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Due to the increasing nesting of the recursion, the $A_i$ represent the successive graduations in a hierarchy of all primitive recursive functions due to Grzegorczyk (see e.g. [30]).

The functions $A_i$ are all strictly increasing and hence injective, so have partial inverses:

(I) $A_0^{-1}: \mathbb{Z} \rightarrow \mathbb{Z}$ mapping $n \mapsto n - 1$,
(II) $A_1^{-1}: 2\mathbb{Z} \rightarrow \mathbb{Z}$ mapping $n \mapsto n/2$, and
(III) $A_i^{-1}: \text{Img } A_i \rightarrow \mathbb{N}$ for all $i > 1$.

Starting with zero and successively applying a few Ackermann functions and their inverses can produce an enormous integer. For example,

$$A_3 A_0 A_1^2 A_0(0) = A_3 A_0 A_1^2(1) = A_3 A_0 A_1(2) = A_3 A_0(4) = A_3(5) = 2^{65536}$$

because

$$A_3(5) = A_3^2 A_3(0) = A_3^2(1) = 2^{2^{2^2}} = 2^{65536}.$$

Thus Ackermann functions give highly compact representations for some very large numbers.

More precisely, here is how a string $w$ of Ackermann functions may represent an integer $w(0)$. For $x_1, \ldots, x_n \in \{A_0^{\pm 1}, \ldots, A_k^{\pm 1}\}$, we say the word $w = x_n x_{n-1} \cdots x_1$ is valid if $x_m x_{m-1} \cdots x_1(0)$ is defined for all $0 \leq m \leq n$. That is, if we evaluate $w(0)$ by starting with 0 and proceeding through $w$ from right to left applying successive $x_i$, we never encounter the problem that we are trying to apply $x_i$ to an integer outside its domain.

For example, $w := A_2^{-1} A_1 A_0$ is valid, and $w(0) = \log_2(2 \cdot 2 \cdot (0 + 1)) = 2$. But $A_2 A_0^{-1}$ and $A_1 A_1^{-1} A_0$ are not valid because $A_0^{-1}(0) = -1$ is not in $\mathbb{N}$ (the domain of $A_2$) and because $A_0(0) = 1$ is not in $2\mathbb{Z}$ (the domain of $A_1^{-1}$).

Motivated by applications in group theory that we will describe in the next section, we wish to compute with these representations in an efficient manner. (Our choices of $Z$ as the domains for $A_0$ and $A_1$ and our definition of $A_0$ represent small variations on the standard definitions of Ackermann functions, which are convenient for our applications.) One could just evaluate $w(0)$ using standard integer arithmetic, but this can be monumentally inefficient because of the sizes of the integers involved. Our first theorem is that it is possible to calculate efficiently in a rudimentary way with these representations of integers:

**Theorem 1.** Fix an integer $k \geq 0$. There is a polynomial-time algorithm, which on input a word $w$ on $A_0^{\pm 1}, \ldots, A_k^{\pm 1}$, declares whether or not $w(0)$ represents an integer, and if so whether $w(0) < 0$, $w(0) = 0$ or $w(0) > 0$.

(In fact our algorithm halts in time bounded above by a polynomial of degree $4 + k$. We have not attempted to optimize the degrees of the polynomial bounds on time complexity here or elsewhere in this work.)
Elements of a group $\Gamma$ with a generating set $A$ can be represented by words—that is, products of elements of $A$ and their inverses. To work with $\Gamma$, it is useful to have an algorithm which, on input a word, declares whether that word represents the identity element in $\Gamma$. After all, if we can recognize when a word represents the identity, then we can recognize when two words represent the same group element, and thereby begin to compute in $\Gamma$. The issue of whether there is such an algorithm is known as the word problem for $(\Gamma, A)$ and was first posed by Dehn [9, 10] in 1912. (He did not precisely ask for an algorithm, of course, rather ‘eine Methode angeben, um mit einer endlichen Anzahl von Schritten zu entscheiden...’—that is, ‘specify a method to decide in a finite number of steps...’)

Suppose a group $\Gamma$ has a finite presentation $\langle a_1, \ldots, a_m \mid r_1, \ldots, r_n \rangle$. The Dehn function $\text{Area} : \mathbb{N} \to \mathbb{N}$ quantifies the difficulty of a direct attack on the word problem: roughly speaking $\text{Area}(n)$ is the minimal $N$ such that if a word of length at most $n$ represents the identity, then it does so ‘as a consequence of’ at most $N$ defining relations.

Here is some notation that we will use to make this more precise. Associated to a set $\{a_1, a_2, \ldots\}$ (an alphabet) is the set of inverse letters $\{a_1^{-1}, a_2^{-1}, \ldots\}$. The inverse map is the involution defined on $\{a_1^{\pm 1}, a_2^{\pm 1}, \ldots\}$ that maps $a_i \mapsto a_i^{-1}$ and $a_i^{-1} \mapsto a_i$ for all $i$. The inverse map extends to words by sending $w = x_1 \cdots x_n \mapsto x_n^{-1} \cdots x_1^{-1} = w^{-1}$ when each $x_i \in \{a_1^{\pm 1}, a_2^{\pm 1}, \ldots\}$. Words $u$ and $v$ are cyclic conjugates when $u = \alpha \beta$ and $v = \beta \alpha$ for some subwords $\alpha$ and $\beta$. Freely reducing a word means removing all $a_i^{\pm 1} a_j^{\mp 1}$ subwords. For $\Gamma$ presented as above, applying a relation to a word $w = w(a_1, \ldots, a_m)$ means replacing some subword $\tau$ with another subword $\sigma$ such that some cyclic conjugate of $\tau \sigma^{-1}$ is one of $r_1^{\pm 1}, \ldots, r_n^{\pm 1}$.

