Quasi-4-Connected Components

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Abstract

We introduce a new decomposition of a graphs into *quasi-4-connected components*, where we call a graph *quasi-4-connected* if it is 3-connected and it only has separations of order 3 that separate a single vertex from the rest of the graph. Moreover, we give a cubic time algorithm computing the decomposition of a given graph.

Our decomposition into quasi-4-connected components refines the well-known decompositions of graphs into biconnected and triconnected components. We relate our decomposition to Robertson and Seymour's theory of tangles by establishing a correspondence between the quasi-4-connected components of a graph and its tangles of order 4.

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1 Introduction

Decompositions of graphs into their connected, biconnected and triconnected components are fundamental in structural graph theory, and they also belong to the basic toolbox of algorithmic graph theory. The existence of such decompositions goes back to work of MacLane [12] from the 1930s (also see Tutte [21]). In the 1970s, Hopcroft and Tarjan [10, 20] showed that the decompositions can be computed in linear time.

In modern terms, the decompositions into biconnected and triconnected components are best described as tree decompositions. To state the decomposition theorems and also our main result, a few technical definitions are unavoidable. Recall that a *tree decomposition* of a graph G is a pair (T, β), where T is a tree and β a mapping that associates a set $\beta(t) \subseteq V(G)$, called the *bag* at t, with every node t of the tree T (subject to certain conditions; see Section 2). The *adhesion* of the decomposition is the maximum of the sizes $|\beta(t) \cap \beta(u)|$ for tree edges tu, which intuitively is the order of the separations of the decomposition. Now the decomposition into biconnected components can be phrased as follows: every graph G has a tree decomposition (T, β) of adhesion at most 1 such that for all tree nodes t the induced subgraph $G[\beta(t)]$ is either 2-connected or a complete graph of order at most 2. The decomposition into triconnected components is more complicated, mainly because the triconnected components of a graph are no longer induced subgraphs, but just topological subgraphs. We say that the *torso* of a set $X \subseteq V(G)$ of vertices of a graph G is the graph $G[\![X]\!]$ obtained from the induced subgraph $G[X]$ by adding edges vw for all distinct $v, w \in X$ such that there is a connected component C of $G \setminus X$ with $v, w \in N(C)$, the neighbourhood of C in G. For example, the torso of the set $X = \{x_1, \dots, x_4\}$ in the graph G shown in Figure 1(a) is the complete graph on X. Now the decomposition into triconnected components can be phrased as follows: every graph G has a tree decomposition (T, β) of adhesion at most 2 such that for all tree nodes t the torso $G[\![\beta(t)]\!]$ is a topological subgraph of G that is either 3-connected or a complete graph of order at most 3.

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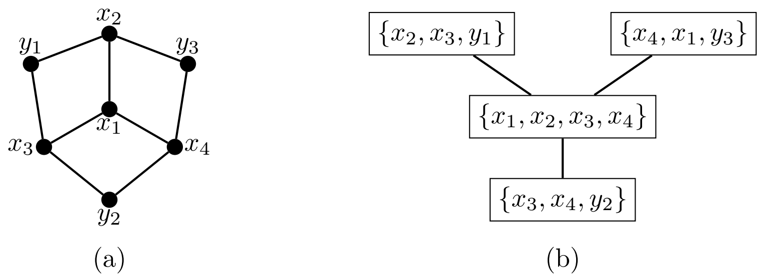


Figure 1 A graph and its decomposition into triconnected components.

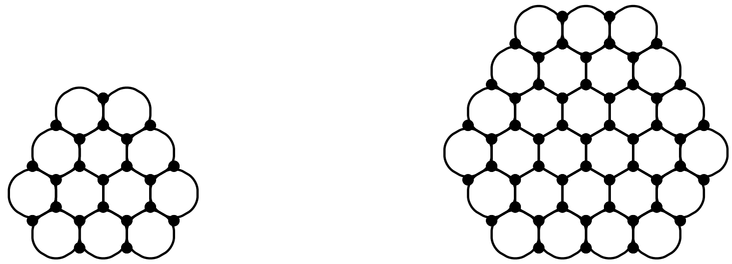


Figure 2 Hexagonal grids of radius 2 and 3.

How about decompositions into 4-connected components, or k-connected components for $k \geq 4$? At least in the clean form of the above decomposition theorems, they simply do not exist. Consider, for example, a hexagonal grid (see Figure 2). Even though the grid is not 4-connected, and it does not even have a nontrivial 4-connected subgraph, there is no good way of decomposing it in a tree-like fashion by separations of order 3. In fact, the only separations of the grid of order 3 are those splitting off a single vertex. If we ignore such separations, we may view the whole grid as one highly connected region. Let us call a graph G *quasi-4-connected* if it is 3-connected and for all separations (Y, S, Z) of order 3 (that is, $|S| = 3$ and Y, S, Z form a partition of $V(G)$ and there are no edges between Y and Z), either $|Y| \leq 1$ or $|Z| \leq 1$. Surprisingly, with this mild relaxation of 4-connectivity we get a nice decomposition theorem along the lines of the decompositions into biconnected and triconnected components.

▶ **Theorem 1** (Decomposition Theorem). *Every graph G has a tree decomposition (T, β) of adhesion at most 3 such that for all tree nodes t the torso $G[\![\beta(t)]\!]$ is a minor of G that is either quasi-4-connected or a complete graph of order at most 4.*

Furthermore, this decomposition can be computed in cubic time.

The decomposition is not unique, but the isomorphism types of the *quasi-4-connected components* into which we decompose are.

There have been earlier generalisations of the decomposition of graphs into triconnected components. The most prominent of these are Robertson and Seymour's tangles [17], which play an important role in the structure theory for graphs with excluded minors [16]. Intuitively, a tangle of order k describes a "k-connected region" in a graph by "pointing to it", that is, by assigning a direction to each separation of order less than k in such a way that "most" of the region described by the tangle is on the side the separation is directed towards. It is known that the tangles of orders 1, 2, 3 are in one-to-one correspondence to the connected, biconnected and triconnected components of a graph [17, 8]. We establish a similar correspondence between the tangles of order 4 and the quasi-4-connected components. This is our second main theorem, which I think is interesting in its own right, but which is also essential for the proof of Theorem 1. We defer the precise technical statement of this Correspondence Theorem to Section 4 (Theorem 4).

