : U \\
\Gamma, x : F \vdash G = E : U \\
\Gamma & \vdash \Pi(x:F) G = \Pi(x:H) E : U \\
\Gamma & \vdash F : U \\
\Gamma, x : F \vdash G : U \\
\Gamma & \vdash \Sigma(x:F) G : U \\
\Gamma & \vdash F : H : U \\
\Gamma, x : F \vdash G = E : U \\
\Gamma & \vdash \Sigma(x:F) G = \Sigma(x:H) E : U
\end{align*}
\]
Congruence:

\[
\begin{align*}
\Gamma \vdash t : F &\quad \Gamma \vdash F = G \quad \Gamma \vdash F = G &\quad \Gamma \vdash F = H \\
\Gamma \vdash t : F &\quad \Gamma \vdash F = G \quad \Gamma \vdash G = F &\quad \Gamma \vdash t = u : F \\
\Gamma \vdash t = u : F &\quad \Gamma \vdash u = v : F &\quad \Gamma \vdash t = v : F
\end{align*}
\]

For brevity we omitted the rules for the types \( N_0 \) and \( N_1 \).

The following four rules are admissible in the this type system [1]:

\[
\begin{align*}
\Gamma \vdash a : A &\quad \Gamma \vdash a = b : A &\quad \Gamma \vdash G[a] = G[b] &\quad \Gamma \vdash t[a] = t[b] : G[a] \\
\end{align*}
\]

2.2 Markov’s principle

Markov’s principle can be represented in type theory by the type

\[
\text{MP} := \Pi(h : N \rightarrow N_2)[\neg\neg (\Sigma(x : N) \text{IsZero}(h \, x)) \rightarrow \Sigma(x : N) \text{IsZero}(h \, x)]
\]

where \( \text{IsZero} : N_2 \rightarrow U \) is defined by \( \text{IsZero} := \lambda y. \text{boolrec}(\lambda x. U) \, N_1 \, N_0 \, y \).

Note that \( \text{IsZero}(h \, n) \) is inhabited when \( h \, n = 0 \) and empty when \( h \, n = 1 \). Thus \( \Sigma(x : N) \text{IsZero}(h \, x) \) is inhabited if there is \( n \) such that \( h \, n = 0 \).

The main result of this paper is the following:

Theorem 2.1. There is no term \( t \) such that \( \text{MLTT} \vdash t : \text{MP} \).

An extension of MLTT is given by introducing new objects, judgment forms and derivation rules. This means in particular that any judgment valid in MLTT is valid in the extension. A consistent extension is one in which the type \( N_0 \) is uninhabited.

To show Theorem 2.1 we will form a consistent extension of MLTT with a new constant \( f \) where \( \vdash f : N \rightarrow N_2 \) and \( \neg\neg (\Sigma(x : N) \text{IsZero}(f \, x)) \rightarrow \Sigma(x : N) \text{IsZero}(f \, x) \) is not derivable. Thus MP is not derivable in this extension and consequently not derivable in MLTT.

While this is sufficient to establish independence in the sense of non-derivability of MP. To establish the independence of MP in the stronger sense one also needs to show that \( \neg\neg \text{MP} \) is not derivable in MLTT. This can achieved by reference to the work of Aczel [2] where it is shown that MLTT extended with \( \vdash \text{dne} : \Pi(A : U)((\neg\neg A \rightarrow A)) \) is consistent. Since \( h : N \rightarrow N_2, x : N \vdash \text{IsZero}(h \, x) : U \) we have \( h : N \rightarrow N_2 \vdash \Sigma(x : N) \text{IsZero}(h \, x) : U \). Thus

\[
h : N \rightarrow N_2 \vdash \text{dne}(\Sigma(x : N) \text{IsZero}(h \, x)) : \neg\neg (\Sigma(x : N) \text{IsZero}(h \, x)) \rightarrow \Sigma(x : N) \text{IsZero}(h \, x)
\]

By \( \lambda \) abstraction we have \( \vdash \lambda h. \text{dne}(\Sigma(x : N) \text{IsZero}(h \, x)) : \text{MP} \). We can then conclude that there is no term \( t \) such that \( \text{MLTT} \vdash t : \neg\neg \text{MP} \).

Finally, we will reﬁne the result of Theorem 2.1 by building a consistent extension of MLTT where \( \neg\neg \text{MP} \) is derivable.

2.3 Forcing extension

A condition \( p \) is a graph of a partial ﬁnite function from \( \mathbb{N} \) to \{0, 1\}. We denote by \( \langle \rangle \) the empty condition. We write \( p(n) = b \) when \( (n, b) \in p \). We say \( q \) extends \( p \) (written \( q \leq p \)) if \( p \) is a subset of \( q \). A condition can be thought of as a compact open in Cantor space \( 2^{\mathbb{N}} \). Two conditions \( p \) and \( q \) are compatible if \( p \cup q \) is a condition and we write \( pq \) for \( p \cup q \),
otherwise they are incompatible. If \( n \notin \text{dom}(p) \) we write \( p(n, 0) \) for \( p \cup \{(n, 0)\} \) and \( p(n, 1) \) for \( p \cup \{(n, 1)\} \). We define the notion of partition corresponding to the notion of finite covering of a compact open in Cantor space.

**Definition 2.2 (Partition).** We write \( p < p_1, \ldots, p_n \) to say that \( p_1, \ldots, p_n \) is a partition of \( p \) and we define it as follows:

1. \( p < p \).
2. If \( n \notin \text{dom}(p) \) and \( p(n, 0) < \ldots, q_i, \ldots \) and \( p(n, 1) < \ldots, r_j, \ldots \) then \( p < \ldots, q_i, \ldots, r_j, \ldots \). Note that if \( p < p_1, \ldots, p_n \) then \( p_i \) and \( p_j \) are incompatible whenever \( i \neq j \). If moreover \( q \leq p \) then \( q < q_1, \ldots, q_p, \ldots \) where \( p_i \) is compatible with \( q \).

We extend the given type theory by annotating the judgments with conditions, i.e. replacing each judgment \( \Gamma \vdash J \) in the given type system with a judgment \( \Gamma \vdash_p J \).

In addition we add the locality rule: \( \frac{\Gamma \vdash p_1 J \ldots \Gamma \vdash p_n J}{\Gamma \vdash_p J \ p < p_1 \ldots p_n} \).

We add a term \( f \) for the generic point along with the introduction and conversion rules:

\[
\begin{align*}
\Gamma \vdash_p f : N \Rightarrow N_2 & \quad \Gamma \vdash_p f \pi = p(n) : N_2 \ n \in \text{dom}(p) .
\end{align*}
\]

We add a term \( w \) and the rule: \( \Gamma \vdash_p w : \neg\neg(\Sigma(x : N) \text{IsZero}(f(x))) \).

Since \( w \) inhabits \( \neg\neg(\Sigma(x : N) \text{IsZero}(f(x))) \), our goal is then to show that no term inhabits \( \Sigma(x : N) \text{IsZero}(f(x)) \).

It follows directly from the description of the forcing extension that:

**Lemma 2.3.** If \( \Gamma \vdash J \) then \( \Gamma \vdash_p J \) for all \( p \). In particular, if \( \vdash t : A \) then \( \vdash_p t : A \) for all \( p \).

Note that if \( q \leq p \) and \( \Gamma \vdash_p J \) then \( \Gamma \vdash_q J \) (monotonicity). A statement \( A \) (represented as a closed type) is derivable in this extension if \( \vdash_p t : A \) for some \( t \), which in turn implies \( \vdash_p t : A \) for all \( p \).

Similarly to [7] we can state a conservativity result for this extension. Let \( \vdash g : N \Rightarrow N_2 \) and \( \vdash v : \neg\neg(\Sigma(x : N) \text{IsZero}(g(x))) \) be two terms of standard type theory. We say that \( g \) is compatible with a condition \( p \) if \( g \) is such that \( \vdash g \pi = b : N_2 \) whenever \( (n, b) \in p \) and \( \vdash g \pi = 0 : N_2 \) otherwise. We say that \( v \) is compatible with a condition \( p \) if \( g \) is compatible with \( p \) and \( v \) is given by \( v = \lambda x.x(\tilde{n}_p, 0) \) where \( n_p \) is the smallest natural number such that \( n_p \notin \text{dom}(p) \). To see that \( v \) is well typed, note that by design \( \Gamma \vdash g \tilde{n}_p = 0 : N_2 \) thus \( \Gamma \vdash \text{IsZero}(g(\tilde{n}_p)) = N_1 \) and \( \Gamma \vdash (\tilde{n}_p, 0) : \Sigma(x : N) \text{IsZero}(g(x)) \). We have then \( \Gamma, x : \neg\neg(\Sigma(y : N) \text{IsZero}(g(y))) \vdash x(\tilde{n}_p, 0) : N_0 \) thus \( \Gamma \vdash \lambda x.x(\tilde{n}_p, 0) : \neg\neg(\Sigma(y : N) \text{IsZero}(g(y))) \).