For a word $w$ representing the identity in $\Gamma$, $\text{Area}(w)$ is the minimal $N \geq 0$ such that there is a sequence of freely reduced words $w_0, \ldots, w_N$ with $w_0$ the freely reduced form of $w$, and $w_N$ is the empty word, such that for all $i$, $w_{i+1}$ can be obtained from $w_i$ by applying a relation and then freely reducing. The Dehn function $\text{Area} : \mathbb{N} \to \mathbb{N}$ is defined by

$$\text{Area}(n) := \max \{ \text{Area}(w) \mid \text{words } w \text{ with } \ell(w) \leq n \text{ and } w = 1 \text{ in } \Gamma \}.$$ 

This is one of a number of equivalent definitions of the Dehn function. While a Dehn function is defined for a particular finite presentation for a group, its growth type—quadratic, polynomial, exponential etc.—does not depend on this choice. Dehn functions are important from a geometric point-of-view and have been studied extensively. There are many places to find background, for example [4, 5, 6, 10, 15, 16, 29, 31].

If $\text{Area}(n)$ is bounded above by a recursive function $f(n)$, then it is possible to solve the word problem by an exhaustive search: to tell whether or not a given word $w$ represents the identity, try all the possible ways of applying at most $f(n)$ defining relations and see whether one reduces $w$ to the empty word. (There are finitely presented groups for which there is no algorithm to solve the word problem [3, 27].) Conversely, when a finitely presented group admits an algorithm to solve its word problem, $\text{Area}(n)$ is bounded above by a recursive function (in fact $\text{Area}(n)$ is a recursive function) [14].

There are finitely presented groups for which an extrinsic algorithm is far more efficient than this intrinsic brute-force approach. A simple example is $\mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle$ (which has Dehn function $\text{Area}(n) \simeq n^2$). Given a word on $a^{\pm 1}, b^{\pm 1}$, the extrinsic approach amounts to searching exhaustively through all the ways of shuffling letters $a^{\pm 1}$ past letters $b^{\pm 1}$ to see if there is one which brings each $a^{\pm 1}$ together with an $a^{\mp 1}$ to be cancelled, and likewise each $b^{\pm 1}$ together with a $b^{\mp 1}$. It is much more efficient to read through the word and check that
the number of $a$ is the same as the number of $a^{-1}$, and the number of $b$ is the same as the number of $b^{-1}$.

There are more dramatic examples of groups where $\text{Area}(n)$ is a fast growing recursive function (so the ‘brute force’ algorithm succeeds but is extremely inefficient), but there are efficient ways to solve the word problem. Cohen, Madlener & Otto built extraordinary examples in a series of papers [7, 8, 25] where Dehn functions were first introduced (under then name derivational complexity). They designed their groups in such a way that the ‘intrinsic’ method of solving the word problem involves running a very slow algorithm which has been suitably ‘embedded’ in the presentation. But running this algorithm to see whether it halts on a given input is pointless as it is constructed to halt (eventually) on all inputs and so presents no obstacle to the word representing the identity. Their examples all admit algorithms to solve the word problem in running times that are at most $n \mapsto \exp(\exp(\ldots \exp(n)))$ for some $\ell$. But for each $k \in \mathbb{N}$ they have examples which have Dehn functions growing like $n \mapsto A_k(n)$. Indeed, better, they have examples with Dehn function growing like $n \mapsto A_n(n)$.

Recently, yet more extreme examples were constructed by Kharlampovich, Miasnikov & Sapir [20]. By simulating Minsky machines in groups, for every recursive function $f : \mathbb{N} \to \mathbb{N}$, they construct a finitely presented group (which also happens to be residually finite and solvable of class 3) with Dehn function growing faster than $f$, but with word problem solvable in polynomial time.

There are also ‘naturally arising’ groups which have fast growing Dehn function but an efficient (that is, polynomial-time) solution to the word problem. A first example is $\langle a, b \mid b^{-1}ab = a^2 \rangle$. Its Dehn function grows exponentially (see, for example, [4]), but the group admits a faithful matrix representation

$$a \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix},$$

so it is possible to check efficiently when a word on $a^{\pm 1}$ and $b^{\pm 1}$ represents the identity by multiplying out the corresponding string of matrices.

A celebrated 1-relator group due to Baumslag [1] provides a more dramatic example:

$$\langle a, b \mid (b^{-1}a^{-1}b)a(b^{-1}ab) = a^2 \rangle.$$\\

Platonov [28] proved its Dehn function grows like $n \mapsto \exp_2(\exp_2(\ldots (\exp_2(1)) \cdots)$, where $\exp_2(n) := 2^n$. (Earlier results in this direction are in [2, 14, 15].) Nevertheless, Miasnikov, Ushakov & Won [26] solve its word problem in polynomial time. (In unpublished work I. Kapovich and Schupp showed it is solvable in exponential time [33].)

Higman’s group

$$\langle a, b, c, d \mid b^{-1}ab = a^2, c^{-1}bc = b^2, d^{-1}cd = c^2, a^{-1}da = d^2 \rangle$$

from [19] is another example. Diekert, Laun & Ushakov [11] recently gave a polynomial time algorithm for its word problem and, citing a 2010 lecture of Bridson, claim it too has Dehn function growing like a tower of exponentials.

The groups we focus on here are yet more extreme ‘natural examples.” They arose in the study of hydra groups by Dison & Riley [13]. Let $\theta : F(a_1, \ldots, a_k) \to F(a_1, \ldots, a_k)$ be the automorphism of the free group of rank $k$ such that $\theta(a_i) = a_i$ and $\theta(a_i) = a_i a_{i-1}$ for $i = 2, \ldots, k$. The family

$$G_k := \langle a_1, \ldots, a_k, t \mid t^{-1}a_i t = \theta(a_i) \forall i > 1 \rangle,$$
are called hydra groups. Define

$$\Gamma_k := \langle a_1, \ldots, a_k, t, p \mid t^{-1}a_it = \theta(a_i), [p, a_i] = 1 \ \forall i \geq 1 \rangle,$$

which is an HNN-extension of $G_k$ in which an additional stable letter $p$ commutes with all elements of the subgroup $H_k := \langle a_1t, \ldots, a_kt \rangle$. It is shown in [13] that for $k = 1, 2, \ldots$, the subgroup $H_k$ is free of rank $k$ and $\Gamma_k$ has Dehn function growing like $n \mapsto A_k(n)$. Our second theorem is that nevertheless:

- **Theorem 2.** For all $k$, the word problem of $\Gamma_k$ is solvable in polynomial time.