This paper grew out of my work on descriptive complexity theory for graph classes with excluded minors [6, 5], and this may also serve as an illustration of potential applications of our Decomposition Theorem. Separations of order 3 play a special, but somewhat annoying role in the main structure theorems for graph classes with excluded minors such as the "Flat Grid Theorem" of [18] and the structure theorem of [19], and the theorems simplify for quasi-4-connected graphs. In [5] I exploited some of the main ideas underlying our Decomposition Theorem to obtain such simplifications in the context of logical definability, and I believe the Decomposition Theorem proved here may turn out to be similarly useful in an algorithmic context.[1]

Due to space limitations, I had to omit many proof details, examples, and remarks from this conference version of the paper. They can be found in the full version [7] (available on arXiv).

1.1 Related work

It was shown in [17, 1] that for every k, every graph admits a canonical decomposition into its tangles of order k. Related to this is the decomposition into so-called $(k-1)$-blocks due to [3]. An important difference between these results and ours, or rather an additional feature of our decomposition, is that the pieces of our decomposition are quasi-4-connected graphs in their own right and can be dealt with separately (for example in an algorithmic context), whereas tangles of order 4 or 3-blocks are only defined relative to the surrounding graph.

In [15, 14], a notion of k-edge connected component is considered. It is similar to the $(k-1)$-blocks, but with respect to edge connectivity.

On the algorithmic side, it was shown in [9] that the decomposition of a graph into its tangles of order k can be computed in time $n^{O(k)}$. I believe that our techniques can be used to improve this to cubic time for $k = 4$.

There is a different line of work on "k-connected components" that, as far as I can see, is unrelated to ours. There, k-connected components are simply defined as maximal k-connected subgraphs (see, for example, [13]). This leads to completely different decompositions. For example, a graph of maximum degree 3 will only have trivial 4-connected components in this framework. However, what I see as the crucial difference between our form of decomposition and this line of work is that we get *tree* decompositions. This is important for typical dynamic-programming or divide-and-conquer algorithms on the decomposition.

2 Preliminaries

We assume basic knowledge of graph theory and refer the reader to [4] for background. Our notation is standard, let us just review the most important and frequently used notations.

[1] Let me clarify the relation of this work to Chapter 10 of the forthcoming monograph [5]. The basic ideas are the same, and actually my original motivation for the present paper was to make these ideas accessible to readers not interested in logic. However, when I started to work on this paper I noticed the connection to tangles, and it is this connection that provides the right framework and also makes the decomposition much simpler. On the other hand, the main goal of [5] is to obtain a decomposition that is definable in fixed-point logic with counting, and the decomposition we obtain here is not. So, except for some of the basic lemmas underlying the proof of the Correspondence Theorem, the results are incomparable.

All graphs considered in this paper are finite and simple. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The *order* of G is $|G| := |V(G)|$. For a set $W \subseteq V(G)$, we denote the induced subgraph of G with vertex set W by $G[W]$ and the induced subgraph with vertex set $V(G) \setminus W$ by $G \setminus W$. For a vertex v, we denote the set of neighbours of v in G by $N^G(v)$. In this and similar notations, we omit the index ^{G} if G is clear from the context. For a set $W \subseteq V(G)$, we define $N(W) := \left(\bigcup_{v \in W} N(v)\right) \setminus W$, and for a subgraph $H \subseteq G$ we let $N^G(H) := N^G(V(H))$.

A *tree decomposition* of a graph G is a pair (T, β), where T is a tree and $\beta : V(T) \to 2^{V(G)}$ such that for all $v \in V(G)$ the set $\{t \in V(T) \mid v \in \beta(t)\}$ is connected in T and for all $vw \in E(G)$ there is a $t \in V(T)$ such that $v, w \in \beta(t)$.

A *minor* of G is a graph obtained from G by deleting vertices and edges and contracting edges. A *model* of H in G consists of a family $(M_w)_{w \in V(H)}$ of mutually disjoint connected subsets of $V(G)$ and a family $(e_f)_{f \in E(H)}$ of edges of G such that for every edge $f = ww'$ of H the edge e_f has one endvertex in M_w and one endvertex in $M_{w'}$. Then H is a *minor* of G if and only if there is a model of H in G. We call the sets M_w, for $w \in V(H)$, the *branch sets* of the model $\mathcal{M}$. When reasoning about a model, it is often enough to know the branch sets.

A *faithful model* of H in G is a model $\big((M_w)_{w \in V(H)}, (e_f)_{f \in E(H)}\big)$ such that $w \in M_w$ for all $w \in V(H)$. We say that H is a *faithful minor* of G if $V(H) \subseteq V(G)$ and there is a faithful model of H in G.

Separations of a graph G are usually defined as pairs of subgraphs. However, in this paper it will be more convenient to view them as partitions of the vertex set. We say that a *separation* of G is a triple (Y, S, Z) of (possibly empty) mutually disjoint subsets of $V(G)$ such that $Y \cup S \cup Z = V(G)$ and there is no edge $vw \in E(G)$ such that $v \in Y$ and $w \in Z$. The order of the separation (Y, S, Z) is $|S|$, and the separation is *proper* if both Y and Z are nonempty. The set of all separations of G is denoted by $\mathrm{Sep}(G)$, and the subset of all separations of order less than k (at most k, exactly k) by $\mathrm{Sep}_{<k}(G)$ (resp. $\mathrm{Sep}_{\leq k}(G)$, $\mathrm{Sep}_{=k}(G)$).

A set $S \subseteq V(G)$ is a *separator* of G of *order* $k := |S|$, or a *k-separator*, if there are two vertices $v, w \in V(G) \setminus S$ such that there is a path from v to w in G, but no path from v to w in $G \setminus S$. Note that if G is connected then S is a separator if and only if there is a proper separation (Y, S, Z) of G.

A graph G is *k-connected* if $|G| > k$ and G has no proper $(k-1)$-separation.

A subset $X \subseteq V(G)$ of the vertex set of a graph G is *k-inseparable* if $|X| > k$ and there is no separation (Y, S, Z) of G of order at most k such that $X \cap Y \neq \emptyset$ and $X \cap Z \neq \emptyset$.

3 Tangles

Let G be a graph. Deviating from Robertson and Seymour's [17] original definition, we define tangles as families of separations of the vertex set (as we defined them in Section 2) rather than separations viewed as pairs of graphs or partitions of the edge set. (We show that the two notions are equivalent in the full version [7].) A *G-tangle* of order k is a family $\mathcal{T} \subseteq \mathrm{Sep}_{<k}(G)$ of separations of G of order less than k satisfying the following conditions.