**Lemma 2.4 (Conservativity).** Let \( \vdash g : N \Rightarrow N_2 \) and \( \vdash v : \neg\neg(\Sigma(x : N) \text{IsZero}(g(x))) \) be compatible with \( p \). If \( \Gamma \vdash_p J \) then \( \Gamma[g/f, v/w] \vdash J[g/f, v/w] \), i.e. replacing \( f \) with \( g \) then \( w \) with \( v \) we obtain a valid judgment in standard type theory. In particular, if \( \Gamma \vdash_p J \) where neither \( f \) nor \( w \) occur in \( J \) then \( \Gamma \vdash J \) is a valid judgment in standard type theory.

**Proof.** The proof is by induction on the type system and it is straightforward for all the standard rules. For \((f\text{-eval})\) we have \( (f \pi)g/f, v/w) := g \pi \) and since \( g \) is compatible with \( p \) we have \( \Gamma[g/f, v/w] \vdash g \pi = p(n) : N_2 \) whenever \( n \in \text{dom}(p) \). For \((w\text{-term})\) we have \( (w : \neg\neg(\Sigma(x : N) \text{IsZero}(f(x))))g/f, v/w) := (w : \neg\neg(\Sigma(x : N) \text{IsZero}(g(x))))v/w := v : \neg\neg(\Sigma(x : N) \text{IsZero}(g(x))) \). For \((\text{loc})\) the statement follows from the observation that when \( g \) is compatible with \( p \) and \( p < p_1, \ldots, p_n \) then \( g \) is compatible with exactly one \( p_i \) for \( 1 \leq i \leq n \).
3 A Semantics of the forcing extension

In this section we outline a semantics for the forcing extension given in the previous section. We will interpret the judgments of type theory by computability predicates and relations defined by reducibility to computable weak head normal forms.

3.1 Reduction rules

We extend the $\beta, \iota$ conversion with $f \mathord{\mapsto_p} b$ whenever $(n, b) \in p$. In order to ease the presentation of the proofs and definitions we introduce evaluation contexts following [20].

$$
\begin{align*}
E & ::= [ ] \mid E \, u \mid E.1 \mid E.2 \mid \Sigma E \mid f E \\
\downarrow \text{rec} (\lambda x. C) \mid \text{unitrec} (\lambda x. C) \mid a E \mid \text{boolrec} (\lambda x. C) a_0 \, a_1 \mid \text{natrec} (\lambda x. C) c \, g \, E
\end{align*}
$$

An expression $E[e]$ is then the expression resulting from replacing the hole $[ ]$ by $e$. We reserve the symbols $E$ and $C$ for evaluation contexts. We have the following reduction rules:

- $\text{unitrec} (\lambda x. C) c \, 0 \rightarrow c$
- $\text{boolrec} (\lambda x. C) c_0 \, c_1 \, 0 \rightarrow c_0$
- $\text{boolrec} (\lambda x. C) c_0 \, c_1 \, 1 \rightarrow c_1$
- $\text{natrec} (\lambda x. C) c \, g \, 0 \rightarrow c$
- $\text{natrec} (\lambda x. C) c \, g \, (S \, k) \rightarrow g \, (\text{natrec} (\lambda x. C) c \, g \, k)$

\[
\begin{align*}
(\lambda x. t) & \rightarrow t[a/x] & (u, v).1 & \rightarrow u & (u, v).2 & \rightarrow v \\
e & \rightarrow e' & t \text{-red} & k \in \text{dom}(p) & e & \rightarrow_p e' \\
e & \rightarrow_p e' & \text{f-red} & f \, k \rightarrow_p p(k) & E[e] & \Rightarrow_p E[e']
\end{align*}
\]

Note that we reduce under $S$.

The relation $\Rightarrow$ is monotone, that is if $q \leq p$ and $t \Rightarrow_p u$ then $t \Rightarrow_q u$. We will also need to show that the reduction is local, i.e. if $p < p_1, \ldots, p_n$ and $t \Rightarrow_p u$ then $t \Rightarrow_p u$.

**Lemma 3.1.** If $m \notin \text{dom}(p)$ and $t \Rightarrow_p (m, 0) \, u$ and $t \Rightarrow_p (m, 1) \, u$ then $t \Rightarrow_p u$.

**Proof.** By induction on the derivation of $t \Rightarrow_p (m, 0) \, u$. If $t \Rightarrow_p (m, 0) \, u$ is derived by (f-red) then $t := f \, k$ and $u := p(m, 0)(k)$ for some $k \in \text{dom}(p(m, 0))$. But since we also have a reduction $f \, k \rightarrow_p (m, 1) \, u$, we have $p(m, 1)(k) := u := p(m, 0)(k)$ which could only be the case if $k \in \text{dom}(p)$. Thus we have a reduction $f \, k \rightarrow_p u := p(k)$. Alternatively, we have a derivation $t \rightarrow_p u$ in which case we have $t \rightarrow_p u$ directly.

**Lemma 3.2.** If $m \notin \text{dom}(p)$ and $t \Rightarrow_p (m, 0) \, u$ and $t \Rightarrow_p (m, 1) \, u$ then $t \Rightarrow_p u$.

**Proof.** From the reduction $t \Rightarrow (m, k) \, u$ we have $t := E[e], u := E[e']$ and $e \rightarrow_p E[e']$ for some context $E$. But then we also have a reduction $E[e] \Rightarrow_p E[e']$, thus $e \rightarrow_p E[e']$. By Lemma 3.1, we have $e \rightarrow_p E[e']$ and thus $E[e] \Rightarrow_p E[e']$.

**Lemma 3.3.** Let $q \leq p$. If $t \Rightarrow_q u$ then either $t \Rightarrow_p u$ or $t$ has the form $E[f \, m]$ for some $m \in \text{dom}(q) \setminus \text{dom}(p)$.

**Proof.** By induction on the derivation of $t \Rightarrow_q u$. If the reduction $t \Rightarrow_q u$ has the form $f \, k \rightarrow_q q(k)$ then either $k \notin \text{dom}(p)$ and the statement follows or $k \in \text{dom}(p)$ and we have $t \rightarrow_p u$. Alternatively, we have $t \rightarrow u$ and immediately $t \rightarrow_p u$.

**Lemma 3.4.** Let $q \leq p$. If $t \Rightarrow_q u$ then either $t \Rightarrow_p u$ or $t$ has the form $E[f \, m]$ for some $m \in \text{dom}(q) \setminus \text{dom}(p)$.
Proof. If \( t \Rightarrow q u \) then \( t := \mathbb{E}[e], u := \mathbb{E}[e'] \) and \( e \Rightarrow e' \) for some context \( E \). By Lemma 3.3 either \( e := \mathbb{C}[m] \) for \( m \notin \text{dom}(p) \) and the statement follows or \( e \Rightarrow e' \) in which case we have \( t \Rightarrow_p u \). 

Corollary 3.5. For any condition \( p \) and \( m \notin \text{dom}(p) \). Let \( t \Rightarrow_{p(m,0)} u \) and \( t \Rightarrow_{p(m,1)} v \). If \( u := v \) then \( t \Rightarrow_p u \); otherwise, \( t \) has the form \( \mathbb{E}[f m] \).

Proof. Follows by Lemma 3.2 and Lemma 3.4.

Next we define the relation \( p \vdash t \Rightarrow u : A \) to mean \( t \Rightarrow_p u \) and \( \vdash_p t = u : A \) and we write \( p \vdash A \Rightarrow B \) for \( p \vdash A \Rightarrow B : U \). We note that it holds that if \( p \vdash t \Rightarrow u : \Pi(x : F)G \) and \( \vdash a : F \) then \( p \vdash ta \Rightarrow ua : G[a] \) and if \( p \vdash t \Rightarrow u : \Sigma(x : F)G \) then \( p \vdash t.1 \Rightarrow u.1 : F \) and \( p \vdash t.2 \Rightarrow u.2 \Sigma[t,1] \). We define a closure for this relation as follows:

\[
\begin{align*}
\vdash_p t & : A \\
p \vdash t \Rightarrow^* u : A & \quad p \vdash t \Rightarrow u : A \quad p \vdash u \Rightarrow^* v : A \\
p \vdash p_A & \quad p \vdash A \Rightarrow B \\
p \vdash A \Rightarrow^* B & \quad p \vdash A \Rightarrow B \quad p \vdash B \Rightarrow^* C \\
p \vdash A \Rightarrow^* C & 
\end{align*}
\]

A term \( t \) is \( \text{in-\mathcal{P}hnf} \) if whenever \( t \Rightarrow_p u \) then \( t := u \). A \( \text{hnf} \) is canonical if it has the form \( 0,1,\pi, \lambda x.t, f, \text{w}, \text{rec}(\lambda x.C), \text{unitrec}(\lambda x.C) a, \text{boolrec}(\lambda x.C) a_0 a_1, \text{natrec}(\lambda x.C) c, g, \text{N}_0, \text{N}_1, \text{N}_2, N, U, \Pi(x : F)G, \text{or } \Sigma(x : F)G \). A \( \text{p-\mathcal{P}hnf} \) is proper if it is canonical or it is of the form \( \mathbb{E}[f m] \) for \( k \notin \text{dom}(p) \).

We have the following corollaries to Lemma 3.2 and Corollary 3.5.