  (In fact, our algorithm halts in time at most a polynomial of degree $3k^2 + k + 2$.)

3 The membership problem, subgroup distortion, and our third theorem

A geometric feature known as distortion is the root cause of the Dehn function of the group $\Gamma_k$ of the previous section growing like $n \mapsto A_k(n)$. The massive gap described in Theorem 2 between Dehn function and the time-complexity of the word problem for $\Gamma_k$ is attributable to a similarly massive gap between a distortion function and the time-complexity of a membership problem. Here are more details.

Suppose $H$ is a subgroup of a group $G$ and $G$ and $H$ have finite generating sets $S$ and $T$, respectively. So $G$ has a word metric $d_S(g, h)$, the length of a shortest word on $S^{\pm 1}$ representing $g^{-1}h$, and $H$ has a word metric $d_T$ similarly. The distortion of $H$ in $G$ is

$$\text{Dist}^G_H(n) := \max \{ d_T(1, g) \mid g \in H \text{ with } d_S(1, g) \leq n \}.$$  

(Distortion is defined here with respect to specific $S$ and $T$, but their choices do not affect the qualitative growth of $\text{Dist}^G_H(n)$.) A fast growing distortion function signifies that $H$ ‘folds back on itself’ dramatically as a metric subspace of $G$.

The membership problem for $H$ in $G$ is to find an algorithm which, on input of a word on $S^{\pm 1}$, declares whether or not it represents an element of $H$.

If the word problem of $G$ is decidable (as it is for all $G_k$, because, for instance, they are free-by-cyclic) and we have a recursive upper bound on $\text{Dist}^G_H(n)$, then there is a brute-force solution to the membership problem for $H$ in $G$. If the input word $w$ has length $n$, then search through all words on $T^{\pm 1}$ of length at most $\text{Dist}^G_H(n)$ for one representing the same element as $w$. This is, of course, likely to be extremely inefficient, and especially so for $H_k$ in $G_k$ as the distortion $\text{Dist}^G_H(n)$ grows like $n \mapsto A_k(n)$.

Nevertheless:

- **Theorem 3.** For all $k$, the membership problem for $H_k$ in $G_k$ is solvable in polynomial time.

  (The algorithm we construct to prove this halts in time at most polynomial of degree $3k^2 + k$.)

Reducing Theorem 2 to Theorem 3 is straight-forward, requiring little more than a standard result about HNN-extensions. We detail this in Section 5 of [12].

4 Comparing our methods for Theorem 1 with power circuits and straight-line programs

Our strategy compares and contrasts with those used to solve the word problem for Baumslag’s group in [26] and Higman’s group in [11], where power circuits are the key tool. Power
circuits provide concise representations of integers: power circuits of ‘size’ \( n \) represent (some) integers up to a height-\( n \) tower of powers of 2. There are efficient algorithms to perform addition, subtraction, and multiplication and division by 2 with power-circuit representations of integers, and to declare which of two power circuits represents the larger integer.

We too use concise representations of large integers, but in place of power circuits we use strings of Ackermann functions. These have the advantage that they may represent much larger integers. After all, \( A_3(n) = \exp_2^{n-1}(1) \) already produces a tower of exponents, and the higher rank Ackermann functions grow far faster. However, we are aware of fewer efficient algorithms to perform operations with strings of Ackermann functions than are available for power circuits: we only have Theorem 1.

Our methods also bear comparison with the work of Lohrey, Schleimer and their coauthors [17, 18, 21, 22, 23, 24, 32] on efficient computation in groups and monoids where words are given in compressed forms using \textit{straight-line programs} and are compared and manipulated using polynomial-time algorithms due to Hagenah, Plandowski and Lohrey. For instance Schleimer obtained polynomial-time algorithms solving the word problem for free-by-cyclic groups and automorphism groups of free groups and the membership problem for the handlebody subgroup of the mapping class group in [32].

5 The hydra phenomenon: connecting the group theory to Ackermann’s functions

The reason \( G_k \) are named hydra groups is that the extreme distortion of \( H_k \) in \( G_k \) stems from a string-rewriting phenomenon which is a reimagining of the battle between Hercules and the Lernean Hydra, a mythical beast which grew two new heads for every one Hercules severed. Think of a hydra as a word \( w \) on \( a_1, a_2, a_3, \ldots \). Hercules fights \( w \) as follows. He removes its first letter, then the remaining letters regenerate in that for all \( i > 1 \), each remaining \( a_i \) becomes \( a_i a_{i-1} \) (and each remaining \( a_1 \) is unchanged). This repeats. An induction on the highest index present shows that every hydra eventually becomes the empty word. (Details are in [13].) Hercules is then declared victorious. For example, the hydra \( a_3a_2a_1 \) is annihilated in 5 steps:

\[
a_2a_3a_1 \rightarrow a_3a_2a_1 \rightarrow a_2a_1a_1 \rightarrow a_1a_1 \rightarrow a_1 \rightarrow \text{empty word.}
\]

Define \( \mathcal{H}(w) \) to be the number of steps required to reduce a hydra \( w \) to the empty word. (So \( \mathcal{H}(a_3a_3a_1) = 5 \). Then, for \( k = 1, 2, \ldots \), define functions \( \mathcal{H}_k : \mathbb{N} \rightarrow \mathbb{N} \) by \( \mathcal{H}_k(n) = \mathcal{H}(a_k^n) \). It is shown in [13] that \( \mathcal{H}_k \) and \( A_k \) grow at the same rate for all \( k \), since the two families of functions exhibit a similar recursion relation.