(T.1) For all separations $(Y, S, Z) \in \mathrm{Sep}_{<k}(G)$ either $(Y, S, Z) \in \mathcal{T}$ or $(Z, S, Y) \in \mathcal{T}$.

(T.2) If $(Y_1, S_1, Z_1), (Y_2, S_2, Z_2), (Y_3, S_3, Z_3) \in \mathcal{T}$ then either $Z_1 \cap Z_2 \cap Z_3 \neq \emptyset$ or there is an edge $e \in E(G)$ that has an endvertex in each Z_i.

(T.3) If $(Y, S, Z) \in \mathcal{T}$ then $Z \neq \emptyset$.

For background on tangles and examples, I refer the reader to [17, 8].

Let G be a graph. We define a partial order $\preceq$ on $\text{Sep}(G)$ by letting

$$(Y, S, Z) \preceq (Y', S', Z') \ :\Longleftrightarrow\ S \cup Z \subset S' \cup Z' \text{ or } \big(S \cup Z = S' \cup Z' \text{ and } S \subseteq S'\big). \tag{1}$$

For a G-tangle $\mathcal{T}$, we let $\mathcal{T}_{\min}$ be the set of minimal elements of $\mathcal{T}$ with respect to the partial order $\preceq$. The minimal elements of a tangle will play an important role later. It can be shown that if $(Y, S, Z) \in \mathcal{T}_{\min}$ then Z is connected in G and $S = N(Z)$.

It is shown in [17, 8] that the tangles of order at most 3 are in one-to-one correspondence to the connected, biconnected, and triconnected components of a graph. The following characterisation of the triconnected components motivates our definition of quasi-4-connected regions in the next section.

▶ **Proposition 2.** *Let G be a graph and $R \subseteq V(G)$. The the following are equivalent.*

1. *R is an inclusionwise maximal subset of G such that $G[\![R]\!]$ is 3-connected and a topological subgraph of G.*
2. *$G[\![R]\!]$ is 3-connected and a topological subgraph of G, and for every connected component C of $G \setminus R$ we have $|N(C)| \leq 2$.*

We call sets R satisfying the conditions of this proposition the *triconnected regions* of a graph and the graphs $G[\![R]\!]$ the *triconnected components*.

We can "lift" a tangle from a minor of a graph to the original graph. Let G be a graph, H a minor of G, and $\mathcal{M}$ a model of H in G, say, with branch sets $(M_w)_{w \in V(H)}$. For a separation $(Y, S, Z) \in \text{Sep}(G)$, the *$\mathcal{M}$-projection* of (Y, S, Z) to H is the triple $\pi_{\mathcal{M}}(Y, S, Z) = (Y', S', Z')$ of subsets of $V(H)$ defined by $Y' := \big\{w \in V(H) \bigm| V(M_w) \subseteq Y\big\}$, $S' := \big\{w \in V(H) \bigm| V(M_w) \cap S \neq \emptyset\big\}$, $Z' := \big\{w \in V(H) \bigm| V(M_w) \subseteq Z\big\}$. It is easy to see that (Y', S', Z') is a separation of H of order $|S'| \leq |S|$.

▶ **Lemma 3** ([17]). *Let G be a graph, H a minor of G, and $\mathcal{M}$ a model of H in G. Let $\mathcal{T}'$ be an H-tangle of order k. Then the set $\mathcal{T}$ of all separations $(Y, S, Z) \in \text{Sep}_{<k}(G)$ such that $\pi_{\mathcal{M}}(Y, S, Z) \in \mathcal{T}'$ is a G-tangle of order k.*

We call $\mathcal{T}$ be the *lifting* of $\mathcal{T}'$ to G with respect to the model $\mathcal{M}$. Clearly, the lifting may depend on the model. This is even the case if we only consider faithful minors and models.

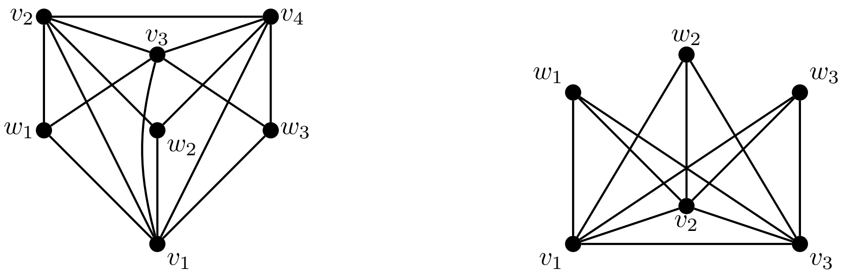
4 Tangles of Order 4

Let us now look at tangles of order 4. We restrict our attention to 3-connected graphs. This is natural; in the full version of the paper we also give a formal justification that we can do this without loss of generality. *For the rest of this section, we assume that G is a 3-connected graph.*

The main result of this section is a correspondence between tangles of order 4 and what we will call *quasi-4-connected regions* of a graph. This correspondence holds for all but a small number of *exceptional* regions, which we shall completely characterise. We first state the theorem; the necessary definitions follow.

▶ **Theorem 4** (Correspondence Theorem). *With every non-exceptional quasi-4-connected region R of G we can associate a G-tangle $\mathcal{T}_R$ of order 4 and with every G-tangle $\mathcal{T}$ of order 4 a non-exceptional quasi-4-connected region $R_{\mathcal{T}}$ such that $\mathcal{T} = \mathcal{T}_{R_{\mathcal{T}}}$.*

We shall call the torsos $G[\![R_{\mathcal{T}}]\!]$ for the G-tangles of order 4 the *quasi-4-connected components* of G.



(a) the graph TH_{+3} (b) the graph TR_{+3}

Figure 3 The exceptional quasi-4-connected graphs.

In general, the mapping $R \mapsto \mathcal{T}_R$ is not injective; the mapping $\mathcal{T} \mapsto R_{\mathcal{T}}$ is (otherwise the theorem could not hold). The mapping $R \mapsto \mathcal{T}_R$ is *canonical* (or *isomorphism invariant*). This means that for any two graphs G, G' and regions R, R', if f is an isomorphism from G to G' that maps R to R' then f also maps $\mathcal{T}_R$ to $\mathcal{T}_{R'}$. The mapping $\mathcal{T} \mapsto R_{\mathcal{T}}$ is not canonical. However, the mapping from $\mathcal{T}$ to the quasi-4-connected component $G[\![R_{\mathcal{T}}]\!]$, viewed as an abstract graph, is.