Corollary 3.6. Let \( m \notin \text{dom}(p) \). Let \( p(m,0) \vdash t \Rightarrow_{p(m,0)} u : A \) and \( p(m,1) \vdash t \Rightarrow_{p(m,1)} v : A \). If \( u := v \) then \( p \vdash t \Rightarrow u : A \); otherwise \( t \) has the form \( \mathbb{E}[f m] \).

Corollary 3.7. If \( p \vdash t \Rightarrow u : A \) and \( q \leq p \) then \( q \vdash t \Rightarrow u : A \). If \( p < p_1, \ldots, p_n \) and \( p_i \vdash t \Rightarrow u : A \) for all \( i \) then \( p \vdash t \Rightarrow u : A \).

Proof. Let \( q \leq p \). If \( t \Rightarrow_p u \) we have \( t \Rightarrow_q u \) and if \( \vdash_p t = u : A \) then \( \vdash_q t = u : A \). Thus \( q \vdash t \Rightarrow u : A \) whenever \( p \vdash t \Rightarrow u : A \). Thus \( p \vdash t \Rightarrow u : A \) for all \( i \) then \( \vdash_p t = u : A \). Thus we have \( p \vdash t \Rightarrow u : A \) whenever \( p_i \vdash t \Rightarrow u : A \) for all \( i \).

From the above we can show that closure \( \Rightarrow^* \) is monotone, it is not however local.

For a closed term \( \vdash p \vdash t : A \), we say that \( t \) has a \( p\)-\( \text{hnf} \) if \( p \vdash t \Rightarrow^* u : A \) and \( u \) is \( p\)-\( \text{hnf} \). If moreover \( u \) is canonical, respectively proper, we say that \( t \) has a canonical, respectively proper, \( p\)-\( \text{hnf} \). Since the reduction relation is deterministic we have

Lemma 3.8. A term \( p \vdash t : A \) has at most one \( p\)-\( \text{hnf} \).

Corollary 3.9. Let \( p \vdash t : A \) and \( m \notin \text{dom}(p) \). If \( t \) has proper \( p(m,0)-\text{\mathcal{P}hnf} \) and a proper \( p(m,1)-\text{\mathcal{P}hnf} \) then \( t \) has a proper \( p\)-\( \text{\mathcal{P}hnf} \).

Proof. Let \( p(m,0) \vdash t \Rightarrow^* u : A \) and \( p(m,1) \vdash t \Rightarrow^* v : A \) with \( u \) in proper \( p(m,0)-\text{\mathcal{P}hnf} \) and \( v \) in proper \( p(m,1)-\text{\mathcal{P}hnf} \). If \( t := u \) or \( t := v \) then \( t \) is already in proper \( p\)-\( \text{\mathcal{P}hnf} \). Alternatively we have reductions \( p(m,0) \vdash t \Rightarrow u_1 : A \) and \( p(m,1) \vdash t \Rightarrow v_1 : A \). By Corollary 3.6 either \( t \) is in proper \( p\)-\( \text{\mathcal{P}hnf} \) or \( u_1 := v_1 \) and \( p \vdash t \Rightarrow u_1 : A \). It then follows by induction that \( u_1 \), and thus \( t \), has a proper \( p\)-\( \text{\mathcal{P}hnf} \).
3.2 Computability predicate and relation

We define inductively a forcing relation $p \vDash A$ to express that a type $A$ is computable at $p$. Mutually by recursion we define relations $p \vDash a : A$, $p \vDash A = B$, and $p \vDash a = b : A$. The definition fits the generalized mutual induction-recursion schema [8]4.

**Definition 3.10** (Computability predicate and relation).

(FNa) If $p \vDash A \Rightarrow \ast \text{ } N_0$ then $p \vDash A$.
1. $p \vDash t : A$ does not hold for all $t$.
2. $p \vdash t = u : A$ does not hold for all $t$ and $u$.
3. If $p \vDash B$ then $p \vDash A = B$ if
   (i) $p \vDash B \Rightarrow \ast \text{ } N_0$.
   (ii) $p \vDash B \Rightarrow \ast \text{ } A$ for some $m \notin \text{dom}(p)$ and $p(m,i) \vdash A = B$ for all $i \in \{0,1\}$.

(FNb) If $p \vDash A \Rightarrow \ast \text{ } N_1$ then $p \vdash A$.
1. $p \vDash t : A$ if
   (i) $p \vDash t \Rightarrow \ast \text{ } 0 : A$.
   (ii) $p \vDash t \Rightarrow \ast \text{ } A$ for some $m \notin \text{dom}(p)$ and $p(m,i) \vdash t : A$ for all $i \in \{0,1\}$.
3. If $p \vDash t : A$ and $p \vDash u : A$ then $p \vDash t = u : A$ if
   (i) $p \vDash t \Rightarrow \ast \text{ } 0 : A$ and $p \vdash u \Rightarrow \ast \text{ } 0 : A$.
   (ii) $p \vDash t \Rightarrow \ast \text{ } A$ for some $m \notin \text{dom}(p)$ and $p(m,i) \vdash t = u : A$ for all $i \in \{0,1\}$.
   (iii) $p \vDash u \Rightarrow \ast \text{ } A$ for some $m \notin \text{dom}(p)$ and $p(m,i) \vdash t = u : A$ for all $i \in \{0,1\}$.
4. If $p \vDash B$ then $p \vDash A = B$ if
   (i) $p \vdash B \Rightarrow \ast \text{ } N_1$.
   (ii) $p \vdash B \Rightarrow \ast \text{ } A$ for some $m \notin \text{dom}(p)$ and $p(m,i) \vdash A = B$ for all $i \in \{0,1\}$.

(FNc) If $p \vDash A \Rightarrow \ast \text{ } N_2$ then $p \vDash A$.
1. $p \vDash t : A$ if
   (i) $p \vDash t \Rightarrow \ast \text{ } 1 : A$ for some $b \in \{0,1\}$.
   (ii) $p \vDash t \Rightarrow \ast \text{ } A$ for some $m \notin \text{dom}(p)$ and $p(m,i) \vdash t : A$ for all $i \in \{0,1\}$.
3. If $p \vDash t : A$ and $p \vDash u : A$ then $p \vDash t = u : A$ if
   (i) $p \vDash t \Rightarrow \ast \text{ } 1 : A$ and $p \vdash u \Rightarrow \ast \text{ } 1 : A$ for some $b \in \{0,1\}$.
   (ii) $p \vDash t \Rightarrow \ast \text{ } A$ for some $m \notin \text{dom}(p)$ and $p(m,i) \vdash t = u : A$ for all $i \in \{0,1\}$.
   (iii) $p \vDash u \Rightarrow \ast \text{ } A$ for some $m \notin \text{dom}(p)$ and $p(m,i) \vdash t = u : A$ for all $i \in \{0,1\}$.
4. If $p \vDash B$ then $p \vDash A = B$ if
   (i) $p \vdash B \Rightarrow \ast \text{ } N_2$.
   (ii) $p \vdash B \Rightarrow \ast \text{ } A$ for some $m \notin \text{dom}(p)$ and $p(m,i) \vdash A = B$ for all $i \in \{0,1\}$.

(FNd) If $p \vDash A \Rightarrow \ast \text{ } N$ then $p \vDash A$.
1. $p \vDash t : A$ if
   (i) $p \vDash t \Rightarrow \ast \text{ } 0 : A$ for some $n \in \mathbb{N}$.
   (ii) $p \vDash t \Rightarrow \ast \text{ } A$ for some $m \notin \text{dom}(p)$ and $p(m,i) \vdash t : A$ for all $i \in \{0,1\}$.
3. If $p \vDash t : A$ and $p \vDash u : A$ then $p \vDash t = u : A$ if
   (i) $p \vDash t \Rightarrow \ast \text{ } 0 : A$ and $p \vdash u \Rightarrow \ast \text{ } 0 : A$ for some $n \in \mathbb{N}$.
   (ii) $p \vDash t \Rightarrow \ast \text{ } A$ for some $m \notin \text{dom}(p)$ and $p(m,i) \vdash t = u : A$ for all $i \in \{0,1\}$.
   (iii) $p \vDash u \Rightarrow \ast \text{ } A$ for some $m \notin \text{dom}(p)$ and $p(m,i) \vdash t = u : A$ for all $i \in \{0,1\}$.
4. If $p \vDash B$ then $p \vDash A = B$ if
   (i) $p \vdash B \Rightarrow \ast \text{ } N$.
   (ii) $p \vdash B \Rightarrow \ast \text{ } A$ for some $m \notin \text{dom}(p)$ and $p(m,i) \vdash A = B$ for all $i \in \{0,1\}$.

