Extension Complexity, MSO Logic, and Treewidth

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Abstract

We consider the convex hull \( P_\varphi(G) \) of all satisfying assignments of a given MSO₂ formula \( \varphi \) on a given graph \( G \). We show that there exists an extended formulation of the polytope \( P_\varphi(G) \) that can be described by \( f(|\varphi|, \tau) \cdot n \) inequalities, where \( n \) is the number of vertices in \( G \), \( \tau \) is the treewidth of \( G \) and \( f \) is a computable function depending only on \( \varphi \) and \( \tau \).

In other words, we prove that the extension complexity of \( P_\varphi(G) \) is linear in the size of the graph \( G \), with a constant depending on the treewidth of \( G \) and the formula \( \varphi \). This provides a very general yet very simple meta-theorem about the extension complexity of polytopes related to a wide class of problems and graphs.

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1 Introduction

In the ’70s and ’80s, it was repeatedly observed that various NP-hard problems are solvable in polynomial time on graphs resembling trees. The graph property of resembling a tree was eventually formalized as having bounded treewidth, and in the beginning of the ’90s, the class of problems efficiently solvable on graphs of bounded treewidth was shown to contain the class of problems definable by the Monadic Second Order Logic (MSO₂) (Courcelle [11], Arnborg et al. [1], Courcelle and Mosbah [13]). Using similar techniques, analogous results for weaker logics were then proven for wider graph classes such as graphs of bounded cliquewidth and rankwidth [12]. Results of this kind are usually referred to as Courcelle’s theorem for a specific class of structures.

In this paper we study the class of problems definable by the MSO logic from the perspective of extension complexity. While small extended formulations are known for various special classes of polytopes, we are not aware of any other result in the theory of

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extended formulations that works on a wide class of polytopes the way Courcelle’s theorem works for a wide class of problems and graphs.

Our Contribution. We prove that satisfying assignments of an MSO$_2$ formula $\varphi$ on a graph of bounded treewidth can be expressed by a “small” linear program. More precisely, there exists a computable function $f$ such that the convex hull $P_\varphi(G)$ of satisfying assignments of $\varphi$ on a graph $G$ on $n$ vertices with treewidth $\tau$ can be obtained as the projection of a polytope described by $f(|\varphi|, \tau) \cdot n$ linear inequalities; we call $P_\varphi(G)$ the MSO polytope. All our results can be extended to general finite structures where the restriction on treewidth applies to the treewidth of their Gaifman graph [30].

Our proof essentially works by “merging the common wisdom” from the areas of extended formulations and fixed parameter tractability. It is known that dynamic programming can be turned into a compact extended formulation [32, 18], and that Courcelle’s theorem can be seen as an instance of dynamic programming [26]; therefore one can expect the polytope of satisfying assignments of an MSO formula of a bounded treewidth graph to be compact.

However, there are a few roadblocks in trying to merge these two folklore wisdoms. For one, while Courcelle’s theorem being an instance of dynamic programming in some sense may be obvious to an FPT theorist, it is far from clear to anyone else what that sentence may even mean. On the other hand, being able to turn a dynamic program into a compact polytope may be a theoretical possibility for an expert on extended formulations, but it is by no means an easy statement for an outsider to comprehend. What complicates the matters even further is that the result of Martin et al. [32] is not a result that can be used in a black box fashion. That is, a certain condition must be satisfied to get a compact extended formulation out of a dynamic program. This is far from a trivial task, especially for a theorem like Courcelle’s theorem.

The rest of the article is organized as follows. In Section 2 we review some previous work related to Courcelle’s theorem and extended formulations. In Section 3 we describe the relevant notions related to polytopes, extended formulations, graphs, treewidth and MSO logic. In Section 4 we prove the existence of compact extended formulations for MSO polytopes parameterized by the length of the given MSO formula and the treewidth of the given graph. In Section 5 we describe how to efficiently construct such a polytope given a tree decomposition of a graph.

2 Related Work

2.1 MSO Logic vs. Treewidth

Because of the wide relevance of the treewidth parameter in many areas (cf. the survey of Bodlaender [5]) and the large expressivity of the MSO and its extensions (cf. the survey of Langer et al. [27]), considerable attention was given to Courcelle’s theorem by theorists from various fields, reinterpreting it into their own setting. These reinterpretations helped uncover several interesting connections.

The classical way of proving Courcelle’s theorem is constructing a tree automaton $A$ in time only dependent on $\varphi$ and the treewidth $\tau$, such that $A$ accepts a tree decomposition of a graph of treewidth $\tau$ if and only if the corresponding graph satisfies $\varphi$; this is the automata theory perspective [11]. Another perspective comes from finite model theory where one can prove that a certain equivalence on the set of graphs of treewidth at most $\tau$ has only finitely many (depending on $\varphi$ and $\tau$) equivalence classes and that it behaves well [16]. Another approach proves that a quite different equivalence on so-called extended model checking
games has finitely many equivalence classes \cite{23} as well; this is the game-theoretic perspective. It can be observed that the finiteness in either perspective stems from the same roots.

Another related result is an expressivity result: Gottlob et al. \cite{16} prove that on bounded treewidth graphs, a certain subset of the database query language Datalog has the same expressive power as the MSO. This provides an interesting connection between the automata theory and the database theory.

2.2 Extended Formulations

Sellmann, Mercier, and Leventhal \cite{34} claimed to show compact extended formulation for binary Constraint Satisfaction Problems (CSP) for graphs of bounded treewidth, but their proof is not correct \cite{33}. The first two authors of this paper gave extended formulations for CSP that has polynomial size for instances whose constraint graph has bounded treewidth \cite{25} using a different technique. Bienstock and Munoz \cite{3} prove similar results for the approximate and exact version of the problem. In the exact case, Bienstock and Munoz’s bounds are slightly worse than those of Kolman and Koutecký \cite{25}. It is worth noting that CSPs are a restricted subclass of problems that can be modeled using MSO logic. Laurent \cite{28} provides extended formulations for the independent set and max cut polytopes of size $O(2^{\tau} n)$ for $n$-vertex graphs of treewidth $\tau$ and, independently, Buchanan and Butenko \cite{8} provide an extended formulation for the independent set polytope of the same size.

A lot of recent work on extended formulations has focused on establishing lower bounds in various settings: exact, approximate, linear vs. semidefinite, etc. (See for example \cite{15, 2, 6, 29}). A wide variety of tools have been developed and used for these results including connections to nonnegative matrix factorizations \cite{37}, communication complexity \cite{14}, information theory \cite{7}, and quantum communication \cite{15} among others.

For proving upper bounds on extended formulations, several authors have proposed various tools as well. Kaibel and Loos \cite{19} describe a setting of branched polyhedral systems which was later used by Kaibel and Pashkovich \cite{20} to provide a way to construct polytopes using reflection relations.