We can only give a very high-level outline of the proof of the Correspondence Theorem.

4.1 Quasi-4-Connected Graphs and Regions

Recall from the introduction that a graph G is *quasi-4-connected* if G is 3-connected and for all separations $(Y, S, Z) \in \mathrm{Sep}_{=3}(G)$, either $|Y| \leq 1$ or $|Z| \leq 1$. A quasi-4-connected graph G is *exceptional* if it is isomorphic to a subgraph of one of the graphs TH_{+3} or TR_{+3} shown in Figure 3.

▶ **Theorem 5.** *Let G be a quasi-4-connected graph. Then G has a tangle of order 4 if and only if it is not exceptional. Furthermore, if G has a tangle of order 4, it has exactly one such tangle, which consists of all separations $(Y, S, Z) \in \mathrm{Sep}_{<4}(G)$ such that $|Y| < |Z|$.*

A *quasi-4-connected region* of G is a subset $R \subseteq V(G)$ satisfying the following conditions.

(Q.1) $G[\![R]\!]$ is a faithful minor of G.

(Q.2) $G[\![R]\!]$ is quasi-4-connected.

(Q.3) For every connected component C of $G \setminus R$ it holds that $N(C) = 3$.

While conditions (Q.1) and (Q.2) are, to some extent, natural, condition (Q.3) may seem less so. It is a (weak) maximality condition: if $R' \supset R$ such that $G[\![R']\!]$ is quasi-4-connected, then $R' \setminus R$ contains at most one vertex of every connected component C of $G \setminus R$ (unless $|R| = 4$); otherwise $N(C)$ would be separator of $G[\![R']\!]$ witnessing that it is not quasi-4-connected. Conditions (Q.1)–(Q.3) are motivated by the characterisation of the triconnected components given in Proposition 2(2). The reason for choosing these conditions instead of adding some maximality condition is simply that they work best in combination with tangles and for the Decomposition Theorem; it is condition (Q.3) which guarantees that our decomposition will have adhesion 3.

Let R be a quasi-4-connected region of G. If $G[\![R]\!]$ is a non-exceptional quasi-4-connected graph, then it has a unique tangle of order 4, and using Lemma 3, we can lift this tangle to a

G-tangle of order 4. It can be proved that the lifted tangle does not depend on the model of $G[\![R]\!]$ in G, as long as it is faithful. In this case we let $\mathcal{T}_R$ (of the Correspondence Theorem) be this lifted tangle.

However, sometimes we can even associate a tangle with a quasi-4-connected region R if $G[\![R]\!]$ is exceptional. A *non-exceptional extension* of R is a graph $\widehat{H}$ satisfying the following conditions.

(X.1) $\widehat{H}$ is a faithful minor of G.

(X.2) $\widehat{H}$ is non-exceptional quasi-4-connected.

(X.3) $R \subseteq V(\widehat{H})$, and for each connected component C of $G \setminus R$ we have $|V(\widehat{H}) \cap V(C)| \leq 1$.

(X.4) Subject to (X.1)–(X.3), $V(\widehat{H})$ is inclusionwise minimal.

We call the region R *non-exceptional* if it has a non-exceptional extension. Note that if $G[\![R]\!]$ is a non-exceptional quasi-4-connected graph, then $G[\![R]\!]$ is a non-exceptional extension of R.

If $G[\![R]\!]$ is exceptional and $\widehat{H}$ is a non-exceptional extension of R, then there is a unique $\widehat{H}$-tangle $\widehat{\mathcal{T}}$ of order 4. Let $\mathcal{I}$ be a faithful image of $\widehat{H}$ in G. Using Lemma 3, we can lift this tangle to a G-tangle $\mathcal{T}(\widehat{H}, \mathcal{I})$ of order 4. It turns out that this tangle neither depends on the choice of $\widehat{H}$ nor on the choice of $\mathcal{I}$. We let $\mathcal{T}_R := \mathcal{T}(\widehat{H}, \mathcal{I})$.

4.2 The Region of a Tangle

The goal of this section is to define the mapping $\mathcal{T} \mapsto R_{\mathcal{T}}$ from G-tangles of order 4 to quasi-4-connected regions. This is much more difficult than defining the mapping $R \mapsto \mathcal{T}_R$; technically it is clearly the most difficult part of the paper. We can only give the basic idea here. *We fix a G-tangle $\mathcal{T}$ of order 4 for the rest of the section.*

We call two separations $(Y_1, S_1, Z_1), (Y_2, S_2, Z_2) \in \mathrm{Sep}(G)$ *orthogonal* if $(Y_1 \cup S_1) \cap (Y_2 \cup S_2) \subseteq S_1 \cap S_2$. It is not hard to show that the minimal separations of a tangle of order 3 in a graph are mutually orthogonal. The minimal separations of a tangle of order 4 are not necessarily orthogonal, but the next lemma shows that they can only "cross" in a very restricted way.

▶ **Lemma 6** (Crossing Lemma). *Let $(Y_1, S_1, Z_1), (Y_2, S_2, Z_2) \in \mathcal{T}_{\min}$ be distinct. Then either (Y_1, S_1, Z_1) and (Y_2, S_2, Z_2) are orthogonal or $Y_1 \cap Y_2 = \emptyset$ and $S_1 \cap S_2 = \emptyset$ and there is an edge $s_1 s_2 \in E(G)$ such that for $i = 1, 2$ we have $S_i \cap Y_{3-i} = \{s_i\}$.*

In the latter case, we call the edge $s_1 s_2$ the crossedge *of (Y_1, S_1, Z_1) and (Y_2, S_2, Z_2).*

We call a proper separation $(Y, S, Z) \in \mathrm{Sep}_{=3}(G)$ *degenerate* if $|Y| = 1$ and S is an independent set of G. It can be shown that if (Y, S, Z) is non-degenerate then $G[\![Z]\!]$ is a faithful minor of G. We call a crossedge e of separations $(Y_1, S_1, Z_1), (Y_2, S_2, Z_2) \in \mathcal{T}_{\min}$ *non-degenerate* if the two separations are non-degenerate. The key to our proof is the following lemma (which is actually easy to prove).