---

4 However, for the canonical proof below we actually need something weaker than an inductive-recursive definition (arbitrary fixed-point instead of least fixed-point), reflecting the fact that the universe is defined in an open way [15].
(FΠ) If \( p \vdash A \Rightarrow^* \Pi(x:F)G \) then \( p \Vdash A \) if \( p \Vdash F \) and for all \( q \leq p \), \( q \Vdash G[a] \) whenever \( q \Vdash a : F \) and \( q \Vdash G[b] \) whenever \( q \Vdash a = b : F \).
1. If \( p \vdash f : A \) then \( p \vdash f : A \) if for all \( q \leq p \), \( q \vdash f : G[a] \) whenever \( q \vdash a : F \) and \( q \vdash f = b : G[a] \) whenever \( q \vdash a = b : F \).
2. If \( p \vdash f : A \) and \( p \vdash g : A \) then \( p \vdash f = g : A \) if for all \( q \leq p \), \( q \vdash f = g : G[a] \) whenever \( q \vdash a : F \).
3. If \( p \vdash B \) then \( p \vdash A = B \) if
   (i) \( p \vdash A = B \) and \( p \vdash B \Rightarrow^* \Pi(x:H)E \) and \( p \vdash F = H \) and for all \( q \leq p \), \( q \vdash G[a] = E[a] \) whenever \( q \vdash a : F \).
   (ii) \( p \vdash B \Rightarrow^* \mathbb{E}[f \overline{m}] \) for some \( m \notin \text{dom}(p) \) and \( p(m,i) \Vdash A = B \) for all \( i \in \{0,1\} \).
(FΣ) If \( p \vdash A \Rightarrow^* \Sigma(x:F)G \) then \( p \vdash A \) if \( p \vdash F \) and for all \( q \leq p \), \( q \vdash G[a] \) whenever \( q \vdash a : F \) and \( q \vdash G[b] \) whenever \( q \vdash a = b : F \).
1. If \( p \vdash t : A \) then \( p \vdash t : A \) if \( p \vdash t.1 : F \) and \( p \vdash t.2 : G[t.1] \).
2. If \( p \vdash t : A \) and \( p \vdash u : A \) then \( p \vdash t = u : A \) if \( p \vdash t.1 = u.1 : F \) and \( p \vdash t.2 = u.2 : G[t.1] \).
3. If \( p \vdash B \) then \( p \vdash A = B \) if
   (i) \( p \vdash A = B \) and \( p \vdash B \Rightarrow^* \Sigma(x:H)E \) and \( p \vdash F = H \) and for all \( q \leq p \), \( q \vdash G[a] = E[a] \) whenever \( q \vdash a : F \).
   (ii) \( p \vdash B \Rightarrow^* \mathbb{E}[f \overline{m}] \) for some \( m \notin \text{dom}(p) \) and \( p(m,i) \Vdash A = B \) for all \( i \in \{0,1\} \).
(FU) If \( p \vdash A \Rightarrow^* U \) then \( p \vdash A \).
1. \( p \vdash C : A \) if
   (i) \( p \vdash C \Rightarrow^* M : A \) for \( M \in \{N_0,N_1,N_2,N\} \).
   (ii) \( p \vdash C \Rightarrow^* \Pi(x:F)G : A \) and \( p \vdash F : A \) and for all \( q \leq p \), \( q \vdash G[a] : A \) whenever \( q \vdash a : F \) and \( q \vdash G[b] : A \) whenever \( q \vdash a = b : F \).
   (iii) \( p \vdash C \Rightarrow^* \Sigma(x:F)G : A \) and \( p \vdash F : A \) and for all \( q \leq p \), \( q \vdash G[a] : A \) whenever \( q \vdash a : F \) and \( q \vdash G[b] : A \) whenever \( q \vdash a = b : F \).
   (iv) \( p \vdash C \Rightarrow^* \mathbb{E}[f \overline{m}] : A \) for some \( m \notin \text{dom}(p) \) and \( p(m,i) \Vdash C : A \) for all \( i \in \{0,1\} \).
5. If \( p \vdash C : A \) and \( p \vdash D : A \) then \( p \vdash C = D : A \) if
   (i) \( p \vdash C \Rightarrow^* M : A \) and \( D \Rightarrow^* M : A \) for \( M \in \{N_0,N_1,N_2,N\} \).
   (ii) \( p \vdash C \Rightarrow^* \Pi(x:F)G : A \) and \( p \vdash D \Rightarrow^* \Pi(x:H)E : A \) and \( p \vdash F = H : A \) and for all \( q \leq p \), \( q \vdash G[a] = E[a] : A \) whenever \( q \vdash a : F \).
   (iii) \( p \vdash C \Rightarrow^* \Sigma(x:F)G : A \) and \( p \vdash D \Rightarrow^* \Sigma(x:H)E : A \) and \( p \vdash F = H : A \) and for all \( q \leq p \), \( q \vdash G[a] = E[a] : A \) whenever \( q \vdash a : F \).
   (iv) \( p \vdash C \Rightarrow^* \mathbb{E}[f \overline{m}] : A \) for some \( m \notin \text{dom}(p) \) and \( p(m,i) \Vdash C = D : A \) for all \( i \in \{0,1\} \).
   (v) \( p \vdash D \Rightarrow^* \mathbb{E}[f \overline{m}] : A \) for some \( m \notin \text{dom}(p) \) and \( p(m,i) \Vdash C = D : A \) for all \( i \in \{0,1\} \).
6. If \( p \vdash B \) then \( p \vdash A = B \) if \( p \vdash B \Rightarrow^* U \).
(FLoc) If \( p \vdash A \Rightarrow^* \mathbb{E}[f \overline{m}] \) for some \( m \notin \text{dom}(p) \) and \( p(m,i) \Vdash A \) for all \( i \in \{0,1\} \) then \( p \vdash A \).
1. If \( p(m,i) \Vdash t : A \) for all \( i \in \{0,1\} \) then \( p \vdash t : A \).
2. If \( p \vdash t : A \) and \( p \vdash u : A \) and \( p(m,i) \vdash t : A \) for all \( i \in \{0,1\} \) then \( p \vdash t = u : A \).
3. If \( p \vdash B \) then \( p \vdash A = B \) if \( p(m,i) \vdash A = B \) for all \( i \in \{0,1\} \).

We note from the definition that when \( p \vdash A = B \) then \( p \vdash A \) and \( p \vdash B \), when \( p \vdash a : A \) then \( p \vdash a = b : A \) and when \( p \vdash a : A \) and \( p \vdash b : A \). We remark also if \( p \vdash A \Rightarrow^* U \) then \( A := U \) since we have only one universe.

The clause (FLoc) gives semantics to variable types. For example, if \( p := \{(0,0)\} \) and \( q := \{(0,1)\} \) the type \( R := \text{boolrec} (\lambda x.U) N_1 N (f0) \) has reductions \( p \vdash R \Rightarrow^* N_1 \) and \( q \vdash R \Rightarrow^* N. \) Thus \( p \vdash R \) and \( q \vdash R \) and since \( \{\} \prec p,q \) we have \( \{\} \vdash R. \)
Immediately from Definition 3.10 we get:

- **Lemma 3.11.** If \( p \vdash A \) then \( \vdash_p A \). If \( p \vdash a:A \) then \( \vdash_p a:a.A \).

- **Lemma 3.12.** If \( p \vdash A \) then there is a partition \( p < p_1, \ldots, p_n \) where \( A \) has a canonical \( p_i\)-whnf for all \( i \).

**Proof.** The statement follows from the definition by induction on the derivation of \( p \vdash A \).

- **Corollary 3.13.** Let \( p < p_1, \ldots, p_n \). If \( p_i \vdash A \) for all \( i \) then \( A \) has a proper \( p\)-whnf.

**Proof.** Follows from Lemma 3.12 and Corollary 3.9 by induction on the partition.

- **Lemma 3.14.** If \( p \vdash A \) and \( q \leq p \) then \( q \vdash A \).

**Proof.** Let \( p \vdash A \) and \( q \leq p \). By induction on the derivation of \( p \vdash A \):

- (F\(_N\)) Since \( p \vdash A \Rightarrow^* N \) and the reduction relation is monotone we have \( q \vdash A \Rightarrow^* N \), thus \( q \vdash A \). The statement follows similarly for (F\(_N_1\)), (F\(_N_2\)), (F\(_U\)), and (F\(_V\)).

- (F\(_I\)) Let \( p \vdash A \Rightarrow^* \Pi(x:F)G \). Since \( p \vdash F \), by induction \( q \vdash F \). Let \( s \leq q \), we have then \( s \leq q \). It then follows from \( p \vdash A \) that \( s \vdash G[a] \) whenever \( s \vdash a:a.F \) and \( s \vdash G[a] = G[b] \) whenever \( s \vdash a = b:F \). Thus \( q \vdash A \). The statement follows similarly for (F\(_S\)).

- (F\(_L\)) Let \( p \vdash A \Rightarrow^* E[f;m] \). If \( m \in \text{dom}(q) \) then \( q \leq p(m,0) \) or \( q \leq p(m,1) \) and since \( p(m,i) \vdash A \), by induction \( q \vdash A \). Alternatively, \( q \vdash A \Rightarrow^* E[f;m] \). But \( q < q(m,0), q(m,1) \) and \( q(m,i) \leq p(m,i) \). By induction \( q(m,i) \vdash A \) for all \( i \in \{0,1\} \) and thus \( q \vdash A \).

- **Lemma 3.15.** If \( p \vdash t:A \) and \( q \leq p \) then \( q \vdash t:A \).