A particularly specific composition rule, which we term glued product (cf. Subsection 3.1), was studied by Margot in his PhD thesis \cite{31}. Margot showed that a property called the projected face property suffices to glue two polytopes efficiently. Conforti and Pashkovich \cite{10} describe and strengthen Margot’s result to make the projected face property to be a necessary and sufficient condition to describe the glued product in a particularly efficient way.

Martin et al. \cite{32} have shown that under certain conditions, an efficient dynamic programming based algorithm can be turned into a compact extended formulation. Kaibel \cite{18} summarizes this and various other methods.

3 Preliminaries

3.1 Polytopes, Extended Formulations and Extension Complexity

For background on polytopes we refer the reader to Grünbaum \cite{17} and Ziegler \cite{38}. To simplify reading of the paper for the audience that is not working often in the area of polyhedral combinatorics, we provide here a brief glossary of common polyhedral notions that are used in this article.

A hyperplane in $\mathbb{R}^n$ is a closed convex set of the form $\{x | a^\top x = b\}$ where $a \in \mathbb{R}^n, b \in \mathbb{R}$. A halfspace in $\mathbb{R}^n$ is a closed convex set of the form $\{x | a^\top x \leq b\}$ where $a \in \mathbb{R}^n, b \in \mathbb{R}$. The inequality $a^\top x \leq b$ is said to define the corresponding halfspace. A polytope $P \subseteq \mathbb{R}^n$
is a bounded subset defined by intersection of finite number of halfspaces. A result of Minkowsky-Weyl states that equivalently, every polytope is the convex hull of a finite number of points. Let \( h \) be a halfspace defined by an inequality \( a^\top x \leq b \); the inequality is said to be valid for a polytope \( P = P \cap h \). Let \( h \) be a halfspace defined by a valid inequality \( a^\top x \leq b \); then, \( P \cap \{x | a^\top x = b\} \) is said to be a face of \( P \).

Note that, taking \( a \) to be the zero vector and \( b = 0 \) results in the face being \( P \) itself. Also, taking \( a \) to be the zero vector and \( b = 1 \) results in the empty set. These two faces are often called the trivial faces and they are polytopes “living in” dimensions \( n \) and \(-1\), respectively. Every face - that is not trivial - is itself a polytope of dimension \( d \) where \( 0 \leq d \leq n - 1 \). The zero dimensional faces of a polytope are called its vertices, and the \((n - 1)\)-dimensional faces are called its facets.

It is not uncommon to refer to three separate (but related) objects as a face: the actual face as defined above, the valid inequality defining it, and the equation corresponding to the valid inequality. While this is clearly a misuse of notation, the context usually makes it clear as to exactly which object is being referred to.

Let \( P \) be a polytope in \( \mathbb{R}^d \). A polytope \( Q \) in \( \mathbb{R}^{d+r} \) is called an extended formulation or an extension of \( P \) if \( P \) is a projection of \( Q \) onto the first \( d \) coordinates. Note that for any linear map \( \pi : \mathbb{R}^{d+r} \to \mathbb{R}^d \) such that \( P = \pi(Q) \), a polytope \( Q' \) exists such that \( P \) is obtained by dropping all but the first \( d \) coordinates on \( Q' \) and, moreover, \( Q \) and \( Q' \) have the same number of facets.

The size of a polytope is defined to be the number of its facet-defining inequalities. Finally, the extension complexity of a polytope \( P \), denoted by \( xc(P) \), is the size of its smallest extended formulation. We refer the readers to the surveys \([9, 35, 18, 36]\) for details and background of the subject and we only state three basic propositions about extended formulations here.

▶ **Proposition 1.** Let \( P \) be a polytope with a vertex set \( V = \{v_1, \ldots, v_n\} \). Then \( xc(P) \leq n \).

**Proof.** Let \( P = \text{conv}(\{v_1, \ldots, v_n\}) \) be a polytope. Then, \( P \) is the projection of

\[
Q = \left\{ (x, \lambda) \mid x = \sum_{i=1}^{n} \lambda_i v_i; \sum_{i=1}^{n} \lambda_i = 1; \lambda_i \geq 0 \text{ for } i \in \{1, \ldots, n\} \right\}.
\]

It is clear that \( Q \) has at most \( n \) facets and therefore \( xc(P) \leq n \). ▶

▶ **Proposition 2.** Let \( P \) be a polytope obtained by intersecting a set \( H \) of hyperplanes with a polytope \( Q \). Then \( xc(P) \leq xc(Q) \).

**Proof.** Note that any extended formulation of \( Q \), when intersected with \( H \), gives an extended formulation of \( P \). Intersecting a polytope with hyperplanes does not increase the number of facet-defining inequalities (and only possibly reduces it). ▶

The (cartesian) product of two polytopes \( P_1 \) and \( P_2 \) is defined as

\[
P_1 \times P_2 = \text{conv}(\{(x, y) \mid x \in P_1, y \in P_2\}).
\]

▶ **Proposition 3.** Let \( P_1, P_2 \) be two polytopes. Then

\[
xc(P_1 \times P_2) \leq xc(P_1) + xc(P_2)
\]
Proof. Let $Q_1$ and $Q_2$ be extended formulations of $P_1$ and $P_2$, respectively. Then, $Q_1 \times Q_2$ is an extended formulation of $P_1 \times P_2$. Now assume that $Q_1 = \{x \mid Ax \leq b\}$ and $Q_2 = \{y \mid Cy \leq d\}$ and that these are the smallest extended formulations of $P_1$ and $P_2$, resp. Then

$$Q_1 \times Q_2 = \{(x,y) \mid Ax \leq b, Cy \leq d\}.$$ 

That is, we have an extended formulation of $P_1 \times P_2$ of size at most xc($P_1$) + xc($P_2$). ▶

We are going to define the glued product of polytopes, a slight generalization of the usual product of polytopes. We use a case where the extension complexity of the glued product of two polytopes is upper bounded by the sum of the extension complexities of the two polytopes, and use it in Section 4 to describe a small extended formulation for the MSO polytope $P_\varphi(G)$ on graphs with bounded treewidth.