▶ **Lemma 7** (Crossedge Independence Lemma[2]). *The set of non-degenerate crossedges is a matching of G.*

Let $e^1, \ldots, e^m$ be the non-degenerate crossedges of $\mathcal{T}$, and suppose that $e^i = s_1^i s_2^i$. We contract all these edges to their endvertex s_1^i. The order of the contractions is irrelevant

2 Actually, this is only a corollary to what we call the "Crossedge Independence Lemma" in the full version [7].

because the edges form a matching. Up to isomorphism, it is also irrelevant whether we contract e^i to s_1^i or s_2^i. Let $G^{(m)}$ be the resulting graph. We show that $G^{(m)}$ is still 3-connected and has a tangle $\mathcal{T}^{(m)}$ of order 4 such that $\mathcal{T}$ is the lifting of $\mathcal{T}^{(m)}$ to G. Furthermore, $\mathcal{T}^{(m)}$ has no non-degenerate crossedges. Hence the non-degenerate separations in $\mathcal{T}^{(m)}_{\min}$ are mutually orthogonal. We let

$$R_{\mathcal{T}} := V(G^{(m)}) \setminus \bigcup_{\substack{(Y,S,Z)\in\mathcal{T}^{(m)}_{\min}\\ \text{non-degenerate}}} Y. \tag{2}$$

We show that $R_{\mathcal{T}}$ is a non-exceptional quasi-4-connected region of G and that $\mathcal{T} = \mathcal{T}_{R_{\mathcal{T}}}$.

5 Decomposition into Quasi-4-Connected Components

With the Correspondence Theorem at hand, it is now relatively easy to prove the Decomposition Theorem 1.

▶ **Theorem 8.** *Let G be a 3-connected graph. Then G has a tree decomposition (T,β) of adhesion at most 3 such that for all $t \in V(T)$, the torso $G[\![\beta(t)]\!]$ is either a complete graph K_3 or K_4 or a quasi-4-connected component of G.*

Furthermore, such a decomposition can be computed in time $O(n^2(n+m))$.

Here, and throughout this section, we denote the numbers of vertices and edges of the input graph G of our algorithms by n and m, respectively.

The Decomposition Theorem 1 follows by combining the decomposition of Theorem 8 with the standard decomposition of a graph into its triconnected components.

The proof of Theorem 8 requires some preparation. For the rest of this section, we assume that G is a 3-connected graph. Let $(Y,S,Z) \in \mathrm{Sep}_{=3}(G)$ be non-degenerate. A *split vertex* of (Y,S,Z) is a vertex $z \in Z$ such that for every connected component C of $G \setminus (S \cup \{z\})$ it holds that $|N(C)| = 3$.

▶ **Lemma 9.** *Let $(Y_0,S_0,Z_0) \in \mathrm{Sep}_{=3}(G)$ be a non-degenerate proper separation such that Z_0 is connected and (Y_0,S_0,Z_0) has no split vertex. Then the set $\mathcal{T}(Y_0,S_0,Z_0)$ of all separations $(Y,S,Z) \in \mathrm{Sep}_{<4}(G)$ such that either $Z_0 \subseteq Z$ or $|Z\cap S_0| > |Y \cap S_0|$ is a G-tangle of order 4.*

Proof. Let $\mathcal{T} := \mathcal{T}(Y_0,S_0,Z_0)$. To see that $\mathcal{T}$ satisfies (T.1), let $(Y,S,Z) \in \mathrm{Sep}_{<4}(G)$. If $S \subseteq Y_0 \cup S_0$, then the connected set Z_0 is either a subset of Z or of Y, and thus either $(Y,S,Z) \in \mathcal{T}$ or $(Z,S,Y) \in \mathcal{T}$. Suppose next that $|S \cap Z_0| = 1$. Let z be the unique vertex in $S \cap Z_0$. Then z is not a split vertex of (Y_0,S_0,Z_0), and hence there is a connected component C of $G \setminus (S_0 \cup \{z\})$ such that $N(C) = S_0 \cup \{z\}$. Then $V(C) \subseteq Z_0$, because $z \in Z_0$, and thus $V(C) \cap S = \emptyset$. It follows that either $V(C) \subseteq Y$ or $V(C) \subseteq Z$. Without loss of generality we may assume that $V(C) \subseteq Z$. As $S_0 \subseteq N(C)$, this implies $S_0 \setminus S \subseteq Z$. As $S_0 \setminus S \neq \emptyset$, it follows that $(Y,S,Z) \in \mathcal{T}$. Finally, suppose that $|S \cap Z_0| \geq 2$. If $S \cap S_0 = \emptyset$, then either $|Z \cap S_0| \geq 2$ or $|Y \cap S_0| \geq 2$, and thus either $(Y,S,Z) \in \mathcal{T}$ or $(Z,S,Y) \in \mathcal{T}$. If $|S \cap S_0| = 1$, then $S \cap Y_0 = \emptyset$, and as G is 3-connected and $Y_0 \neq \emptyset$, the vertices in $S_0 \setminus S$ belong to the same connected component of $G \setminus S$. Hence either both are in Z or both are in Y, and again it follows that either $(Y,S,Z) \in \mathcal{T}$ or $(Z,S,Y) \in \mathcal{T}$.

Observe next that $|V(G)| \geq 6$, because $|Y_0| \geq 1$ and $|S_0| = 3$ and $|Z_0| \geq 2$ (otherwise the unique vertex in Z_0 would be a split vertex).

▶ **Claim 10.** *For all $(Y,S,Z) \in \mathcal{T}$ we have $|S \cup Z| \geq 4$.*

Proof. It follows from the definition of $\mathcal{T}$ that $Z \neq \emptyset$. If $Y = \emptyset$, then $|S \cup Z| = |V(G)| \geq 6$. Otherwise, (Y, S, Z) is a proper separation and thus $|S| = 3$, which implies $|S \cup Z| \geq 4$. ◀

The claim implies that $\mathcal{T}$ satisfies (T.3).

To prove that $\mathcal{T}$ satisfies (T.2), let $(Y_i, S_i, Z_i) \in \mathcal{T}$ for $i = 1, 2, 3$. Suppose for contradiction $Z_1 \cap Z_2 \cap Z_3 = \emptyset$ and that there is no edge that has an endvertex in each Z_i.