Here is an outline of the argument from [13] as to why \( \text{Dist}^G_k \) grows at least as fast as \( n \mapsto \mathcal{H}_k(n) \) (and so as fast as \( n \mapsto A_k(n) \)). When \( k \geq 2 \) and \( n \geq 1 \), there is a reduced word \( u_{k,n} \) on \( \{a_1t, \ldots, a_kt\}^{\pm 1} \) of length \( \mathcal{H}_k(n) \) representing \( a_k^n \mathcal{H}_k(n) \) in \( G_k \) on account of the hydra phenomenon. (For example, \( u_{2,3} = (a_2t)^2(a_1t)(a_2t)(a_1t)^3 \) equals \( a_2^2t^7 \) in \( G_2 \) since \( a_2, a_2, a_1, a_2, a_1, a_1, a_1 \) and \( a_1 \) are the \( \mathcal{H}_2(3) = 7 \) initial letters removed by Hercules as he vanquishes the hydra \( a_2^3 \).) It follows that in \( G_k \)

\[
a_k^n a_2 t a_1 a_2^{-1} a_k^{-n} = u_{k,n} (a_2t) (a_1t) (a_2t)^{-1} u_{k,n}^{-1}.
\]

The word on the left is a product of length \( 2n + 4 \) of the generators \( a_1^\pm 1, \ldots, a_k^\pm 1 \) of \( G_k \) and that on the right is a product of length \( 2\mathcal{H}_k(n) + 3 \) of the generators \( (a_1t)^\pm 1, \ldots, (a_kt)^\pm 1 \) of \( H_k \). As \( \mathcal{H}_k \) is free of rank \( k \) and this word is reduced, it is not equal to any shorter word on these generators.
Hydra functions and Ackermann functions grow at the same rates, but do not precisely agree. So for Theorem 3 we, in fact, need a variation of Theorem 1, namely Proposition 3.4 in [12] which concerns a recursively defined family of functions $\psi$, we call $\psi$-functions. Like strings of Ackermann functions, strings of $\psi$-functions (which we call $\psi$-words) can concisely represent extremely large integers. We do not have a direct proof of the equivalence of this proposition to Theorem 1, but they can be proved in essentially the same ways as the defining recurrence for the $\psi_i$ is very similar to that for the $A_i$. We prefer to highlight Theorem 1 here because Ackermann functions have a long history and so are of intrinsic interest.

6 An outline of our strategy for Theorem 1

Here is a sketch of the algorithm we construct in Section 2 of [12] to prove Theorem 1. A more detailed high-level description is in Section 2.2 of [12].

Suppose we have a word $w$ on $A_0^{\pm 1}, \ldots, A_k^{\pm 1}$ and we seek to determine in polynomial time whether it is valid and, if so, whether the integer $w(0)$ is negative, zero, or positive.

We will attempt to pass to successive new words $w = w_0, w_1, \ldots$ that are equivalent to $w$ (denoted $w \sim w_j$) in that each $w_j$ is valid if and only if $w$ is, and when they both are, $w(0) = w_j(0)$. These words are obtained by making substitutions such as replacing a letter $A_{i+1}$ by a subword $A_i A_{i+1} A_0^{-1}$ (the recursion defining the Ackermann functions), or deleting a subword $A_i A_j^{-1}$ or $A_i A_0$. The lengths of these $w_j$ will all be at most a constant times the length of $w$, which is important for our proof that our algorithm halts in polynomial time. The aim of the substitutions is to reach a $w' \sim w$ which contains no $A_i^{-1}, \ldots, A_k^{-1}$. Eliminating these letters represents progress because they denote functions which have sparse domains and so present the greatest obstacle to checking whether a word is valid.

We will look at how to make these substitutions momentarily, but first here’s what happens when we have reached such a $w'$. Consider calculating a succession of integers beginning with 0 and ending with $w'(0)$ by evaluating $w'(0)$ letter-by-letter starting from the right. Only $A_0^{\pm 1}$ can trigger decreases in absolute value. So, to determine the sign of $w'(0)$, we can stop our evaluation if the integer calculated ever exceeds the length of $w'$: after all, whatever sign our evaluation then has will be the sign of $w'(0)$. This threshold for the integers in our calculation allow for a polynomial time bound.

So how do we reach this $w'$? The rough idea is to ‘cancel’ each $A_i^{-1}$ (where $i \geq 1$) in $w$ with some $A_i$ (if present) further to the right in $w'$. We do this inductively on $i$ by manipulating suffixes of the form $\sigma = A_i^{-1} u A_i v$ such that $u$ is a word on $A_0^{\pm 1}, \ldots, A_{i-1}^{\pm 1}$ and $v$ a word on $A_0, \ldots, A_{i-1}$. A number of complications may arise. For instance, there are exceptional cases when substituting a $A_{i+1}$ with $A_i A_{i+1} A_0^{-1}$ fails to preserve validity. Another issue is that we may have to introduce an $A_i$ ‘artificially’ to cancel with an $A_i^{-1}$.

It is only possible to give a few details of our algorithm in the space available here. We choose to present a subroutine BasePinch, which serves as the base case of this inductive process of manipulating suffixes (the instance where $u$ only contains letters $A_0^{\pm 1}$). It displays the crucial idea that allows us to operate within polynomial time: because the gaps between elements of $\text{Img} A_i$ are large, we can either recognize efficiently that $\sigma$ (and hence $w$) is invalid on account of $u$ not being able to carry $A_i v(0) \in \text{Img} A_i$ to another element of $\text{Img} A_i$ (this is what the commentary on line 12 below is about), or $\sigma$ is long enough that computing letter-by-letter by usual integer arithmetic is possible in polynomial time.

BasePinch will call two other subroutines (from Section 2.3 of [12]):
- Bounds which, on input $\ell \in \mathbb{N}$ (expressed in binary), returns in time $O(\ell)$ a list of all the (at most $(\log_2 \ell)^2$) triples of integers $(r, n, A_r(n))$ such that $r \geq 2, n \geq 3$, and $A_r(n) \leq \ell$. 

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Algorithm 1 BasePinch.