Let $P \subseteq \mathbb{R}^{d_1+k}$ and $Q \subseteq \mathbb{R}^{d_2+k}$ be 0/1-polytopes defined by $m_1$ and $m_2$ inequalities and with vertex sets $\text{vert}(P)$ and $\text{vert}(Q)$, respectively. Let $I_P \subseteq \{1, \ldots, d_1 + k\}$ be a subset of coordinates of size $k$, $I_Q \subseteq \{1, \ldots, d_2 + k\}$ be a subset of coordinates of size $k$, and let $I_P' = \{1, \ldots, d_1 + k\} \setminus I_P$. For a vector $x$, and a subset $I$ of coordinates, we denote by $x_I$ the subvector of $x$ specified by the coordinates $I$. The glued product of $P$ and $Q$, (glued) with respect to the $k$ coordinates $I_P$ and $I_Q$, denoted by $P \times_k Q$, is defined as

$$P \times_k Q = \text{conv}\left(\{(x|_{I_P'}, y) \in \mathbb{R}^{d_1+k} \mid x \in \text{vert}(P), y \in \text{vert}(Q), x|_{I_P'} = y|_{I_Q}\}\right).$$

We adopt the following convention while discussing glued products in the rest of this article. In the above scenario, we say that $P \times_k Q$ is obtained by gluing $P$ and $Q$ along the $k$ coordinates $I_P$ of $P$ with the $k$ coordinates $I_Q$ of $Q$. If, for example, these coordinates are named $z$ in $P$ and $w$ in $Q$, then we also say that $P$ and $Q$ have been glued along the $z$ and $w$ coordinates and we refer to the coordinates $z$ and $w$ as the glued coordinates. In the special case that we glue along the last $k$ coordinates, the definition of the glued product simplifies to

$$P \times_k Q = \text{conv}\left(\{(x,y,z) \in \mathbb{R}^{d_1+d_2+k} \mid (x,z) \in \text{vert}(P), (y,z) \in \text{vert}(Q)\}\right).$$

This notion was studied by Margot [31] who provided a sufficient condition for being able to write the glued product in a specific (and efficient) way from the descriptions of $P$ and $Q$. We will use this particular way in Lemma 1. The existing work [31, 10], however, is more focused on characterizing exactly when this particular method works. We do not need the result in its full generality and therefore we only state a very specific version of it that is relevant for our purposes; for the sake of completeness, we also provide a proof of it.

**Lemma 1 (Gluing lemma).** Let $P$ and $Q$ be 0/1-polytopes and let the $k$ (glued) coordinates in $P$ be labeled $z_1, \ldots, z_k$, and the $k$ (glued) coordinates in $Q$ be labeled $w_1, \ldots, w_k$. Suppose that $1^T z \leq 1$ is valid for $P$ and $1^T w \leq 1$ is valid for $Q$. Then xc($P \times_k Q$) \leq xc($P$) + xc($Q$).

**Proof.** Let $(x',z',y',w')$ be a point from $P \times Q \cap \{(x,z,y,w) \mid z = w\}$. Observe that the point $(x',z')$ is a convex combination of points $(x',0),(x',e_1),\ldots,(x',e_k)$ from $P$ with coefficients $(1-\sum_{i=1}^k z'_i), z'_1, z'_2, \ldots, z'_k$ where $e_i$ is the $i$-th unit vector. Similarly, the point $(y',w')$ is a convex combination of points $(0,y',0),(y',e_1),\ldots,(y',e_k)$ from $Q$ with coefficients $(1-\sum_{i=1}^k w'_i), w'_1, w'_2, \ldots, w'_k$. Notice that for every $j \in [k]$, $(x'_j,e_j,y'_j)$ is a point from the glued product. As $w_i = z_i$ for every $i \in [k]$, we conclude that $(x',w',z') \in P \times_k Q$. Thus, by Proposition 2 the extension complexity of $P \times_k Q$ is at most that of $P \times Q$ which is at most xc($P$) + xc($Q$) by Proposition 3. ▶
3.2 Graphs and Treewidth

For notions related to the treewidth of a graph and nice tree decomposition, in most cases we stick to the standard terminology as given in the book by Kloks [22]; the only deviation is in the leaf nodes of the nice tree decomposition where we assume that the bags are empty. For a vertex \( v \in V \) of a graph \( G = (V, E) \), we denote by \( \delta(v) \) the set of neighbors of \( v \) in \( G \), that is, \( \delta(v) = \{ u \in V : \{ u, v \} \in E \} \).

A tree decomposition of a graph \( G = (V, E) \) is a tree \( T \) in which each node \( a \in T \) has an assigned set of vertices \( B(a) \subseteq V \) (called a bag) such that \( \bigcup_{a \in T} B(a) = V \) with the following properties:
- for any \( \{ u, v \} \in E \), there exists a node \( a \in T \) such that \( u, v \in B(a) \).
- if \( v \in B(a) \) and \( v \in B(b) \), then \( v \in B(c) \) for all nodes \( c \) on the path from \( a \) to \( b \) in \( T \).

The treewidth \( tw(T) \) of a tree decomposition \( T \) is the size of the largest bag of \( T \) minus one. The treewidth \( tw(G) \) of a graph \( G \) is the minimum treewidth over all possible tree decompositions of \( G \).

A nice tree decomposition is a tree decomposition with one special node \( r \) called the root in which each node is one of the following types:
- Leaf node: a leaf \( a \) of \( T \) with \( B(a) = \emptyset \).
- Introduce node: an internal node \( a \) of \( T \) with one child \( b \) for which \( B(a) = B(b) \cup \{ v \} \) for some \( v \in B(a) \).
- Forget node: an internal node \( a \) of \( T \) with one child \( b \) for which \( B(a) = B(b) \setminus \{ v \} \) for some \( v \in B(b) \).
- Join node: an internal node \( a \) with two children \( b \) and \( c \) with \( B(a) = B(b) = B(c) \).

For a vertex \( v \in V \), we denote by \( top(v) \) the topmost node of the nice tree decomposition \( T \) that contains \( v \) in its bag. For any graph \( G \) on \( n \) vertices, a nice tree decomposition of \( G \) with at most \( 8n \) nodes can be computed in time \( O(n) \) [4, 22].

Given a graph \( G = (V, E) \) and a subset of vertices \( \{ v_1,\ldots,v_d \} \subseteq V \), we denote by \( G[v_1,\ldots,v_d] \) the subgraph of \( G \) induced by the vertices \( v_1,\ldots,v_d \). Given a tree decomposition \( T \) and a node \( a \in V(T) \), we denote by \( T_a \) the subtree of \( T \) rooted in \( a \), and by \( G_a \) the subgraph of \( G \) induced by all vertices in bags of \( T_a \); that is, \( G_a = G[\bigcup_{b \in V(T_a)} B(b)] \). Throughout this paper we assume that for every graph, its vertex set is a subset of \( \mathbb{N} \). We define the following operator \( \sigma \): for any set \( U = \{ v_1, v_2, \ldots, v_l \} \subseteq \mathbb{N} \), \( \sigma(U) = (v_1, v_2, \ldots, v_l) \) such that \( v_1 < v_2 < \cdots < v_l \).