**Proof.** Let \( p \vdash t:A \) and \( q \leq p \). By induction on the derivation of \( p \vdash A \):

- (F\(_N\)) Since \( p \vdash A \Rightarrow^* N \) then \( q \vdash A \Rightarrow^* N \). By induction on the derivation of \( p \vdash t:A \).

- (F\(_N_1\)) Let \( p \vdash t \Rightarrow^* \Pi a:A \) for \( n \in \mathbb{N} \) then \( q \vdash t \Rightarrow^* \Pi a:A \), hence, \( q \vdash t:A \). Alternatively, let \( p \vdash t \Rightarrow^* E[f\[k\]]:A \) for some \( k \notin \text{dom}(p) \) and \( p(k,b) \vdash t:A \) for all \( b \in \{0,1\} \). If \( k \in \text{dom}(q) \) then \( q \leq p(k,1) \) or \( q \leq p(k,0) \) and in either case, by induction, \( q \vdash t:A \). Otherwise, we have \( q(k,b) \leq p(k,b) \) and by induction \( q(k,b) \vdash t:A \) for all \( b \). By definition the \( q \vdash t:A \).

- (F\(_U\)) We can show the statement by a proof similar to that of Lemma 3.14.

- (F\(_I\)) Let \( p \vdash A \Rightarrow^* \Pi(x:F)G \). We have \( q \vdash A \Rightarrow^* \Pi(x:F)G \). From \( p \vdash t:A \) we have \( q \vdash t:A \).

- (F\(_S\)) Let \( p \vdash A \Rightarrow^* \Sigma(x:F)G \). We have \( q \vdash A \Rightarrow^* \Sigma(x:F)G \). From \( p \vdash t:A \) we have \( q \vdash t:A \).

- (F\(_L\)) Let \( p \vdash A \Rightarrow^* E[f;m] \). Since \( p \vdash t:A \) we have \( p(k,b) \vdash t:A \) for all \( b \in \{0,1\} \). If \( k \in \text{dom}(q) \) then \( q \leq p(k,0) \) or \( q \leq p(k,1) \) and by induction \( q \vdash t:A \). Otherwise, \( q \vdash A \Rightarrow^* E[f;m] \) and since \( q(k,b) \leq p(k,b) \), by induction, \( q(k,b) \vdash t:A \) for all \( b \). By definition \( q \vdash t:A \).

Using similar arguments we can also show the following two statements:

- **Lemma 3.16.** Let \( p \vdash A \). If \( p \vdash A = B \) and \( q \leq p \) then \( q \vdash A = B \).

- **Lemma 3.17.** Let \( p \vdash A \). If \( p \vdash t = u:A \) and \( q \leq p \) then \( q \vdash t = u:A \).

We collect the results of Lemmas 3.14, Lemma 3.15, Lemma 3.17, and Lemma 3.16 in the following corollary.
Corollary 3.18 (Monotonicity). If \( p \vdash J \) and \( q \leq p \) then \( q \vdash J \).

We write \( \vdash J \) when \( \emptyset \vdash J \). By monotonicity \( \vdash J \) iff \( p \vdash J \) for all \( p \).

Lemma 3.19. If \( p(m,0) \vdash A \) and \( p(m,1) \vdash A \) for some \( m \notin \text{dom}(p) \) then \( p \vdash A \).

Proof. By Corollary 3.13, either \( A \) has a canonical \( p \)-whnf or \( p \vdash A \Rightarrow \exists \{f \} \) for \( k \notin \text{dom}(p) \).

If \( p \vdash A \Rightarrow \exists \{f \} \) with \( M \in \{{N_0}, {N_1}, {N_2}, N\} \) then we have immediately that \( p \vdash A \).

If \( p \vdash A \Rightarrow \exists \{f \} \) with \( M \) of the form \( \Pi(x:F)G \) or \( \Sigma(x:F)G \) then \( p(m,b) \vdash A \Rightarrow \exists \{f \} \) for all \( b \in \{0,1\} \). Since \( p(m,b) \vdash A \) we have \( p(m,b) \vdash F \) for all \( b \) and by induction \( p \vdash F \). Let \( q \leq p \) and \( q \vdash a:F \). If \( m \in \text{dom}(q) \) then \( q \leq p(m,b) \) for some \( b \in \{0,1\} \). Assume, w.l.o.g, \( q \leq p(m,0) \). Since \( p(m,0) \vdash A \) we have by the definition that \( q \vdash G[a] \).

Alternatively, if \( m \notin \text{dom}(q) \) we have a partition \( q \leq q(m,0), q(m,1) \). By monotonicity \( q(m,b) \vdash a:F \), and since \( q(m,b) \leq p(m,b) \), we have \( q(m,b) \vdash G[a] \) for all \( b \in \{0,1\} \). By induction \( q \vdash G[a] \).

Similarly we can show \( q \vdash G[a] = G[b] \) whenever \( q \vdash a = b:F \).

Alternatively, \( p \vdash A \Rightarrow \exists \{f \} \) for some \( k \notin \text{dom}(p) \). If \( k = m \) then by the definition \( p \vdash A \).

Otherwise, \( p(m,0) \vdash A \Rightarrow \exists \{f \} \) and by the definition \( p(m,0)(k,b) \vdash A \) for all \( b \in \{0,1\} \). Similarly, \( p(m,1)(k,b) \vdash A \) for all \( b \in \{0,1\} \). But \( p(k,b) \vdash p(m,0)(k,b), p(m,1)(k,b) \).

By induction \( p(k,b) \vdash A \) for all \( b \in \{0,1\} \) and thus \( p \vdash A \).

Similarly we can show the following two statements:

Lemma 3.20. 1. If \( p(m,0) \vdash t : A \) and \( p(m,1) \vdash t : A \) for some \( m \notin \text{dom}(p) \) then \( p \vdash t : A \).

2. If \( p(m,0) \vdash t = u : A \) and \( p(m,1) \vdash t = u : A \) then \( p \vdash t = u : A \).

Lemma 3.21. If \( p(m,0) \vdash A = B \) and \( p(m,1) \vdash A = B \) for some \( m \notin \text{dom}(p) \) then \( p \vdash A = B \).

Corollary 3.22 (Local character). If \( p \vdash p_1, \ldots , p_n \) and \( p_i \vdash J \) for all \( i \) then \( p \vdash J \).


Lemma 3.23. Let \( p \vdash A \Rightarrow \exists \{f \} \) with \( M \in \{N_1, N_2, N\} \). If \( p \vdash a : A \) then there is a partition \( p \vdash q_1, \ldots , q_m \) where \( a \) has a canonical \( p_i \)-whnf for all \( i \). If \( p \vdash a = b : A \) then there is a partition \( p \vdash q_1, \ldots , q_m \) where \( a \) and \( b \) have the same canonical \( q_j \)-whnf for each \( j \).

Proof. Follows by induction from the definition.

Lemma 3.24. Let \( p \vdash A = B \).

1. If \( p \vdash t : A \) then \( p \vdash t : B \) and if \( p \vdash u : A \) then \( p \vdash u : B \).

2. If \( p \vdash t = u : A \) then \( p \vdash t = u : B \) and if \( p \vdash v = w : A \) then \( p \vdash v = w : B \).

Proof. By induction on the derivation of \( p \vdash A \).

\((F_N)\) By induction on the derivation of \( p \vdash A = B \). (i) Let \( p \vdash B \Rightarrow \exists \{f \} \) for \( m \notin \text{dom}(p) \) and \( p(m,b) \vdash A = B \) for all \( b \in \{0,1\} \). Let \( p \vdash t : A \). By monotonicity \( p(m,b) \vdash t : A \) and by induction \( p(m,b) \vdash t : B \) for all \( b \). By the definition \( p \vdash t : B \). Let \( p \vdash u : B \). By monotonicity \( p(m,b) \vdash u : B \) and \( p(m,b) \vdash A = B \) by induction \( p(m,b) \vdash u : A \) for all \( b \). By local character \( p \vdash u : A \). Similarly we can show the second statement. The statement follows similarly for \((F_{N_1})\) and \((F_{N_2})\).
(F1) Let \( p \vdash A \Rightarrow \Pi(x : F) G \). By induction on the derivation of \( p \vdash A = B \). (i) Let \( \Gamma \vdash A = B \) and \( p \vdash B \Rightarrow \Pi(x : H) E \) and \( \vdash F = H \) and for all \( q \leq p \), \( q \vdash G[a] = E[a] \) whenever \( q \vdash a : F \). If \( p \vdash f : A \) then \( \Gamma \vdash f : B \). Let \( q \leq p \) and \( q \vdash u : H \). By monotonicity \( q \vdash u : F \). By induction \( q \vdash u : G[a] \) and by induction \( q \vdash f : E[a] \). Similarly, \( q \vdash f u = f v : E[a] \) whenever \( q \vdash u = v : H \). Thus \( p \vdash f : B \). Similarly, if \( p \vdash g : B \) we get \( p \vdash g : A \). (ii) Let \( p \vdash B \Rightarrow \* \Sigma(x : F) \) and \( p(k, b) \vdash A = B \) for all \( b \in \{0, 1\} \). If \( p \vdash f : A \) then by monotonicity \( p(k, b) \vdash f : A \) and by induction \( p(k, b) \vdash f : B \) for all \( b \). By the definition \( p \vdash f : B \). If on the other hand \( p \vdash g : B \) then by definition \( p(k, b) \vdash g : B \) and by induction \( p(k, b) \vdash g : A \) for all \( b \). By local character \( p \vdash g : A \). Similarly we can show the second statement.