- Input a word \( w = A_r^{-1}a_r A_r v \) where \( r \geq 1 \), \( a_r \) is a word on \( A_r^{\pm 1} \), and \( v \) is a word on \( A_r^{\pm 1} \), \( \ldots \), \( A_k \).
- Either return that \( w \) is invalid, or return a valid word \( \sigma' = A_r^0 v \sim \sigma \) such that \( \ell(\sigma') \leq \ell(\sigma) - 2 \).
- Halting time \( O(\ell(\sigma)^4) \).

\[
\ell := u(0) \quad (\text{so } A_r^{-1} A_r^0 A_r v \sim w)
\]

if \( \text{Positive}(A_r, v) = \text{Invalid} \), halt and return invalid

run Positive\((v)\) to determine whether \( v(0) < 0 \)

if \( r \geq 2 \) and \( v(0) < 0 \), halt and return invalid

if \( \ell = 0 \), halt and return \( \sigma' := v \)

if \( r = 1 \), halt and return \( \sigma' := A_r^{-2} v \) if \( i \) is even or invalid otherwise

Now we have \( \ell \neq 0 \) and \( r > 1 \)

run Positive\((A_r^0 A_r v)\) to determine if \( A_r^0 A_r v(0) \leq 0 \) (so \( \ell \) domain of \( A_r^{-1} \))

if so, halt and return invalid

run Positive\((A_r^{-2} l A_r v)\) to determine whether \( A_r v(0) > 2 |l| \)

if so, halt and return invalid

we now have that \( 0 \leq v(0) \leq |l| \) and \( 0 < A_r v(0) \leq 2 |l| \) and \( A_r v(0) + l \leq 3 |l| \)

calculate \( v(0) \) by running Positive\((A_r^{-1} v)\) for \( i = 0, 1, \ldots, |l| \)

run Bounds\((3 |l|)\)

search the output of Bounds\((3 |l|)\) to find \( A_r v(0) \)

set \( m := A_r v(0) + l \)

search the output of Bounds\((3 |l|)\) for \( c \) with \( A_r(c) = m \)

\( \text{so } c = A_r^{-1} A_r^0 A_r v(0) = \sigma(0) \)

if such a \( c \) exists, halt and return \( \sigma' := A_r^{-w(0)} v \)

else halt and return invalid

Positive which, on input a word \( w \) on \( A_r^{\pm 1}, A_1, \ldots, A_k \) in time \( O(\ell(\sigma)^3) \) either declares \( w \) invalid or declares whether \( w(0) < 0 \), \( w(0) = 0 \), or \( w(0) > 0 \).

We use these properties of Ackermann functions:

Lemma 4.

\[
A_r(n) + m \leq A_r(n+m) \quad \forall i, n, m \geq 0, \quad (1)
\]

\[
|A_r(n) - A_r(m)| \geq \frac{1}{2} A_r(n) \quad \forall i \geq 2 \text{ and } n \neq m. \quad (2)
\]

The proofs follow by inductive arguments applied to the definition of an Ackermann function. Refer to Lemma 2.1 of [12] for details.

Correctness of BasePinch. Here are the salient points line-by-line.

4: If \( v(0) < 0 \), then \( \sigma \) is invalid.

5: If \( r < 2 \) or \( v(0) \geq 0 \), \( A_r^{-1} A_r v \sim v \).

6: Since \( A_1 \) is the function \( n \mapsto 2n \), the parity of \( A_r^0 A_r v(0) \) is the parity of \( l \) when \( r = 1 \), and determines the validity of \( \sigma \).

9, 11: We know \( A_r^0 A_r v \) and \( A_r^{-2} l A_r v \) are valid at these points because \( A_r v \) is valid.

12: Let \( q = v(0) \). For all \( p \neq q \) we have \( |A_r(q) - A_r(p)| \geq \frac{1}{2} A_r(q) \) by Lemma 4, and so \( |A_r(q) - A_r(p)| > |l| \). If \( A_r^{-1} A_r^0 A_r v \) is valid, then there exists \( p \in \mathbb{N} \) such that \( A_r(p) = A_r^0 A_r v(0) + l + A_r(q) \), but then \( |A_r(p) - A_r(q)| = |l| \) for some \( p \neq q \) (since \( l \neq 0 \)), contradicting \( |A_r(q) - A_r(p)| > l \). Thus \( w \) is invalid.
14: The reason $0 < A_r(v(0))$ is that $r > 1$ and so $\text{Img}A_r$ contains only positive integers. And $A_r(v(0)) \leq 2|l|$ because of lines 11 and 12. It follows that $v(0) \leq |l|$ because $2v(0) = A_1v(0) \leq A_rv(0) \leq 2|l|$. And $v(0) \geq 0$ since $v(0)$ is in the domain of $A_r$, which is $\mathbb{N}$ when $r > 1$. We have $A_0^lA_rv(0) \leq 3|l|$ here because $A_rv(0) \leq 2|l|$ and so $A_0^lA_rv(0) \leq |l| + 2|l|$.  

20: If $m = A_rv(0) + l = A_0^lA_rv(0)$ is in the domain of $A_r^{-1}$, then $m > 0$. And, from line 14, we know $m \leq 3|l|$, so this will find $c$ if it exists. If no such $c$ exists, $\sigma$ is invalid.  

21: $A_0^{2v(0)}v(0) = c = A_r^{-1}(l + A_rv(0)) = A_r^{-1}A_0^lA_rv(0)$.  

We must show that $\ell(\sigma') \leq \ell(\sigma) - 2$. In the cases of lines 5 and 6, this is immediate, so suppose $r \geq 2$. As for line 21, by Lemma 4:  

$$|c - v(0)| \leq |A_r(v(0) + c - v(0)) - A_rv(0)| = |A_r(c) - A_rv(0)| = |l|$$  

from which $\ell(\sigma') \leq \ell(\sigma) - 2$ follows immediately.  

The integer calculations performed by the algorithm involve integers of absolute value at most $3\ell(\sigma)$. See [12] for details.  