For an integer \( m \geq 0 \), an \([m] \)-colored graph is a pair \((G, \bar{V})\) where \( G = (V, E) \) is a graph and \( \bar{V} = (V_1,\ldots,V_m) \) is an \( m \)-tuple of subsets of vertices of \( G \) called an \([m] \)-coloring of \( G \). For integers \( m \geq 0 \) and \( \tau \geq 0 \), an \([m] \)-colored \( \tau \)-boundary graph is a triple \((G, \bar{V}, \bar{p})\) where \((G, \bar{V})\) is an \([m] \)-colored graph and \( \bar{p} = (p_1,\ldots,p_r) \) is a \( \tau \)-tuple of vertices of \( G \) called a boundary of \( G \). If the tuples \( \bar{V} \) and \( \bar{p} \) are clear from the context or if their content is not important, we simply denote an \([m] \)-colored \( \tau \)-boundary graph by \( G^{[m] \tau} \). For a tuple \( \bar{p} = (p_1,\ldots,p_r) \), we denote by \( p \) the corresponding set, that is, \( p = \{ p_1,\ldots,p_r \} \).

Two \([m] \)-colored \( \tau \)-boundary graphs \((G_1, \bar{V}, \bar{p})\) and \((G_2, \bar{U}, \bar{q})\) are compatible if the function \( h : \bar{p} \rightarrow \bar{q} \), defined by \( h(p_i) = q_i \) for each \( i \), is an isomorphism of the induced subgraphs \( G_1[p_1,\ldots,p_r] \) and \( G_2[q_1,\ldots,q_r] \), and if for each \( i \) and \( j \), \( p_i \in V_j \iff q_i \in U_j \).

Given two compatible \([m] \)-colored \( \tau \)-boundary graphs \( G_1^{[m] \tau} = (G_1, \bar{U}, \bar{p}) \) and \( G_2^{[m] \tau} = (G_2, \bar{W}, \bar{q}) \), the join of \( G_1^{[m] \tau} \) and \( G_2^{[m] \tau} \), denoted by \( G_1^{[m] \tau} \oplus G_2^{[m] \tau} \), is the \([m] \)-colored \( \tau \)-boundary graph \( G^{[m] \tau} = (G, \bar{V}, \bar{p}) \) where \( G \) is the graph obtained by taking the disjoint union of \( G_1 \) and \( G_2 \), and for each \( i \), identifying the vertex \( p_i \) with the vertex \( q_i \) and keeping the label \( p_i \) for it;
- \( \bar{V} = (V_1,\ldots,V_m) \) with \( V_j = U_j \cup W_j \) and every \( q_i \) replaced by \( p_i \), for each \( j \);
which is an extension of first order logic that allows quantification over monadic variables. If they satisfy the same MSO formula, without loss of generality (cf. [21]) we assume that the input formulae are given in a variant of MSO into the prenex form (i.e., all quantifiers are at the beginning of the formula). We denote by \( \mathcal{A} \) the vocabulary of all free variables in \( \phi \). In addition quantification over sets of edges. As every MSO formula \( \phi \), viewed as a \( \sigma \)-structure, one can view a graph \( G = (V,E) \) as an \( \sigma \)-structure \( \sigma \) which satisfies \( G \models \phi \) if and only if \( I(G) \models \phi \) [folklore], for the sake of presentation we restrict our attention, without loss of generality, to MSO formulae over the \( \sigma_2 \) vocabulary. To further simplify the presentation, without loss of generality (cf. [21]) we assume that the input formulae are given in a variant of MSO that uses only set variables (and no element variables).

An important kind of structures that are necessary in the proofs in this paper are the \( [m] \)-colored \( \tau \)-boundaried graphs. An \( [m] \)-colored \( \tau \)-boundaried graph \( G = (V,E) \) with boundary \( p_1, \ldots, p_r \) colored with \( V_1, \ldots, V_m \) is viewed as a structure \( \sigma_2 \) which consists of a universe \( \sigma \) together with an interpretation of the constant and relation symbols: each constant symbol \( c_i \) from \( \sigma \) is associated with an element \( c_i \in A \) and each relation symbol \( P_i \) from \( \sigma \) is associated with an \( r_i \)-ary relation \( P_i \subseteq A^{r_i} \). To give an example, a graph \( G = (V,E) \) can be viewed as a \( \sigma_1 \)-structure \( \sigma_1 \) which satisfies \( G \models \phi \) if and only if \( I(G) \models \phi \) for every graph \( G \), \( G \models \phi \) if and only if \( I(G) \models \phi \) [folklore], for the sake of presentation we restrict our attention, without loss of generality, to MSO formulae over the \( \sigma_2 \) vocabulary. To further simplify the presentation, without loss of generality (cf. [21]) we assume that the input formulae are given in a variant of MSO that uses only set variables (and no element variables).

The main subject of this paper are formulas for graphs in monadic second order logic (MSO) which is an extension of first order logic that allows quantification over monadic variables (i.e., over sets of vertices). By MSO \( \sigma_2 \) we denote the extension of MSO that allows in addition quantification over sets of edges. As every MSO formula \( \phi \) over \( \sigma_1 \) can be turned into an MSO formula \( \phi' \) over \( \sigma_2 \) such that for every graph \( G \), \( G \models \phi \) if and only if \( I(G) \models \phi' \) [folklore], for the sake of presentation we restrict our attention, without loss of generality, to MSO formulae over the \( \sigma_2 \) vocabulary. To further simplify the presentation, without loss of generality (cf. [21]) we assume that the input formulae are given in a variant of MSO that uses only set variables (and no element variables).

An important kind of structures that are necessary in the proofs in this paper are the \( [m] \)-colored \( \tau \)-boundaried graphs. An \( [m] \)-colored \( \tau \)-boundaried graph \( G = (V,E) \) with boundary \( p_1, \ldots, p_r \) colored with \( V_1, \ldots, V_m \) is viewed as a structure \( \sigma_2 \) which consists of a universe \( \sigma \) together with an interpretation of the constant and relation symbols: each constant symbol \( c_i \) from \( \sigma \) is associated with an element \( c_i \in A \) and each relation symbol \( P_i \) from \( \sigma \) is associated with an \( r_i \)-ary relation \( P_i \subseteq A^{r_i} \). To give an example, a graph \( G = (V,E) \) can be viewed as a \( \sigma_1 \)-structure \( \sigma_1 \) which satisfies \( G \models \phi \) if and only if \( I(G) \models \phi \) for every graph \( G \), \( G \models \phi \) if and only if \( I(G) \models \phi \) for every graph \( G \), \( G \models \phi \) if and only if \( I(G) \models \phi \) for every graph \( G \), \( G \models \phi \) if and only if \( I(G) \models \phi \) for every graph \( G \).
Let us denote the equivalence classes of the relation $\equiv^\text{MSO}_k$ by $C = \{\alpha_1, \ldots, \alpha_w\}$, fixing an ordering such that $\alpha_1$ is the class containing the empty graph. Note that the size of $C$ depends only on $k$, $m$ and $\tau$, that is, $|C| = f(k, m, \tau)$ for some computable function $f$. For a given MSO formula $\varphi$ with $m$ free variables, we define an indicator function $\rho_{\varphi} : \{1, \ldots, |C|\} \to \{0, 1\}$ as follows: for every $i$, if there exists a graph $G_i^{(m),\tau} \in \alpha_i$ such that $G_i^{(m),\tau} \models \varphi$, we set $\rho_{\varphi}(i) = 1$, and we set $\rho_{\varphi}(i) = 0$ otherwise; note that if there exists a graph $G_i^{(m),\tau} \in \alpha_i$ such that $G_i^{(m),\tau} \models \varphi$, then $G_i^{(m),\tau} \models \varphi$ for every $G_i^{(m),\tau} \in \alpha_i$.