▶ **Claim 11.** *For distinct $i, j, k \in [3]$ and $x \in V(G)$, if $x \in Z_i \cap Z_j$ then $x \in Y_k$.*

Proof. We have $x \notin Z_k$, because $Z_i \cap Z_j \cap Z_k = \emptyset$. Suppose that $x \in S_k$, and let $z \in N(x) \cap Z_k$. Such a z exists, because $Z_k \neq \emptyset$ and $N(Z_k) \subseteq S_k$, and as $|S_k| \leq 3$ and G is 3-connected, this implies $N(Z_k) = S_k$. But the edge xz has an endvertex in every Z_i, which contradicts our assumption that no such edge exists. ◀

Case 1: There is an $i \in [3]$ such that $S_i \subseteq Y_0 \cup S_0$. Without loss of generality, we may assume that $i = 1$ and $(Y_1, S_1, Z_1) = (Y_0, S_0, Z_0)$. We may further assume that $S_i \not\subseteq Y_0 \cup S_0$ for $i = 2, 3$. Then $|Z_i \cap S_0| > |Y_i \cap S_0|$.

By Claim 11 we have $Z_2 \cap Z_3 \cap S_0 = Z_2 \cap Z_3 \cap S_1 = \emptyset$. Thus for some $i \in \{2, 3\}$ $|Z_i \cap S_0| < 2$. Without loss of generality we assume $|Z_2 \cap S_0| < 2$. Then $|Y_2 \cap S_0| = \emptyset$ and thus $|S_2 \cap S_0| = 2$. Since $S_2 \not\subseteq Y_0 \cup S_0$, we have $|S_2 \cap Z_0| = 1$. As the vertex in $S_2 \cap Z_0$ is not a split vertex, there is a connected component C of $G \setminus (S_0 \cup S_2)$ such that $N(C) = S_0 \cup S_2$. Then $V(C) \subseteq Z_0 \cap Z_2 = Z_1 \cap Z_2$. Now let $v \in Z_3 \cap S_0$, and let $w \in V(C)$ be adjacent to v. Then the edge vw has an endvertex in each Z_i.

Case 2: $|S_i \cap Z_0| \neq \emptyset$ for all $i \in [3]$. Then $|Z_i \cap S_0| > |Y_i \cap S_0|$. If $|Z_i \cap Z_j \cap S_0| = \emptyset$ for all $i \neq j$, then $|Z_i \cap S_0| = 1$ and thus $|Y_i \cap S_0| = 0$ for all i. Thus $|S_i \cap S_0| = 2$ and $|S_i \cap Y_0| = \emptyset$, because $S_i \not\subseteq S_0 \cup Y_0$. But this implies $Y_0 \subseteq Z_1 \cap Z_2 \cap Z_3$, which is a contradiction.

Hence without loss of generality we may assume that $Z_1 \cap Z_2 \cap S_0 \neq \emptyset$. Let $s \in Z_1 \cap Z_2 \cap S_0$. Then by Claim 11, $s \in Y_3$. Then $|Y_3 \cap S_0| \geq 1$, and this implies $|Z_3 \cap S_0| \geq 2$. Let $s', s'' \in Z_3 \cap S_0$. Then $S_0 = \{s, s', s''\}$.

If $|S_3 \cap Z_0| \leq 1$, there is a connected component C of $G \setminus (S_0 \cup S_3)$ such that $N(C) = S_0 \cup S_3$. But then there is a path from $s \in Y_3$ to $s' \in Z_3$ in $G \setminus S_3$, which is impossible. Hence $|S_3 \cap Z_0| \geq 2$.

Thus $|S_3 \cap Y_0| \leq 1$. Since G is a 3-connected and $Y_0 \neq \emptyset$, there is a path from s to $\{s', s''\}$ with all internal vertices in Y_0. Hence $|Y_0 \cap S_3| = 1$, and the unique vertex $y \in Y_0 \cap S_3$ separates $s \in Y_3$ from $\{s', s''\} \subseteq Z_3$ in the graph $G[Y_0 \cup S_0]$. Then $ss', ss'' \notin E(G)$. Furthermore, $sy \in E(G)$ and y is the only neighbour of s in $Y_0 \cup S_0$, because otherwise $\{y, s\}$ would be separator of G. By Claim 11 and because $y \in S_3$, we have $y \notin Z_1 \cap Z_2$. Say, $y \notin Z_2$. Then $y \in S_2$, because y is adjacent to $s \in Z_2$. As $S_2 \not\subseteq Y_0 \cup S_0$, it now follows that s' and s'' are not both in S_2. As $|Z_2 \cap S_0| > |Y_2 \cap S_2|$, one of these vertices, say, s' is in Z_2.

By Claim 11, $s' \in Z_2 \cap Z_3$ implies $s' \in Y_1$. Arguing as above with (Y_1, S_1, Z_1) instead of (Y_3, S_3, Z_3), we see that $Z_1 \cap S_0 = \{s, s''\}$ and $|S_1 \cap Z_0| = 2$ and $|S_1 \cap Y_0| = 1$, and the unique vertex $y' \in S_1 \cap Y_0$ separates s' from s, s'' in G. Furthermore, $s's, s's'' \notin E(G)$, and $s'y' \in E(G)$ and y' is the only neighbour of s' in $Y_0 \cup S_0$.

Now we have $s'' \in Z_1 \cap Z_3$, and again by the same argument we see that $s'' \in Y_2$ and $Z_2 \cap S_0 = \{s, s'\}$ and $|S_2 \cap Z_0| = 2$ and $|S_2 \cap Y_0| = 1$ and the unique vertex $y'' \in S_1 \cap Y_0$ separates s' from s, s'' in G. Furthermore, $s''s, s''s' \notin E(G)$, and $s''y'' \in E(G)$ and y'' is the only neighbour of s'' in $Y_0 \cup S_0$.

Let us rename the vertices s, s', s'' to s_{12}, s_{23}, s_{13} and the vertices y, y', y'' to y_{12}, y_{23}, y_{13}. Then for distinct i, j, k we have $s_{ij} \in S_0 \cap Z_i \cap Z_j \cap Y_k$ and $S_k \cap Y_0 = \{y_{ij}\}$ and $N(s_{ij}) \cap$

$(Y_0 \cup S_0) = \{y_{ij}\}$. Note that this implies that $S_0 = \{s_{12}, s_{13}, s_{23}\}$ is an independent set. Moreover, $Y_0 \setminus \{y_{ij}\} \subseteq Z_k$, because all $y \in Y_0 \setminus \{y_{ij}\}$ are reachable in $G[Y_0 \cup S_0] \setminus \{y_{ij}\}$ by a path from $\{s_{ik}, s_{jk}\} \subseteq Z_k$.