That BasePinch halts in time $O(\ell(\sigma)^4)$ follows the following. Positive and Bounds halt in cubic and linear time, respectively. BasePinch may add a pair of positive binary numbers each at most $2\ell(\sigma)$, may determine the parity of a number of absolute value at most $\ell(\sigma)$, and may halve an even positive number less than $\ell(\sigma)$. It calls Positive at most $|l| + 3 \leq \ell(\sigma) + 3$ times, always on a word of length at most $2\ell(\sigma)$. It calls Bounds at most once and on a non-negative integer that is at most $3\ell(\sigma)$. The output of Bounds is then searched at most twice and has size $O((\log_2 \ell(\sigma))^2)$. ▲

7 An outline of our strategy for Theorem 3

Here is an outline of our algorithm solving the membership problem for $H_k$ in $G_k$ from Section 4 of [12], proving Theorem 3. For a more detailed high-level description, see Section 4.1 of [12].

Suppose $w$ is a word on $a_1^{±1}, \ldots, a_k^{±1}, t^{±1}$, so represents an element of $G_k$. To tell whether or not $w$ represents an element of $H_k$, first collect all the $t^{±1}$ at the front by shuffling them to the left through the word, applying $\theta^{±1}$ as appropriate to the intervening $a_i$ so that the element of $G_k$ represented does not change. The result is a word $t^\nu v$ where $|\nu| \leq \ell(w)$ and $v$, a word on $a_1^{±1}, \ldots, a_k^{±1}$ has length at most a constant times $\ell(w)$ since $\theta$ is a free group automorphism of such polynomial growth.

Here is an example (one of a number in Section 4.2 of [12]). Suppose $w = a_3^2a_2a_1a_2^{-1}a_3^{-4}$. This equals $tv$ in $G_3$ where $v = (a_3a_2)^4a_2a_1^2a_2^{-1}a_3^{-4}$ because $a_2t = a_2a_1$ and $a_3t = a_3a_2$.

We next look to carry the $t^\nu$ back through $v$ working from left to right, converting (if possible) what lies to the left of the power of $t$ to a word on the generators $(a_1t)^{±1}, \ldots, (a_kt)^{±1}$ of $H_k$. However the power of $t$ being carried along will vary as this proceeds and, in fact, can get extremely large as a result of the hydra phenomenon. Similarly, the length of the word on the generators of $H_k$ appearing to the left can be impractically long. For instance, in our example, the calculation outlined in Section 5 shows that $w$ equals an element of the subgroup $H_3$ of $G_3$ which has length $2^{47} \cdot 3 - 1$ as a reduced word on the generators $(a_1t)^{±1}$, $(a_2t)^{±1}$, $(a_3t)^{±1}$ of $H_3$.  

So, instead of keeping this power of $t$ directly, we record it as a word on $\psi$-functions (the functions that are analogues of Ackermann functions, as we explained in Section 5). Roughly speaking, checking whether this process ever gets stuck (in which case $w \notin H_k$) amounts to checking whether an associated $\psi$-word is valid. If the end of the word is reached,
we then have a word on \((a_k t)^{\pm 1}, \ldots, (a_1 t)^{\pm 1}\) times some power of \(t\), where the power is represented by a \(\psi\)-word whose length is at most a polynomial function of the length of \(w\). We then determine whether or not \(w \in H_k\) by checking whether or not that \(\psi\)-word represents 0. Both tasks can be accomplished suitably efficiently thanks to Proposition 3.4 in [12] (a variation of Theorem 1 as we explained in Section 5).

A complication is that we do not carry the power of \(t\) through from left to right one letter at a time. Rather, we partition \(v\) into subwords we call \(\text{rank } k\)-pieces and are determined by the locations of the \(a_k\) and \(a_k^{-1}\) in \(v\). Each contains at most one \(a_k\) and at most one \(a_k^{-1}\), and if the \(a_k\) is present in a piece, it is the first letter of that piece, and if the \(a_k^{-1}\) is present, it is the last letter. For instance, in our example \(k = 3\) and \(v = (a_3 a_2) (a_3 a_2) (a_3 a_2 a_3^{-1} a_2^{-1}) (a_3^{-1}) \cdots (a_3^{-1})\). We look to carry the power of \(t\) through one piece at a time. Lemma 6.2 of [13] details how \(a\) can be carried through a piece with respect to the locations of the \(a_k^{-1}\) it contains. The main technical result behind our algorithm is our ‘Piece Criterion’ (Proposition 4.10 in [12]). This determines whether a power \(t^r\) can pass through a piece \(\pi\)—that is, whether \(t^r \pi \in H_k t^s\) for some \(s \in \mathbb{Z}\)—and, if it can, how to represent \(s\) by a \(\psi\)-word. The way this plays out in our example is:

\[
\begin{align*}
t(a_3 a_2) & \in H_k t^{f_1(0)} & \text{where } f_1 = \psi_1^{-1}, \\
t(a_3 a_2) & \in H_k t^{f_2(0)} & \text{where } f_2 = \psi_3 f_1, \\
t(a_3 a_2) & \in H_k t^{f_3(0)} & \text{where } f_3 = \psi_3 f_2, \\
t(a_3^{-1} a_3^{-1}) & \in H_k t^{f_4(0)} & \text{where } f_4 = \psi_3^{-1} \psi_1 f_3, \\
t(a_3^{-1}) & \in H_k t^{f_5(0)} & \text{where } f_5 = \psi_3^{-1} f_4, \\
t(a_3^{-1}) & \in H_k t^{f_6(0)} & \text{where } f_6 = \psi_3^{-1} f_5, \\
t(a_3^{-1}) & \in H_k t^{f_7(0)} & \text{where } f_7 = \psi_3^{-1} f_6.
\end{align*}
\]

(The integers encoded here are \(f_1(0) = 0, f_2(0) = -3, f_3(0) = -45, f_4(0) = -46, f_5(0) = -4, f_6(0) = -1\), and \(f_7(0) = 0\). The conclusion is that \(v \in H_3\) since \(f_7(0) = 0\).)