For every $[m]$-colored $\tau$-boundaried graph $G_i^{(m),\tau}$, its type, with respect to the relation $\equiv^\text{MSO}_k$, is the class to which $G_i^{(m),\tau}$ belongs. We say that types $\alpha_i$ and $\alpha_j$ are compatible if there exist two $[m]$-colored $\tau$-boundaried graphs of types $\alpha_i$ and $\alpha_j$ that are compatible; note that this is well defined as all $[m]$-colored $\tau$-boundaried graphs of a given type are compatible. For every $i \geq 1$, we will encode the type $\alpha_i$ naturally as a binary vector $\{0, 1\}^{|C|}$ with exactly one 1, namely with 1 on the position $i$.

An important property of the types and the join operation is that the type of a join of two $[m]$-colored $\tau$-boundaried graphs depends on their types only.

**Lemma 3** (Lemma 7.11 [30] and Lemma 3.5 [16]). Let $G_a^{(m),\tau}, G_{a'}^{(m),\tau}, G_b^{(m),\tau}$ and $G_{b'}^{(m),\tau}$ be $[m]$-colored $\tau$-boundaried graphs such that $G_a^{(m),\tau} \equiv^\text{MSO}_k G_{a'}^{(m),\tau}$ and $G_b^{(m),\tau} \equiv^\text{MSO}_k G_{b'}^{(m),\tau}$. Then $(G_a^{(m),\tau} \oplus G_b^{(m),\tau}) \equiv^\text{MSO}_k (G_{a'}^{(m),\tau} \oplus G_{b'}^{(m),\tau})$.

The importance of the lemma rests in the fact that for determination of the type of a join of two $[m]$-colored $\tau$-boundaried graphs, it suffices to know only a small amount of information about the two graphs, namely their types. The following two lemmas deal in a similar way with the type of a graph in other situations.

**Lemma 4** (implicitly in [16]). Let $(G_a, \bar{X}, \bar{p})$, $(G_b, \bar{Y}, \bar{q})$ be $[m]$-colored $\tau$-boundaried graphs and let $(G_{a'}, \bar{X}', \bar{p}')$, $(G_{b'}, \bar{Y}', \bar{q}')$ be $[m]$-colored $(\tau + 1)$-boundaried graphs with $G_a = (V, E)$, $G_{a'} = (V', E')$, $G_b = (W, F)$, $G_{b'} = (W', F')$ such that for some $v \notin V$ and $w \notin W$

1. $(G_a, \bar{X}, \bar{p}) \equiv^\text{MSO}_k (G_b, \bar{Y}, \bar{q})$;
2. $V' = V \cup \{v\}$, $\delta(v) \subseteq p$, $\bar{p}$ is a subtuple of $\bar{p}'$ and $(G_{a'}[V], \bar{X}'[V], \bar{p}'[V]) = (G_a, \bar{X}, \bar{p})$;
3. $W' = W \cup \{w\}$, $\delta(w) \subseteq q$, $\bar{q}$ is a subtuple of $\bar{q}'$ and $(G_{b'}[W], \bar{Y}'[W], \bar{q}'[W]) = (G_b, \bar{Y}, \bar{q})$;
4. $(G_{a'}, \bar{X}', \bar{p}')$ and $(G_{b'}, \bar{Y}', \bar{q}')$ are compatible.

Then $(G_{a'}, \bar{X}', \bar{p}') \equiv^\text{MSO}_k (G_{b'}, \bar{Y}', \bar{q}')$.

**Lemma 5** (implicitly in [16]). Let $(G_a, \bar{X}, \bar{p})$, $(G_b, \bar{Y}, \bar{q})$ be $[m]$-colored $\tau$-boundaried graphs and let $(G_{a'}, \bar{X}', \bar{p}')$, $(G_{b'}, \bar{Y}', \bar{q}')$ be $[m]$-colored $(\tau + 1)$-boundaried graphs with $G_a = (V, E)$, $G_{a'} = (V', E')$, $G_b = (W, F)$, $G_{b'} = (W', F')$ such that

1. $(G_{a'}, \bar{X}', \bar{p}') \equiv^\text{MSO}_k (G_{b'}, \bar{Y}', \bar{q}')$;
2. $V \subseteq V'$, $|V'| = |V| + 1$, $\bar{p}$ is a subtuple of $\bar{p}'$ and $(G_{a'}[V], \bar{X}'[V], \bar{p}'[V]) = (G_a, \bar{X}, \bar{p})$;
3. $W \subseteq W'$, $|W'| = |W| + 1$, $\bar{q}$ is a subtuple of $\bar{q}'$ and $(G_{b'}[W], \bar{Y}'[W], \bar{q}'[W]) = (G_b, \bar{Y}, \bar{q})$.

Then $(G_{a'}, \bar{X}, \bar{p}) \equiv^\text{MSO}_k (G_b, \bar{Y}, \bar{q})$.

### 3.4 Feasible Types

Suppose that we are given an MSO formula $\varphi$ over $\sigma_2$ with $m$ free variables and a quantifier rank at most $k$, a graph $G$ of treewidth at most $\tau$, and a nice tree decomposition $T$ of the graph $G$.

For every node of $T$ we are going to define certain types and tuples of types as feasible. For a node $b \in V(T)$ of any kind (leaf, introduce, forget, join) and for $\alpha \in C$, we say that $\alpha$ is a feasible type of the node $b$ if there exist an $[m]$-coloring $\bar{X} = (X_1, \ldots, X_m)$ of $G_b$ such
that $(G_b, X, \sigma(B(b)))$ is of type $\alpha$; we say that $\vec{X}$ realizes type $\alpha$ on the node $b$. We denote the set of feasible types of the node $b$ by $\mathcal{F}(b)$.

For an introduce node $b \in V(T)$ with a child $a \in V(T)$ (assuming that $v$ is the new vertex), for $\alpha \in \mathcal{F}(a)$ and $\beta \in \mathcal{F}(b)$, we say that $(\alpha, \beta)$ is a feasible pair of types for $b$ if there exist $\vec{X} = (X_1, \ldots, X_m)$ and $\vec{X'} = (X'_1, \ldots, X'_m)$ realizing types $\alpha$ and $\beta$ on the nodes $a$ and $b$, respectively, such that for each $i$, either $X'_i = X_i$ or $X'_i = X_i \cup \{v\}$. We denote the set of feasible pairs of types of the introduce node $b$ by $\mathcal{F}_p(b)$.