(F2) Let \( p \vdash A \Rightarrow \* \Sigma(x : F) G \). By induction on the derivation of \( p \vdash A = B \). (i) Let \( \Gamma \vdash A = B \) and \( p \vdash B \Rightarrow \* \Sigma(x : H) E \) and \( \vdash F = H \) and for all \( q \leq p \), \( q \vdash G[a] = E[a] \) whenever \( q \vdash a : F \). If \( p \vdash t : A \) then \( \Gamma \vdash t : B \). Since \( p \vdash t : E \), by induction \( p \vdash t : F \). Thus \( p \vdash t : B \). Similarly if \( p \vdash u : B \) we have \( p \vdash u : A \). (ii) Let \( p \vdash B \Rightarrow \* \Sigma(x : F) \) and \( p(k, b) \vdash A = B \) for all \( b \in \{0, 1\} \). If \( p \vdash t : A \) then by monotonicity \( p(k, b) \vdash t : A \) and by induction \( p(k, b) \vdash t : B \) for all \( b \). By the definition \( p \vdash f : B \). If on the other hand \( p \vdash g : B \) then by definition \( p(k, b) \vdash g : B \) and by induction \( p(k, b) \vdash g : A \) for all \( b \). By local character \( p \vdash g : A \). Similarly we can show the second statement.

(F3) Since \( p \vdash A = B \), we have \( p \vdash B \Rightarrow \* U \) and the statements follow directly.

(F4) Let \( p \vdash A \Rightarrow \* \Sigma(x : F) \) for some \( k \notin \text{dom}(p) \). Since \( p \vdash A = B \), we have \( p(k, b) \vdash A = B \) for all \( b \in \{0, 1\} \). If \( p \vdash t : A \) then \( p(k, b) \vdash t : A \) and by induction \( p(k, b) \vdash t : B \) for all \( b \). By the definition \( p \vdash f : B \). If \( p \vdash u : B \) then \( p(k, b) \vdash u : B \) and by induction \( p(k, b) \vdash u : A \) for all \( b \). By local character \( p \vdash u : A \). Similarly we can show the second statement.

From the above results we can show that the relations \( p \vdash A = B = C = D = \ldots \) are equivalence relations. We omit the proof here.

Lemma 3.25.

Reflexivity: If \( p \vdash A \) then \( p \vdash A = A \) and if \( p \vdash t : A \) then \( p \vdash t = t : A \).

Symmetry: If \( p \vdash A = B \) then \( p \vdash B = A \) and if \( p \vdash t = u : A \) then \( p \vdash u = t : A \).

Transitivity: If \( p \vdash A = B \) and \( p \vdash B = C \) then \( p \vdash A = C \) and if \( p \vdash t = u : A \) and \( p \vdash u = v : A \) then \( p \vdash t = v : A \).

Section 4

Soundness

In this section we show that the type theory described in Section 2 is sound with respect to the semantics described in Section 3. That is, we aim to show that for any judgment \( J \) whenever \( \Gamma \vdash J \) then \( p \vdash J \).

Lemma 4.1. If \( p \vdash A \Rightarrow \* B \) and \( p \vdash B \) then \( p \vdash A \) and \( p \vdash A = B \).

Proof. Follows from the definition.

Lemma 4.2. Let \( p \vdash A \). If \( p \vdash t : u : A \) and \( p \vdash u : A \) then \( p \vdash t : A \) and \( p \vdash t = u : A \).

Proof. Let \( p \vdash t : u : A \) and \( p \vdash u : A \). By induction on the derivation of \( p \vdash A \).

(F1) That is, \( p \vdash A \Rightarrow \* U \). The statement follows similarly to Lemma 4.1.

(F2) By induction on the derivation of \( p \vdash u : A \). If \( p \vdash u : \* \Sigma(N) \) for some \( n \in \mathbb{N} \) then \( p \vdash t \Rightarrow \* \Sigma(N) \) and the statement follows by the definition. If \( p \vdash u : \* \Sigma(F) \) for \( k \notin \text{dom}(p) \) and \( p(k, b) \vdash u : A \) for all \( b \in \{0, 1\} \) then since \( p(k, b) \vdash t \Rightarrow u : A \), by induction,
Thus it’s direct to see that ▶

\( \forall t : A \) and \( p(k, b) \vdash t : A \) and \( p(k, b) \vdash t = u : A \). The statement follows similarly for \( (F_{N1}) \), \( (F_{N2}) \).

\( (F_{11}) \) Let \( p \vdash A \rightarrow \Pi(x : F)G \). Since \( p \vdash t : A \) we have \( p \vdash t : A \). Let \( q \leq p \) and \( q \vdash a : F \). We have \( q \vdash t a = u a : G[a] \). By induction \( q \vdash t a : G[a] \) and \( q \vdash t a = u a : G[a] \). If \( q \vdash a = b : F \) we similarly get \( q \vdash t b : G[b] \) and \( q \vdash t b = u b : G[b] \). Since \( q \vdash G[a] = G[b] \), by Lemma 3.24, \( q \vdash t b = u b : G[a] \). But \( q \vdash u a = u b : G[a] \). By symmetry and transitivity \( q \vdash t a = t b : G[a] \). Thus \( p \vdash t : A \) and \( p \vdash t = u : A \).

\( (F_{22}) \) Let \( p \vdash A \rightarrow^{*} \Sigma(x : F)G \). From \( p \vdash t \Rightarrow u : A \) we have \( p \vdash t : A \) and we have \( p \vdash t.1 : u.1 : F \) and \( p \vdash t.2 : u.2 : G[u.1] \). By induction \( p \vdash t.1 : F \) and \( p \vdash t.1 = u.1 : F \). By induction \( p \vdash t.2 : G[u.1] \) and \( p \vdash t.2 = u.2 : G[u.1] \). But since \( p \vdash A \) and we have shown \( p \vdash t.1 = u.1 : F \) we get \( p \vdash G[t.1] = G[u.1] \). By Lemma 3.24, \( p \vdash t.2 : G[t.1] \) and \( p \vdash t.2 = u.2 : G[t.1] \). Thus \( p \vdash t : A \) and \( p \vdash t = u : A \).

\( (F_{Loc}) \) Let \( p \vdash A \rightarrow^{*} \exists f \in \mathbb{F} \) for \( k \notin \text{dom}(p) \). Since \( p \vdash u : A \) we have \( p(k, b) \vdash u : A \) for all \( b \in \{0, 1\} \). But we have \( p(k, b) \vdash t \Rightarrow u : A \). By induction \( p(k, b) \vdash t : A \) and \( p(k, b) \vdash t = u : A \). By the definition \( p \vdash t : A \) and \( p \vdash t = u : A \).

\( \blacktriangleleft \)

**Corollary 4.3.** Let \( p \vdash t \Rightarrow^{*} u : A \) and \( p \vdash t \). If \( p \vdash u : A \) then \( p \vdash t : A \) and \( p \vdash t = u : A \).

\( \blacktriangleleft \)

**Corollary 4.4.** \( \vdash f : N \rightarrow N_{2} \).

**Proof.** It’s direct to see that \( \vdash N \rightarrow N_{2} \). For an arbitrary condition \( p \vdash n : N \). By Lemma 3.23, we have a partition \( p \vdash p_{1}, \ldots, p_{m} \) where for each \( i, p_{i} \vdash n \Rightarrow^{*} \overline{m}_{i} : N \) for some \( m_{i} \in \mathbb{N} \). We have thus a reduction \( p_{i} \vdash f n \Rightarrow^{*} f \overline{m}_{i} : N_{2} \). If \( \overline{m}_{i} \in \text{dom}(p_{i}) \) then \( p_{i} \vdash f n \Rightarrow^{*} f \overline{m}_{i} \Rightarrow b_{i} : N_{2} \) for some \( b_{i} \in \{0, 1\} \) and by definition \( p_{i} \vdash f n : N_{2} \). If for any \( j, \overline{m}_{j} \notin \text{dom}(p_{j}) \) then \( p_{j}(m_{j}, 0) \vdash f n \Rightarrow^{*} f \overline{m}_{j} \Rightarrow 0 : N_{2} \) and \( p_{j}(m_{j}, 1) \vdash f n \Rightarrow^{*} f \overline{m}_{j} \Rightarrow 1 : N_{2} \). Thus \( p_{j}(m_{j}, 0) \vdash f n : N_{2} \) and \( p_{j}(m_{j}, 1) \vdash f n : N_{2} \). By the definition \( p_{j} \vdash f n : N_{2} \). We thus have that \( p_{i} \vdash f n : N_{2} \) for all \( i \) and by local character \( p \vdash f n : N_{2} \). Similarly we can show \( p \vdash f n_{1} = f n_{2} : N_{2} \) whenever \( p \vdash n_{1} = n_{2} : N \). Hence \( \vdash f : N \rightarrow N_{2} \).