Like in the previous section, we do not have space here to present many of the details, and so will only give an illustrative subroutine, namely ‘\(\text{Back}_m\).’ This attempts to pass a power \(t^r\) through a rank \(m\)-piece which has the special form \(u a_m^{-c}\) where \(c \in \{0, 1\}\), \(u\) is a word \(a_1^{\pm 1} \cdots a_m^{\pm 1}\) and \(m \geq 3\). There are several precursors to the construction of \(\text{Back}_m\):\

- The construction is inductive on \(m\). \(\text{Back}_m\) calls an algorithm \(\text{Push}_{m-1}\) (of Section 4.5 of [12]) which takes as input a word \(v\) on \(a_0^{\pm 1}, \ldots, a_{m-1}^{\pm 1}\) and a \(\psi\)-word \(f\) representing an integer, and declares whether \(t^{f(0)} v \in H_k t^s\) for some \(s \in \mathbb{Z}\); if so, \(\text{Push}_{m-1}\) also returns a \(\psi\)-word \(g\) so that \(t^{f(0)} v \in H_k t^{g(0)}\).

- The Piece Criterion (Proposition 4.10 in [12]) stipulates (in particular) that if \(t^r u a_m^{-c}\) \(\in H_k t^s\) for some \(s \in \mathbb{Z}\), exactly one of the following three conditions must hold:
  (a) \(c = 0\) and \(t^r u a_m^{-c} = t^r u \in H_k t^s\) (the trivial case).
  (b) \(c = 1, s \leq 0\) and \(t^r u \in H_k t^s\).
  (c) \(c = 1, s > 0\), \(t^r u a_m^{-c} \theta^s(a_m) \in H_k t^{s-1}\) and \(\theta^{s-1}(a_m)\) is a suffix of \(u a_m^{-c}\).
(Here, \(\theta\) is the free group automorphism we defined in Section 2.)

- A routine \(\text{Prefix}_m\) (of Section 4.5 of [12]) inputs a rank-\(m\) piece \(\pi = a_m^{-c} u a_m^{-c}\) where \(m \geq 3\). It returns the largest integer \(i > 0\) (if any) such that \(t^{i-1}(a_m)\) is a prefix of \(\pi\) and halts in time in \(O(\ell(\pi)^2)\).
Algorithm 2 Back...

- Input a rank-$m$ piece $π = uα_{m−1}$ with $m ≥ 3$ (so $u$ is a reduced word on $α_1, α_2, ..., α_{m−1}$ and $α_2 ∈ \{0, 1\}$) and a valid $ψ$-word $f$ on $ψ_{1}^±, ..., ψ_{k}^±$. Let $r := f(0)$.
- Declare whether or not $t^\prime π ∈ \bigcup_{s∈Z} H_k t^s$. And, if it is, return a valid $ψ$-word $f'$ such that $t^\langle f \rangle π ∈ H_k t^\langle f \rangle (\ell)(\ell' f) ≤ \ell(f) + 2(m − 1)\ell(π) + 1$ and rank $(f') ≤ \max\{\text{rank}(f), m\}$.
- Halt in time $O((\ell(π) + \ell(f))^{2m + 4})$.

```plaintext
run Push\textsubscript{m−1}(u, f) to test whether or not $t^\prime u ∈ \bigcup_{s∈Z} H_k t^s$
if so, let $g$ be the valid $ψ$-word it outputs such that $t^\prime u ∈ H_k t^g(0)$
if $ε_2 = 0$,
if $t^\prime u ∈ H_k t^g(0)$ (so, (2) of the Piece Criterion holds with $s = g(0)$),
return $f' := g$
else declare $t^\prime π \notin \bigcup_{s∈Z} H_k t^s$
halt
we now have that $ε_2 = 1$
run Psi($ψ_m^i g$) to check validity of $ψ_m^i g$ (so whether $g(0) ∈ \text{Img}_{ψ_m}$)
and, if so, to check $ψ_m^i g(0) ≤ 0$ (i.e. whether (2) of the Criterion holds with $s = ψ_m^i g(0)$)
if so, halt and return $f' := ψ_m^i g$
run Prefix\textsubscript{m}(π) to determine the maximum $i$ (if any)
such that $a_{m−1}^{−1} θ_i(a_m)$ is a suffix of $π$
if there is no such $i$, halt and declare $t^\prime π \notin \bigcup_{s∈Z} H_k t^s$
for $s = 1$ to $i$:
run Push\textsubscript{m−1}(u', f) where $u'$ is the reduced word representing $ua_{m−1}^{−1} θ_i(a_m)$
if it outputs a $ψ$-word $h$, run Psi($ψ_i^−1 h$) to check if $h(0) = s − 1$
if so halt and return $f' := ψ_i h$
declare that $t^\langle f \rangle u ∈ \bigcup_{s∈Z} H_k t^s$
```

- $\text{Psi}$ is our algorithm (of Section 3.3 of [12]) determining in polynomial time whether a $ψ$-word is valid and, if so, whether the integer it represents is negative, zero, or positive. We discussed its Ackermann-function analogue in the previous section.

Proof of correctness. Here is our justification line-by-line.

2: It follows from the workings of Push\textsubscript{m−1} (proved in Section 4.5 of [12]) that $\ell(g) ≤ \ell(u) + \ell(f)$ and $\text{rank}(g) ≤ \max\{\text{rank}(f), m\}$.

3–6: Push\textsubscript{m−1} in lines 1–2 tests whether or not $t^\prime u$ is in $\bigcup_{s∈Z} H_k t^s$ and, if so, it identifies the $s$ such that $t^\prime u ∈ H_k t^s$. The Piece Criterion then tells us that the answer to whether $t^\prime π ∈ \bigcup_{s∈Z} H_k t^s$ is the same, and if affirmative the $s$ agrees. By our comment on line 2, $\ell(f') ≤ \ell(f) + \ell(u) = \ell(f) + \ell(π)$, and rank $(f') ≤ \max\{\text{rank}(f), m\}$, as required.