For a forget node $b \in V(T)$ with a child $a \in V(T)$ and for $\beta \in \mathcal{F}(b)$ and $\alpha \in \mathcal{F}(a)$, we say $(\alpha, \beta)$ is a feasible pair of types for $b$ if there exists $\vec{X}$ realizing $\beta$ on $b$ and $\alpha$ on $a$. We denote the set of feasible pairs of types of the forget node $b$ by $\mathcal{F}_p(b)$.

For a join node $c \in V(T)$ with children $a, b \in V(T)$ and for $\alpha \in \mathcal{F}(c)$, $\gamma_1 \in \mathcal{F}(a)$ and $\gamma_2 \in \mathcal{F}(b)$, we say that $(\gamma_1, \gamma_2, \alpha)$ is a feasible triple of types for $c$ if $\gamma_1, \gamma_2$ and $\alpha$ are mutually compatible and there exist $\vec{X}^1, \vec{X}^2$ realizing $\gamma_1$ and $\gamma_2$ on $a$ and $b$, respectively, such that $\vec{X} = (X_1 \cup X'_1, \ldots, X_m \cup X'_m)$ realizes $\alpha$ on $c$. We denote the set of feasible triples of types of the join node $c$ by $\mathcal{F}_j(c)$.

We define an indicator function $\mu : \mathcal{C} \times V(G) \times \{1, \ldots, m\} \to \{0, 1\}$ such that $\mu(\beta, v, i) = 1$ if and only if there exists $\vec{X} = (X_1, \ldots, X_m)$ realizing the type $\beta$ on the node $top(v) \in V(T)$ and $v \in X_i$.

4 Extension Complexity of the MSO Polytope

For a given MSO formula $\varphi(\vec{X})$ over $\sigma_2$ with $m$ free set variables $X_1, \ldots, X_m$, we define a polytope of satisfying assignments on a given graph $G$, represented as a $\sigma_2$ structure $I(G) = (V_G, \emptyset, \{E_G, L_V, L_E\})$ with domain of size $n$, in a natural way. We encode any assignment of elements of $I(G)$ to the sets $X_1, \ldots, X_m$ as follows. For each $X_i$ in $\varphi$ and each $v \in V_G$, we introduce a binary variable $y_i^v$. We set $y_i^v$ to be one if $v \in X_i$ and zero otherwise.

For a given 0/1 vector $y$, we say that $y$ satisfies $\varphi$ if interpreting the coordinates of $y$ as described above yields a satisfying assignment for $\varphi$. The polytope of satisfying assignments, also called the MSO polytope, is defined as

$$P_\varphi(G) = \text{conv} \{\{y \in \{0, 1\}^m \mid y \text{ satisfies } \varphi\}\}.$$  

Theorem 6 (Extension Complexity of the MSO Polytope). For every graph $G$ and for every MSO formula $\varphi$, $\text{exc}(P_\varphi(G)) = f(|\varphi|, \tau) \cdot n$ where $f$ is some computable function, $\tau = \text{tw}(G)$ and $n = |V_G|$.

Proof. Let $T$ be a fixed nice tree decomposition of treewidth $\tau$ of the given graph $G$ represented as $I(G)$ and let $k$ denote the quantifier rank of $\varphi$ and $m$ the number of free variables of $\varphi$. Let $\mathcal{C}$ be the set of equivalence classes of the relation $\equiv_k^{\text{MSO}}$. For each node $b$ of $T$ we introduce $|\mathcal{C}|$ binary variables that will represent a feasible type of the node $b$; we denote the vector of them by $t_b$ (i.e., $t_b \in \{0, 1\}^{|\mathcal{C}|}$). For each introduce and each forget node $b$ of $T$, we introduce additional $|\mathcal{C}|$ binary variables that will represent a feasible type of the child (descendant) of $b$; we denote the vector of them by $d_b$ (i.e., $d_b \in \{0, 1\}^{|\mathcal{C}|}$). Similarly, for each join node $b$ we introduce additional $|\mathcal{C}|$ binary variables, denoted by $l_b$, that will represent a feasible type of the left child of $b$, and other $|\mathcal{C}|$ binary variables, denoted by $r_b$, that will represent a feasible type of the right child of $b$ (i.e., $l_b, r_b \in \{0, 1\}^{|\mathcal{C}|}$).

We are going to describe inductively a polytope in the dimension given (roughly) by all the binary variables of all nodes of the given nice tree decomposition. Then we show that its extension complexity is small and that a properly chosen face of it is an extension of $P_\varphi(G)$.

First, for each node $b$ of $T$, depending on its type, we define a polytope $P_b$ as follows:
\begin{itemize}
  \item \(b\) is a leaf. \(P_b\) consists of a single point \(P_b = \{100\ldots 0\}\).
  \item \(b\) is an introduce or forget node. For each feasible pair of types \((\alpha_i, \alpha_j) \in \mathcal{F}_b\) of the node \(b\), we create a vector \((d_b, t_b) \in \{0, 1\}^{2|C|}\) with \(d_b[i] = t_b[j] = 1\) and all other coordinates zero. \(P_b\) is defined as the convex hull of all such vectors.
  \item \(b\) is a join node. For each feasible triple of types \((\alpha_i, \alpha_i, \alpha_j) \in \mathcal{F}_b\) of the node \(b\), we create a vector \((r_b, t_b, s_b) \in \{0, 1\}^{3|C|}\) with \(r_b[\ell] = r_b[i] = t_b[j] = 1\) and all other coordinates zero. \(P_b\) is defined as the convex hull of all such vectors.
\end{itemize}

It is clear that for every node \(b\) in \(T\), the polytope \(P_b\) contains at most \(|C|^3\) vertices, and, thus, by Proposition 1 it has extension complexity at most \(\text{xc}(P_b) \leq |C|^3\). Recalling our discussion in Section 3 about the size of \(C\), we conclude that there exists a function \(f\) such that for every \(b \in V(T)\), it holds that \(\text{xc}(P_b) \leq f(|\varphi|, \tau)\).

To obtain an extended formulation for \(P_{\varphi}(G)\), we first glue these polytopes together, starting in the leaves of \(T\) and processing \(T\) in a bottom up fashion. We create polytopes \(Q_b\) for each node \(b\) in \(T\) recursively as follows:

\begin{itemize}
  \item If \(b\) is a leaf then \(Q_b = P_b\).
  \item If \(b\) is an introduce or forget node, then \(Q_b = Q_b \times_{|C|} P_b\) where \(a\) is the child of \(b\) and the gluing is done along the coordinates \(t_a\) in \(Q_a\) and \(d_b\) in \(P_b\).
  \item If \(b\) is a join node, then we first define \(R_b = Q_b \times_{|C|} P_b\) where \(a\) is the left child of \(b\) and the gluing is done along the coordinates \(t_a\) in \(Q_a\) and \(l_b\) in \(P_b\). Then \(Q_b\) is obtained by gluing \(R_b\) with \(Q_c\) along the coordinates \(t_c\) in \(Q_c\) and \(r_b\) in \(R_b\) where \(c\) is the right child of \(b\).
\end{itemize}

The following lemma states the key property of the polytopes \(Q_b\)’s.