As the separation (Y_0, S_0, Z_0) is non-degenerate and S_0 is an independent set, we have $|Y_0| > 1$. Since $N(S_0) = \{y_{12}, y_{23}, y_{13}\}$ and $N(y_{ij}) \cap S_0 = \{s_{ij}\}$ and G is 3-connected, it is easy to see that this implies that the vertices y_{ij} are mutually distinct. Now let $e = vw$ be an arbitrary edge of $G[Y_0]$. Such an edge exists, and it has an endvertex in each Z_k. Again, this is a contradiction. ◀

Let $W, X \subseteq V(G)$. Then a (W, X)*-separation* is a separation (Y, S, Z) such that $W \subseteq Y \cup S$ and $X \subseteq Z \cup S$. It is *proper* if $W \cap Y \neq \emptyset$ and $X \cap Z \neq \emptyset$. A (proper) (W, X)-separation (Y, S, Z) is *minimum* if its order is minimal, that is, there is no (proper) (W, X)-separation (Y', S', Z') such that $|S'| < |S|$. It is *leftmost minimum* if it is minimum and, subject to this condition, Y is inclusionwise minimal. It can be shown by a standard submodularity argument that there always is a unique leftmost minimum (W, X)-separation. There is not necessarily a unique leftmost minimum proper (W, X)-separation, but the number of such separations is (polynomially) bounded in terms of k.

▶ **Lemma 12.** *Let $k \geq 1$. Then there is a linear time algorithm that, given a graph G and sets $W, X \subseteq V(G)$, decides if there is a proper (W, X)-separation of order at most k, and if there is computes the set of all leftmost minimum proper (W, X)-separations.*

Let us say that a separation $(Y_0, S_0, Z_0) \in \mathrm{Sep}_{=3}(G)$ *defines* a tangle if (Y_0, S_0, Z_0) is non-degenerate and Z_0 is connected in G and (Y_0, S_0, Z_0) has no split vertex. Then the tangle *defined by* (Y_0, S_0, Z_0) is $\mathcal{T}(Y_0, S_0, Z_0)$ (of Lemma 9).

▶ **Lemma 13.** *There is an algorithm that, given a 3-connected graph G and a separation (Y_0, S_0, Z_0) of G of order 3 defining the tangle $\mathcal{T} = \mathcal{T}(Y_0, S_0, Z_0)$, computes the set of all non-degenerate separations in $\mathcal{T}_{\min}$ and the set of all non-degenerate crossedges of $\mathcal{T}$ in time $O(n(n + m))$.*

Proof. We show how to compute the set $\mathcal{T}_{\min}$; then we can easily filter out the non-degenerate separations.

Let $x \in Z_0$. Observe that if (Z, S, Y) is a proper $(S_0, \{x\})$-separation of order at most 3, then $(Y, S, Z) \in \mathcal{T}$. This follows immediately from the definition of $\mathcal{T}$. It implies the following equivalence for every separation (Y, S, Z) of G of order 3.

1. $(Y, S, Z) \in \mathcal{T}_{\min}$ and (Y, S, Z) does not cross (Y_0, S_0, Z_0).
2. There is an $x \in Z_0$ such that (Z, S, Y) is a leftmost minimum proper $(S_0, \{x\})$-separation.

We can use this equivalence to compute the set of all $(Y, S, Z) \in \mathcal{T}_{\min}$ such that (Y, S, Z) does not cross (Y_0, S_0, Z_0) (repeatedly applying the algorithm of Lemma 12 to all $x \in Z_0$). Note that the equivalence also gives us a linear bound on the number of such (Y, S, Z).

It remains to deal with the $(Y, S, Z) \in \mathcal{T}_{\min}$ crossing (Y_0, S_0, Z_0). For each $s \in S_0$ that has a unique neighbour $y \in Y_0 \cup S_0$, the edge sy may be a crossedge. This gives us at most three potential crossedges, and we deal with them separately. So let $s \in S$ and $y \in Y_0$ such that $N(s) \cap (Y_0 \cap S_0) = \{y\}$. Then for every separation $(Y, S, Z) \in \mathrm{Sep}_{=3}(G)$ the following are equivalent.

3. $y \in S$ and $\big(Z \cap (S_0 \cup Z_0), S \cap (S_0 \cup Z_0), Y \cap (S_0 \cup Z_0)\big)$ is a leftmost minimum proper $(S \setminus \{s\}, \{s\})$-separation in the graph $G[S_0 \cup Z_0]$.
4. $(Y, S, Z) \in \mathcal{T}_{\min}$ and (Y, S, Z) crosses (Y_0, S_0, Z_0) with crossedge ys.

To see this, note that (3) implies that $|S \cap Z_0| = 2$, because (Y_0, S_0, Z_0) has no split vertex. The equivalence between (3) and (4) allows us to compute the remaining separations in $\mathcal{T}_{\min}$.

As we have an overall linear bound on the number of separations in $\mathcal{T}_{\min}$, we can easily compute the set of non-degenerate crossedges. ◀

Let us a call a 3-separator S of G *degenerate* if there is a connected component C of $G \setminus S$ such that the separation $(G \setminus (S \cup V(C)), S, V(C))$ is degenerate. It is easy to see that this is the case if and only if S is an independent set and $G \setminus S$ has exactly two connected components, one of which has order 1.

▶ **Lemma 14.** *There is an a algorithm that, given a 3-connected graph G, decides if G has a non-degenerate 3-separator and computes one if there is in time $O(n^2(n+m))$.*

Proof. We first test if there is an $S \subseteq V(G)$ such that $|S| = 3$ and all connected components of $G \setminus S$ have order 1. In this case, S is a non-degenerate 3-separator if $|G| \geq 6$ or if $|G| \geq 5$ and S is not an independent set.

In the following, we assume that for every $S \subseteq V(G)$ such that $|S| = 3$ there is at least one connected components C of $G \setminus S$ such that $|C| \geq 2$. Now suppose that S is a non-degenerate 3-separator of G. Let Y be the vertex set of a connected component of G of size $|Y| \geq 2$, and let $Z := V(G) \setminus (S \cup Y)$. Let $y \in Y$ and $z \in Z$.