\( \blacktriangleleft \)

**Lemma 4.5.** \( \vdash p \vdash t : \neg A \) and \( p \vdash t \) then \( p \vdash t : \neg A \) if and only if \( \forall q \leq p \) there is no term \( u \) such that \( q \vdash u : A \).

**Proof.** Let \( p \vdash A \) and \( p \vdash t : \neg A \). We have directly that \( p \vdash \neg A \). Let \( p \vdash t : \neg A \). If \( q \vdash u : A \) then \( q \vdash t u : N_{0} \). Which is impossible. Conversely, assume it is the case that for all \( q \leq p \) there is no \( u \) for which \( q \vdash u : A \). Since \( r \vdash a : A \) and \( r \vdash a = b : A \) never hold for any \( r \leq p \), the statement \( \forall q \vdash t u : N_{0} \) whenever \( r \vdash a : A \) and \( r \vdash t u = t b : N_{0} \) whenever \( r \vdash a = b : A \) holds trivially.

\( \blacktriangleleft \)

**Lemma 4.6.** \( \vdash w : \neg(\Sigma(x : N)\text{IsZero}(f x)) \).

**Proof.** By Lemma 4.5 it is enough to show that for all \( q \) there is no term \( u \) for which \( q \vdash u : \neg(\Sigma(x : N)\text{IsZero}(f x)) \). Assume \( q \vdash u : \neg(\Sigma(x : N)\text{IsZero}(f x)) \). Let \( m \notin \text{dom}(q) \) we have then \( q(m, 0) \vdash (\overline{m}, 0) : \Sigma(x : N)\text{IsZero}(f x) \) thus \( q(m, 0) \vdash u(\overline{m}, 0) : N_{0} \) which is impossible.

\( \blacktriangleleft \)

Let \( \Gamma := x_{1} : A_{1}, \ldots, x_{n} : A_{n}[x_{1}, \ldots, x_{n-1}] \) \( \text{and } \rho := a_{1}, \ldots, a_{n} \). We say \( p \vdash \rho : \Gamma \) if \( p \vdash a_{1} : A_{1}, \ldots, p \vdash a_{n} : A_{n}[a_{1}, \ldots, a_{n-1}] \). If moreover \( \sigma := b_{1}, \ldots, b_{n} \) and \( p \vdash \sigma : \Gamma \), we say \( p \vdash \rho = \sigma : \Gamma \) if \( p \vdash a_{1} = b_{1} : A_{1}, \ldots, p \vdash a_{n} = b_{n} : A_{n}[a_{1}, \ldots, a_{n-1}] \).

**Lemma 4.7.** Let \( \Gamma \vdash p \). For all \( q \leq p \), if \( q \vdash \rho : \Gamma \), \( q \vdash \sigma : \Gamma \) and \( q \vdash \rho = \sigma : \Gamma \) then

- If \( \Gamma \vdash p \ Then \ q \vdash A p = A \sigma \) and if \( \Gamma \vdash p \ Then \ q \vdash A p = B \rho \).
- If \( \Gamma \vdash p \ Then \ q \vdash a p = a \sigma A p \) and if \( \Gamma \vdash p \ a = b : A \ Then \ q \vdash a p = b p A p \).
Proof. The proof is by induction on the rules of the type system. We show that if the statement holds for the premise of the rule it holds for the conclusion. For economy of presentation we only present the proof for few selected rules. For the rest of the rules the proof follows in a similar fashion.

- For the elimination rules \((\beta), (\text{unitrec}-0), (\text{boolrec}-0), (\text{boolrec}-1), (\text{natrec}-0), (\text{natrec-suc}), (pr_1), (pr_2)\) and \(f\)-\text{eval} the statement follows from Corollary 4.3.
- For the congruence rules the statement follows from Lemma 3.24, Lemma 3.25.
- The statement follows for \((I-I)\) by Corollary 4.4, for \((w\text{-term})\) by Lemma 4.6, and for \((L\text{oc})\) by Lemma 3.22.

\((\text{nat-suc})\) By induction \(q \vdash n_\sigma = n_\rho : N\). By Lemma 3.23 there is a partition \(q < q_1, \ldots, q_\ell\) where for each \(i\), \(q_i \vdash n_\sigma \Rightarrow^* m_i : N\) and \(q_i \vdash n_\sigma \Rightarrow^* m_i : N\) for some \(m_i \in \mathbb{N}\). But then \(q_i \vdash S n_\sigma \Rightarrow^* S m_i : N\) and \(q_i \vdash S n_\sigma \Rightarrow^* S m_i : N\) for all \(i\). Thus \(q_i \vdash S n_\sigma = S n_\sigma : N\) for all \(i\) and by local character \(q \vdash S n_\sigma = S n_\sigma : N\).

\((I-I)\) By induction \(q \vdash F_\rho = F_\sigma\). Let \(r \leq q\). We have \(r \vdash (\rho, c) : (\Gamma, x : F)\) whenever \(r \vdash c = b : F\) and by induction \(r \vdash G_\rho[a] = G_\rho[b]\). We have then \(q \vdash \Pi(x : F)G_\rho\) and similarly \(q \vdash \Pi(x : F)G_\sigma\). Whenever \(r \vdash a : F_\rho\) then, by Lemma 3.24, \(r \vdash (\rho, a) = (\sigma, a) : (\Gamma, x : F)\) and by induction \(r \vdash G_\rho[a] = G_\sigma[a]\). Thus \(q \vdash \Pi(x : F)G_\rho = \Pi(x : F)G_\sigma\).

\((\lambda-I)\) From \(\Gamma, x : F \vdash_p t : G\) we have \(\Gamma \vdash_p F\) and \(\Gamma, x : F \vdash_p G\). Similarly to \((I-I)\) we can show \(q \vdash \Pi(x : F)G_\rho, q \vdash \Pi(x : F)G_\sigma\), and \(q \vdash \Pi(x : F)G_\rho = \Pi(x : F)G_\sigma\). Let \(r \leq q\) and \(r \vdash a : F_\rho\). We have \(r \vdash (\rho, a) = (\sigma, a) : (\Gamma, x : F)\) and by induction \(r \vdash t_\rho[a] = t_\sigma[a] : G_\rho[a]\). But \(r \vdash (\lambda x. t_\rho) a \Rightarrow t_\rho[a] : G_\rho[a]\) and \(r \vdash (\lambda x. t_\sigma) a \Rightarrow t_\sigma[a] : G_\sigma[a]\). By Lemma 4.2 one has \(r \vdash (\lambda x. t_\rho) a = t_\rho[a] : G_\rho[a]\) and \(r \vdash (\lambda x. t_\sigma) a = t_\sigma[a] : G_\sigma[a]\). Since by induction we have \(r \vdash G_\rho[a] = G_\sigma[a]\), by Lemma 3.24, \(r \vdash (\lambda x. t_\rho) a = (\lambda x. t_\sigma) a : (\Gamma, x : F)G_\sigma\). By symmetry and transitivity \(r \vdash (\lambda x. t_\rho) a = (\lambda x. t_\sigma) a : G_\rho[a]\). Similarly we can show \(r \vdash (\lambda x. t_\rho) a = (\lambda x. t_\sigma) a : \Pi(x : F)G_\rho\) whenever \(r \vdash a = b : F_\rho\) and \(r \vdash (\lambda x. t_\sigma) a = (\lambda x. t_\sigma) b : \Pi(x : F)G_\sigma\) whenever \(r \vdash a = b : F_\sigma\). Thus \(q \vdash (\lambda x. t_\rho) = (\lambda x. t_\sigma) : \Pi(x : F)G_\rho\).

\((L\text{rec-i-t})\) Follows trivially since \(r \vdash t : N_0\) never holds for any condition \(r\).

\((\text{natrec-i})\) While we omit the proof here the basic idea is as follows: If for some \(r \leq q\) we have \(r \vdash t : N\) then by Lemma 3.23, we have \(r \triangleq r_1, \ldots, r_\ell\) and for each \(i\), \(r_i \vdash t \Rightarrow^* S^{k_i} 0\) for some \(k_i \in \mathbb{N}\). By induction on \(k_i\) we can show \(r_i \vdash (\text{natrec} (\lambda x. F) a_0) g \rho t : F_\rho[t]\) for all \(i\). By local character we will have \(r \vdash (\text{natrec} (\lambda x. F) a_0) g \rho t : F_\rho[t]\). Similarly we can show \(r \vdash (\text{natrec} (\lambda x. F) a_0) g \rho t = (\text{natrec} (\lambda x. F) a_0) g \rho u : F_\rho[t]\) whenever \(r \vdash t = u : N\). By the definition we will have \(q \vdash (\text{natrec} (\lambda x. F) a_0) g \rho : \Pi(x : N) F_\rho\) and similarly we can show \(q \vdash (\text{natrec} (\lambda x. F) a_0) g \rho = (\text{natrec} (\lambda x. F) a_0) g \sigma : \Pi(x : N) F_\rho\).