**Lemma 7.** For every vertex \(y\) of the polytope \(Q_b\) there exist an \([m]\)-coloring \(\vec{X} = (X_1, \ldots, X_m)\) of \(G_b\) such that \((G_b, \vec{X}, \sigma(B(b)))\) is of type \(\alpha\) where \(\alpha\) is the unique type such that the coordinate of \(y\) corresponding to the binary variable \(t_b(\alpha)\) is equal to one.

**Proof.** The proof is by induction, starting in the leaves of \(T\) and going upwards towards the root. For leaves, the lemma easily follows from the definition of the polytopes \(P_b\)’s.

For the inductive step, we consider an inner node \(b\) of \(T\) and we distinguish three cases:

\begin{itemize}
  \item If \(b\) is a join node, then the claim for \(b\) follows from the inductive assumptions for the children of \(b\), definition of a feasible triple, definition of the polytope \(P_b\), Lemma 3 and the construction of the polytope \(Q_b\).
  \item If \(b\) is an introduce node or a forget node, respectively, then, analogously, the claim for \(b\) follows from the inductive assumption for the child of \(b\), definition of a feasible pair, definition of the polytope \(P_b\), Lemma 4 or Lemma 5, respectively, and the construction of the polytope \(Q_b\).
\end{itemize}

Let \(c\) be the root node of the tree decomposition \(T\). Consider the polytope \(Q_c\). From the construction of \(Q_c\), our previous discussion and the Gluing lemma, it follows that \(\text{xc}(Q_c) \leq \sum_{b \in V(T)} \text{xc}(P_b) \leq f(|\varphi|, \tau) \cdot O(n)\). It remains to show that a properly chosen face of \(Q_c\) is an extension of \(P_{\varphi}(G)\). We start by observing that \(\sum_{i=1}^{|C|} t_c[i] \leq 1\) and \(\sum_{i=1}^{|C|} \rho_{\varphi}(i) \cdot t_c[i] \leq 1\), where \(\rho_{\varphi}\) is the indicator function, are valid inequalities for \(Q_c\).

Let \(Q_{\varphi}\) be the face of \(Q_c\) corresponding to the valid inequality \(\sum_{i=1}^{|C|} \rho_{\varphi}(i) \cdot t_c[i] \leq 1\). Then, by Lemma 7, the polytope \(Q_{\varphi}\) represents those \([m]\)-colorings of \(G\) for which \(\varphi\) holds. The corresponding feasible assignments of \(\varphi\) on \(G\) are obtained as follows: for every vertex \(v \in V(G)\) and every \(i \in \{1, \ldots, m\}\) we set \(y^*_i = \sum_{j=1}^{|C|} \mu(\alpha_j, v, i) \cdot t_{\text{top}(v)}[j]\). The sum is 1 if
and only if there exists a type \( j \) such that \( t_{\text{top}(v)}[j] = 1 \) and at the same time \( \mu(\alpha_j, v, i) = 1 \); by the definition of the indicator function \( \mu \) in Subsection 3.4, this implies that \( v \in X_i \). Thus, by applying the above projection to \( Q_\varphi \) we obtain \( P_\varphi(G) \), as desired.

It is worth mentioning at this point that the polytope \( Q_\varphi \) depends only on the quantifier rank \( k \) of \( \varphi \) and the number of free variables of \( \varphi \). The dependence on the formula \( \varphi \) itself only manifests in the choice of the face \( Q_\varphi \) of \( Q_\varphi \) that projects to \( P_\varphi(G) \).

\begin{corollary}
The extension complexity of the convex hull of all satisfying assignments of a given MSO\(_2\) formula \( \varphi \) on a given graph \( G \) of bounded treewidth is linear in the size of the graph \( G \).
\end{corollary}

\section{Efficient Construction of the MSO Polytope}

In the previous section we have proven that \( P_\varphi(G) \) has a compact extended formulation but our definition of feasible tuples and the indicator functions \( \mu \) and \( \rho_\varphi \) did not explicitly provide a way how to actually obtain it efficiently. That is what we do in this section. We also briefly mention some implications of our results for optimization versions of Courcelle’s theorem.

As in the previous section we assume that we are given a graph \( G \) of treewidth \( \tau \) and an MSO formula \( \varphi \) with \( m \) free variables and quantifier rank \( k \). We start by constructing a nice tree decomposition \( T \) of \( G \) of treewidth \( \tau \) in linear time.

Let \( C \) denote the set of equivalence classes of \( \equiv_{\text{MSO}}^{k} \). Because \( C \) is finite and its size is independent of the size of \( G \) (Theorem 2), for each class \( \alpha \in C \), there exists an \([m]\)-colored \( \tau \)-boundaried graph \((G^\alpha, \vec{X}^\alpha, p^\alpha)\) of type \( \alpha \) whose size is upper-bounded by a function of \( k, m \) and \( \tau \). For each \( \alpha \in C \), we fix one such graph, denote it by \( W(\alpha) \) and call it the witness of \( \alpha \). Let \( \mathcal{W} = \{W(\alpha) \mid \alpha \in C\} \). The witnesses make it possible to compute the indicator function \( \rho_\varphi \): for every \( \alpha \in C \), we set \( \rho_\varphi(\alpha) = 1 \) if and only if \( W(\alpha) \models \varphi \), and we set \( \rho_\varphi(\alpha) = 0 \) otherwise.

\begin{lemma}[implicitly in [16] in the proof of Theorem 4.6 and Corollary 4.7] The set \( \mathcal{W} \) and the indicator function \( \rho_\varphi \) can be computed in time \( f(k, m, \tau) \), for some computable function \( f \).
\end{lemma}

It will be important to have an efficient algorithmic test for MSO\([k, \tau] \)-elementary equivalence. This can be done using the Ehrenfeucht-Fraïssé games:

\begin{lemma}[Theorem 7.7 [30]] Given two \([m]\)-colored \( \tau \)-boundaried graph \( G_1^{[m], \tau} \) and \( G_2^{[m], \tau} \), it can be decided in time \( f(m, k, \tau, |G_1|, |G_2|) \) whether \( G_1^{[m], \tau} \equiv_{\text{MSO}}^{k} G_2^{[m], \tau} \), for some computable function \( f \).
\end{lemma}

\begin{corollary} Recognizing the type of an \([m]\)-colored \( \tau \)-boundaried graph \( G^{[m], \tau} \) can be done in time \( f(m, k, \tau, |G|) \), for some computable function \( f \).
\end{corollary}

Now we describe a linear time construction of the sets of feasible types, pairs and triples of types \( \mathcal{F}(b), \mathcal{F}_p(b) \) and \( \mathcal{F}_t(b) \) for all relevant nodes \( b \) in \( T \). In the initialization phase, we construct the set \( \mathcal{W} \) and the indicator function \( \rho_\varphi \) using the algorithm from Lemma 9. The rest of the construction is inductive, starting in the leaves of \( T \) and advancing in a bottom up fashion towards the root of \( T \). The idea is to always replace a possibly large graph \( G_\alpha^{[m], \tau} \) of type \( \alpha \) by the small witness \( W(\alpha) \) when computing the set of feasible types for the parent of the node \( \alpha \).