Then there is a leftmost minimum proper $(\{y\}, \{z\})$-separation (Y', S', Z') with $Y' \cup S' \subseteq Y \cup S$, because (Y, S, Z) is a minimum proper $(\{y\}, \{z\})$-separation. The separator S' is non-degenerate unless $Y' = \{y\}$ and S' is an independent set. Then $S' = N(y)$. However, in this case there is a leftmost minimum proper $(S', \{z\})$ separation (Y'', S'', Z'') such that S'' is non-degenerate. To see this, let $y' \in N(y) \cap Y$. Then there is a proper leftmost minimum $(S', \{z\})$ separation (Y'', S'', Z'') with $y, y' \in Y''$ and $Y'' \cup S'' \subseteq Y \cup S$, because (Y, S, Z) is a minimum proper $(S', \{z\})$ separation with $y, y' \in Y$. The set S'' is a non-degenerate 3-separator.

Thus we can find a non-degenerate 3-separator as follows. For all pairs y, z of distinct vertices, we compute all leftmost minimum proper $(\{y\}, \{z\})$-separations (Y', S', Z') and check if there is one such that S' is a non-degenerate 3-separator. If y has degree 3 and $S' := N(y)$ is an independent set, we also compute all leftmost minimum proper $(S', \{z\})$ separations (Y'', S'', Z'') and check if S'' is a non-degenerate 3-separator. ◀

Proof of Theorem 8. If G has no non-degenerate 3-separator, then G is quasi-4-connected, and we return the trivial tree decomposition with a one-node tree. In the following, we assume that G has at least one non-degenerate 3-separator.

We view the tree T in the tree decomposition as directed with all edges pointing away from the root, and we denote the descendant order in the tree by $\trianglelefteq$. With each (directed) edge $e = (s, t)$ of the tree we associate a separation $\mathrm{sep}(s, t) = (Y, S, Z)$ of order 3 such that Z is connected in G and $S = \beta(t) \cap \beta(s)$ and $S \cup Z = \bigcup_{u \trianglerighteq t} \beta(u)$.

We build the tree decomposition iteratively starting from the root r of the tree. We pick an arbitrary non-degenerate 3-separator S_r of G and let $\beta(r) := S_r$. For every connected component C of $G \setminus S_r$ we create a child t of r, and we let $\mathrm{sep}(r, t) := (V(G) \setminus (S_r \cup V(C)), S_r, V(C))$.

At every step of the construction, we pick a leaf t of the current tree such that $\beta(t)$ is not yet defined. Let s be the parent of t and $\mathrm{sep}(s, t) = (Y_0, S_0, Z_0)$.

Case 1: $|Z_0| \leq 1$. Then $|S_0 \cup Z_0| \leq 4$, and we let $\beta(t) := S_0 \cup Z_0$. The node t will remain a leaf of the final tree.

Case 2: $|Z_0| > 1$ and (Y_0, S_0, Z_0) has a split vertex $z_0 \in Z_0$. Then we let $\beta(t) := S_0 \cup \{z_0\}$. For every connected component C of $G \setminus (S_0 \cup \{z_0\})$ with $V(C) \subseteq Z_0$ we create a child u of t and let $\operatorname{sep}(t,u) := (V(G) \setminus (N(C) \cup V(C)), N(C), V(C))$.

Case 3: $|Z_0| > 1$ and (Y_0, S_0, Z_0) has no split vertex. Let $\mathcal{T} = \mathcal{T}(Y_0, S_0, Z_0)$ be the G-tangle of Lemma 9. Note that $(Y_0, S_0, Z_0) \in \mathcal{T}_{\min}$. Let $R_{\mathcal{T}}$ be the quasi-4-connected region associated with $\mathcal{T}$. When contracting the non-degenerate crossedges that involve S_0, we make sure that we contract them to their endvertices in S_0. Then $S_0 \subseteq R_{\mathcal{T}}$ and $Y_0 \cap R_{\mathcal{T}} = \emptyset$. We let $\beta(t) := R_{\mathcal{T}}$.

For every connected component C of $G \setminus R_{\mathcal{T}}$ we create a child u of t and let $\operatorname{sep}(t,u) := (V(G) \setminus (N(C) \cup V(C)), N(C), V(C))$.

The completes the description of our construction. We need to describe a time $O(n^2(n+m))$-algorithm implementing it. By Lemma 14, we can compute a non-degenerate 3-separator S_r (for the root r) within this time if there is one.

Now we show that we can handle every step of the construction in time $O(n(n+m))$. So let t be a leaf of the current tree, s its parent, and $(Y_0, S_0, Z_0) := \operatorname{sep}(s,t)$. Case 1 is easy. For Case 2, we need to compute all connected components of $G \setminus (S_0 \cup \{z\})$ for all $z \in Z_0$, which we can do in time $O(n(n+m))$. For Case 3, we need to compute $\mathcal{T}_{\min}$ and $R_{\mathcal{T}}$ for the tangle $\mathcal{T} = \mathcal{T}(Y_0, S_0, Z_0)$, and Lemma 13 allows us to do this. ◀

Note that the results of Section 4, in particular the Correspondence Theorem 4, are used in Case 3 of the proof of Theorem 8 (and this is the only place in the proof where they are used).

▶ Remark. Let (T, β) be tree decomposition of G into quasi-4-connected components. The G-tangles of order 4 are associated with all nodes t such that either $|\beta(t)| \geq 5$ or $|\beta(t)| = 4$ and for each subset $S \subseteq \beta(t)$ of size $|S| = 3$ there is a neighbour u of t such that $\beta(u) \cap \beta(t) = S$. In the second case, the neighbours of t allow us to find a non-exceptional extension of the quasi-4-connected region $\beta(t)$.

6 Conclusions

Relaxing 4-connectedness, we introduce the notion of quasi-4-connectedness of graphs and prove that every graph has a decomposition into quasi-4-connected components. We show that the quasi-4-connected components correspond to the tangles of order 4, putting our result in the context of recent work on tangles and decompositions [1, 2, 3, 9, 11, 17]. Furthermore, we prove that our decomposition can be computed in cubic time. Although we do not explore this in the present paper, I believe that the decomposition may turn out to be a useful algorithmic tool, just like the decomposition into 3-connected components (though maybe not quite as broadly applicable).

The most obvious question is whether our result has a generalisation to "quasi-k-connected components", whatever they may be, for $k \geq 5$. I am skeptical, because we exploit many special properties of separators of order 3 here, most importantly the limited way in which they can cross. However, our decomposition is not a straightforward generalisation of the decomposition into 3-connected components either, and it may well be that new ideas lead to perfectly nice decompositions of higher order.

Finally, in particular when thinking of applications, it would be desirable to have a decomposition algorithm working in quadratic or even in linear time. I see no fundamental obstructions to the existence of such an algorithm.

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