\(\text{Theorem 4.8 (Fundamental Theorem).} \ If \vdash_p J \ then \vdash J.\)

5 Markov’s principle

Now we have enough machinery to show the independence of MP from type theory. The idea is that if a judgment \(J\) is derivable in type theory (i.e. \(\vdash J\)) then it is derivable in the forcing extension (i.e. \(\vdash_J J\)) and by Theorem 4.8 it holds in the interpretation (i.e. \(\models J\)). It thus suffices to show that there no \(t\) such that \(\vdash t : MP\) to establish the independence of MP from type theory. First we recall the formulation of MP.

\[
MP := \Pi(h : N \rightarrow N_2)[\neg \neg (\Sigma(x : N) \text{IsZero} (h \, x)) \rightarrow \Sigma(x : N) \text{IsZero} (h \, x)]
\]

where \(\text{IsZero} : N_2 \rightarrow U\) is given by \(\text{IsZero} := \lambda y. \text{boolrec} (\lambda x. U) \, N_1 \, N_0 \, y\).
Lemma 5.1. There is no term t such that \( \vdash t : \Sigma(x:N) \text{IsZero}(f(x)) \).

Proof. Assume \( \vdash t : \Sigma(x:N) \text{IsZero}(f(x)) \) for some t. We then have \( \vdash t.1 : N \) and \( \vdash t.2 : \text{IsZero}(f(t.1)) \). By Lemma 3.23, one has a partition \( \langle \rangle < p_1, \ldots, p_n \) where for each i, \( p_i \vdash t.1 \Rightarrow^* m_i \) for some \( m_i \in \mathbb{N} \). Hence \( p_i \vdash \text{IsZero}(f(t.1)) \Rightarrow^* \text{IsZero}(f(m_i)) \) and by Lemma 4.1, \( p_i \vdash \text{IsZero}(f(t.1)) = \text{IsZero}(f(m_i)) \). But, by definition, a partition of \( \langle \rangle \) must contain a condition, say \( p_j \), such that \( p_j(k) = 1 \) whenever \( k \in \text{dom}(p_j) \) (this holds vacuously for \( \langle \rangle < \langle \rangle \)). Assume \( m_j \in \text{dom}(p_j) \), then \( p_j \vdash \text{IsZero}(f(t.1)) \Rightarrow^* \text{IsZero}(f(m_j)) \Rightarrow^* N_0 \). By monotonicity, from \( \vdash t.2 : \text{IsZero}(f(t.1)) \) we get \( p_j \vdash t.2 : \text{IsZero}(f(t.1)) \). But \( p_j \vdash \text{IsZero}(f(t.1)) \Rightarrow^* N_0 \) thus \( p_j \vdash \text{IsZero}(f(t.1)) = N_0 \). Hence, by Lemma 4.2, \( p_j \vdash t.2 : N_0 \) which is impossible, thus contradicting our assumption. If on the other hand \( m_j \notin \text{dom}(p_j) \) then since \( p_j < p_j(m_j, 0), p_j(m_j, 1) \) we can apply the above argument with \( p_j(m_j, 1) \) instead of \( p_j \). □

Lemma 5.2. There is no term t such that \( \vdash t : \text{MP} \).

Proof. Assume \( \vdash t : \text{MP} \) for some t. From the definition, whenever \( \vdash g : N \rightarrow N_2 \) we have \( \vdash t : (\neg(\Sigma(x:N) \text{IsZero}(g(x))) \rightarrow \Sigma(x:N) \text{IsZero}(g(x)) \). Since by Corollary 4.4, \( \vdash f : N \rightarrow N_2 \) we have \( \vdash t : (\neg(\Sigma(x:N) \text{IsZero}(f(x))) \rightarrow \Sigma(x:N) \text{IsZero}(f(x)) \). Since by Lemma 4.6, \( \vdash w : (\neg(\Sigma(x:N) \text{IsZero}(f(x))) \rightarrow (t f) w : \Sigma(x:N) \text{IsZero}(f(x)) \) which is impossible by Lemma 5.1.

From Theorem 4.8, Lemma 5.2, and Lemma 2.3 we can then conclude:

Theorem 2.1. There is no term t such that MLTT \( \vdash t : \text{MP} \).

5.1 Many Cohen reals

We extend the type system in Section 2 further by adding a generic point \( f_q \) for each condition \( q \).

The introduction and conversion rules for \( f_q \) are given by:

\[
\begin{align*}
\Gamma \vdash p & \quad \Gamma \vdash p q : N \rightarrow N_2 \\
\Gamma \vdash p & \quad \Gamma \vdash q \ p = \ 1 \\
\Gamma \vdash p & \quad \Gamma \vdash q \ p = p(n) \\
& \quad n \notin \text{dom}(q), n \in \text{dom}(p).
\end{align*}
\]

With the reduction rules:

\[
\begin{align*}
\Gamma \vdash p & \quad n \in \text{dom}(q) \\
\Gamma \vdash p & \quad n \notin \text{dom}(q), n \in \text{dom}(p)
\end{align*}
\]

We observe that the reduction relation is still monotone.

For each \( f_q \) we add a term

\[
\Gamma \vdash p \quad w_q : \neg(\Sigma(x:N) \text{IsZero}(f_q(x))).
\]

Finally we add a term \( mw : \neg \text{MP} \) witnessing the negation of \( \text{MP} \)

\[
\Gamma \vdash p \quad \Gamma \vdash p \quad \Gamma \vdash p \quad w_q \quad \neg(\Sigma(x:N) \text{IsZero}(f_q(x))).
\]

By analogy to Corollary 4.4 we have:

Lemma 5.3. \( \vdash f_q : N \rightarrow N_2 \) for all \( q \).

Lemma 5.4. \( \vdash w_q : \neg(\Sigma(x:N) \text{IsZero}(f_q(x))) \) for all \( q \).

Proof. Assume \( p \vdash t : (\neg(\Sigma(x:N) \text{IsZero}(f_q(x))) \) for some \( p \) and \( t \). Let \( m \notin \text{dom}(q) \cup \text{dom}(p) \), we have \( p(m, 0) \vdash t.1 = m \). Thus \( p(m, 0) \vdash (m, 0) : \Sigma(x:N) \text{IsZero}(f_q(x)) \) and \( p(m, 0) \vdash t(m, 0) : N_0 \) which is impossible. □

Lemma 5.5. There is no term \( t \) for which \( q \vdash t : \Sigma(x:N) \text{IsZero}(f_q(x)) \).
Proof. Assume \( q \vdash t : \Sigma(x : N)\text{isZero}(f_q x) \) for some \( t \). We then have \( q \vdash \top.1 : N \) and \( q \vdash \top.2 : \text{isZero}(f_q t.1) \). By Lemma 3.23 one has a partition \( q \ll 1, \ldots , q_n \) where for each \( i \), \( t.1 \Rightarrow \neg\neg \text{isZero}(f_q t.1) \Rightarrow \neg\neg \text{isZero}(f_q m_i) \). But any partition of \( q \) contain a condition, say \( q_i \), where \( q_i(k) = 1 \) whenever \( k \notin \text{dom}(q) \) and \( k \in \text{dom}(q_j) \). Assume \( m_j \in \text{dom}(q_j) \). If \( m_j \notin \text{dom}(q) \) then \( q_j \vdash f_q m_j \Rightarrow 1 : N_2 \) and if \( m_j \notin \text{dom}(q) \) then \( q_j \vdash f_q m_j \Rightarrow q_j(k) \Rightarrow 1 : N_2 \). Thus \( q_j \vdash \text{isZero}(f_q t.1) \Rightarrow N_0 \) and by Lemma 4.1, \( q_j \vdash \text{isZero}(f_q t.1) \Rightarrow N_0 \). From \( \vdash \top.2 : \text{isZero}(f_q t.1) \) by monotonicity and Lemma 3.24 we have \( q_j \vdash \top.2 : N_0 \) which is impossible. If on the other hand \( m_j \notin \text{dom}(q_j) \) then since \( q_j \ll q_j(m_j, 0), q_j(m_j, 1) \) we can apply the above argument with \( q_j(m_j, 1) \) instead of \( q_j \). ▶

Lemma 5.6. \( \vdash \text{m} \vdash \neg \text{MP} \)

Proof. Assume \( \vdash t : \text{MP} \) for some \( p \) and \( \vdash u : N \Rightarrow N_2 \) then \( q \vdash \top.1 : \neg\neg(\Sigma(x : N)\text{isZero}(u x)) \Rightarrow (\Sigma(x : N)\text{isZero}(u x)) \). But we have \( q \vdash f_q : N \Rightarrow N_2 \) by Lemma 5.3. Hence \( q \vdash f_q \vdash \neg\neg(\Sigma(x : N)\text{isZero}(f_q x)) \Rightarrow (\Sigma(x : N)\text{isZero}(f_q x)) \). But \( q \vdash w_q : \neg\neg(\Sigma(x : N)\text{isZero}(f_q x)) \) by Lemma 5.4. Thus \( q \vdash (t f_q w_q) : \Sigma(x : N)\text{isZero}(f_q x) \) which is impossible by Lemma 5.5. ▶

We have then the following result.

Theorem 5.7. There is a consistent extension of MLTT where \( \neg \text{MP} \) is derivable.

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References