**Leaf node.** For every leaf node \( a \in V(T) \) we set \( \mathcal{F}(a) = \{\alpha_1\} \). Obviously, this corresponds to the definition in Section 3.
**Introduction**. Assume that $b \in V(T)$ is an introduce node with a child $a \in V(T)$ for which $\mathcal{F}(a)$ has already been computed, and $v \in V(G)$ is the new vertex. For every $\alpha \in \mathcal{F}(a)$, we first produce a $\tau'$-boundaried graph $H^\tau' = (H^\alpha, \vec{q})$ from $W(\alpha) = (G^\alpha, \vec{X}^\alpha, \vec{p}^\alpha)$ as follows: let $\tau' = |\vec{p}^\alpha| + 1$ and $H^\alpha$ be obtained from $G^\alpha$ by attaching to it a new vertex in the same way as $v$ is attached to $G_a$. The boundary $\vec{q}$ is obtained from the boundary $\vec{p}^\alpha$ by inserting in it the new vertex at the same position that $v$ has in the boundary of $(G_a, \sigma(B(a)))$. For every subset $I \subseteq \{1, \ldots, m\}$ we construct an $[m]$-coloring $Y^\alpha,I$ from $\vec{X}^\alpha$ by setting $Y^\alpha,i_1 = X_1^\alpha \cup \{v\}$, for every $i_1 \in I$, and $Y^\alpha,i_1 = X_i^\alpha$, for every $i_1 \notin I$. Each of these $[m]$-colorings $Y^\alpha,I$ is used to produce an $[m]$-colored $\tau'$-boundaried graph $(H^\alpha, \vec{Y}^\alpha,I, \vec{q})$ and the types of all these $[m]$-colored $\tau'$-boundaried graphs are added to the set $\mathcal{F}(b)$ of feasible types of $b$, and, similarly, the pairs $(\alpha, \beta)$ where $\beta$ is a feasible type of some of the $[m]$-colored $\tau'$-boundaried graph $(H^\alpha, \vec{Y}^\alpha,I, \vec{q})$, are added to the set $\mathcal{F}_b(b)$ of all feasible pairs of types of $b$. The correctness of the construction of the sets $\mathcal{F}(b)$ and $\mathcal{F}_b(b)$ for the node $b$ of $T$ follows from Lemma 4.

**Forget node.** Assume that $b \in V(T)$ is a forget node with a child $a \in V(T)$ for which $\mathcal{F}(a)$ has already been computed and that the $d$-th vertex of the boundary $\sigma(B(a))$ is the vertex being forgotten. We proceed in a similar way as in the case of the introduce node. For every $\alpha \in \mathcal{F}(a)$ we produce an $[m]$-colored $\tau'$-boundaried graph $(H^\alpha, \vec{Y}^\alpha, \vec{q})$ from $W(\alpha) = (G^\alpha, \vec{X}^\alpha, \vec{p}^\alpha)$ as follows: let $\tau' = |\vec{p}^\alpha| + 1$, $H^\alpha = G^\alpha$, $\vec{Y}^\alpha = \vec{X}^\alpha$ and $\vec{q} = (p_1, \ldots, p_d, p_{d+1}, \ldots, p_{\tau' + 1})$. For every $\alpha \in \mathcal{F}(a)$, the type $\beta$ of the constructed graph is added to $\mathcal{F}(b)$, and, similarly, the pairs $(\alpha, \beta)$ are added to $\mathcal{F}_b(b)$. The correctness of the construction of the sets $\mathcal{F}(b)$ and $\mathcal{F}_b(b)$ for the node $b$ of $T$ follows from Lemma 5.

**Join node.** Assume that $c \in V(T)$ is a join node with children $a, b \in V(T)$ for which $\mathcal{F}(a)$ and $\mathcal{F}(b)$ have already been computed. For every pair of compatible types $\alpha \in \mathcal{F}(a)$ and $\beta \in \mathcal{F}(b)$, we add the type $\gamma$ of $W(\alpha) \uplus W(\beta)$ to $\mathcal{F}(c)$, and the triple $(\alpha, \beta, \gamma)$ to $\mathcal{F}_3(c)$. The correctness of the construction of the sets $\mathcal{F}(c)$ and $\mathcal{F}_3(c)$ for the node $b$ of $T$ follows from Lemma 3.

It remains to construct the indicator function $\mu$. We do it during the construction of the sets of feasible types as follows. We initialize $\mu$ to zero. Then, every time we process a node $b$ in $T$ and we find a new feasible type $\beta$ of $b$, for every $v \in B(b)$ and for every $i$ for which $d$-th vertex in the boundary of $W(\beta) = (G^\beta, \vec{X}, \vec{p})$ belongs to $X_i$, we set $\mu(\beta, v, i) = 1$ where $d$ is the order of $v$ in the boundary of $(G_b, \sigma(B(b)))$. The correctness follows from the definition of $\mu$ and the definition of feasible types.

Concerning the time complexity of the inductive construction, we observe, exploiting Corollary 11, that for every node $b$ in $T$, the number of steps, the sizes of graphs that we worked with when dealing with the node $b$, and the time needed for each of the steps, depends on $k$, $m$ and $\tau$ only. We summarize the main result of this section in the following theorem.

**Theorem 12.** Under the assumptions of Theorem 6, the polytope $P_\varphi(G)$ can be constructed in time $f'(|\varphi|, \tau) \cdot n$, for some computable function $f'$.

### 5.1 Courcelle’s Theorem and Optimization

It is worth noting that even though linear time optimization versions of Courcelle’s theorem are known, our result provides a linear size LP for these problems out of the box. Together with a polynomial algorithm for solving linear programming we immediately get the following:
Theorem 13. Given a graph $G$ on $n$ vertices with treewidth $\tau$, a formula $\varphi \in \text{MSO}$ with $m$ free variables and real weights $w_{vi}$, for every $v \in V(G)$ and $i \in \{1, \ldots, m\}$, the problem

$$\text{opt} \left\{ \sum_{v \in V(G)} \sum_{i=1}^{m} w_{vi} \cdot y_{vi} \mid y \text{ satisfies } \varphi \right\}$$

where $\text{opt}$ is $\min$ or $\max$, is solvable in time polynomial in the input size.

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References