10–13: Again, we refer back to lines 1–2 for whether or not $t^\prime u$ is in $\bigcup_{s∈Z} H_k t^s$. Assuming that it is, in fact, in $H_k t^g(0)$, then Condition 2, is satisfied if and only if $g(0) = ψ_m(s)$ for some $s ≤ 0$. And that is checked in line 10. The Piece Criterion then tells us that the answer to this is the same as the answer to whether $t^\prime u ∈ \bigcup_{s∈Z} H_k t^s$, and, if affirmative, the $s$ agrees. By our comment on line 2, $\ell(f') = \ell(g) + 1 ≤ \ell(f) + \ell(u) + 1 = \ell(f) + \ell(π)$ and rank $(f') ≤ \max\{\text{rank}(f), m\}$, as required.

16–21: The aim here is to determine whether Condition 3 holds—that is, whether $t^\prime u a_{m−1}^{−1} θ_i(a_m) ∈ H_k t^s$.
and $a_{m-1}^{-1} \theta_{s-1}(a_{m-1}^{-1})$ is a suffix of $\pi$ for some $s > 0$—and, if so, output a $\psi$-word $f'$ such that $f'(0) = s$. (This $s$ must be unique, because, by the Criterion, it is the $s$ such that $\ell' \pi \in H_k \ell^s$, and we know that is unique.)

The possibilities for $s$ are limited to the range $1, \ldots, i$ by the suffix condition and the requirement that $s > 0$, where $i$ is as found in line 16 and must be at most $\ell(\pi)$. If there is such a suffix $a_{m-1}^{-1} \theta_{s-1}(a_{m-1}^{-1})$ of $\pi$, then $a_{m-1}^{-1} \theta_{s-1}(a_{m-1}^{-1})$ is a suffix of $\pi$ for all $s \in \{1, \ldots, i\}$. If there is no such suffix, then Condition 3 fails, and, as we know at this point that Conditions 1 and 2 also fail, we declare in line 17 that (by the Criterion),

$t' \pi \notin \bigcup_{s \in \mathbb{Z}} H_k \ell^s$.

For each $s$ in the range $1, \ldots, i$, lines 18–21 address the question of whether or not $t' u a_{m-1}^{-1} \theta^s(a_m) \in H_k t^s$. First $\textbf{Push}_{m-1}$ is called, which can be done because, on freely reducing $u a_{m-1}^{-1} \theta^s(a_m)$, the $a_m$ cancels with the $a_m$ at the start of $\theta^s(a_m)$ to give a word of rank at most $m - 1$. $\textbf{Push}_{m-1}$ either tells us that $t' u a_{m-1}^{-1} \theta^s(a_m) \notin \bigcup_{s \in \mathbb{Z}} H_k \ell^s$, or it gives a $\psi$-word $h$ such that $t' u a_{m-1}^{-1} \theta^s(a_m) \in H_k \ell^{h(0)}$. In the latter case, $\textbf{Psi}$ is then used to test whether or not $h(0) = s - 1$.

By the specifications of $\textbf{Push}_{m-1}$, $\ell(h) \leq \ell(f) + 2(m - 1)\ell(u')$. And, as $\pi = u a_{m-1}^{-1}$ has suffix $\theta_{s-1}(a_{m-1}^{-1})$, when we form $u'$ by freely reducing $u a_{m-1}^{-1} \theta^s(a_m)$, at least half of $\theta^s(a_m) = \theta_{s-1}(a_{m-1}) \theta_{s-1}(a_{m-1}^{-1})$ cancels into $\pi$. So $\ell(u') \leq \ell(\pi)$, and

$$
\ell(f') = \ell(h) + 1 \leq \ell(f) + 2(m - 1)\ell(u') + 1 \leq \ell(f) + 2(m - 1)\ell(\pi) + 1,
$$

as required. Also, it is immediate that rank($f'$) $\leq \max\{\text{rank}(f), m\}$, as required.

22: At this point, we know 1, 2 and 3 fail for all $s \in \mathbb{Z}$, so $t' \pi \notin \bigcup_{s \in \mathbb{Z}} H_k \ell^s$.

$\textbf{Back}_m$ runs $\textbf{Push}_{m-1}(u, f)$ once (with $\ell(u) \leq \ell(\pi)$), $\textbf{Psi}(\psi_{m-1}^{-1}g)$ at most once (with $\ell(g) \leq \ell(\pi) + \ell(f)$), $\textbf{Prefix}_m(\pi^{-1})$ at most once, $\textbf{Push}_{m-1}(u', f)$ at most $i \leq \ell(\pi)$ times (with $\ell(u') < \ell(\pi)$), and $\textbf{Psi}(\psi_{m-1}^{-1}h)$ at most $i \leq \ell(\pi)$ times (with $1 \leq s \leq \ell(\pi)$ and $\ell(h) < \ell(f) + \ell(\pi)$). Other operations such as free reductions of words etc. do not contribute significantly to the running time. Referring to the specifications of $\textbf{Push}_{m-1}$, $\textbf{Psi}$, and $\textbf{Prefix}_m$, we see that they (respectively) contribute:

$$
\ell(\pi) O((\ell(\pi) + \ell(f))^{2(m-1)+k+1}) + \ell(\pi) O((\ell(f) + 2\ell(\pi))^{4+k}) + O(\ell(\pi)^2)
$$

$$
= O((\ell(\pi) + \ell(f))^{2m+k})
$$

which is the claimed bound on the halting time of $\textbf{Back}_m$. 

There is also an algorithm $\textbf{Front}_m$ which takes a rank-$m$-piece $\pi$ and a $\psi$-word $f$ and determines (in a manner similar to $\textbf{Back}_m$, see [12] Algorithm 4.2 for details) whether $t^{f(0)}$ can efficiently pass an initial $a_m$ (if it exists) in $\pi$. If so, $\textbf{Front}_m$ outputs a word of the form $u a_{m-1}$ suitable for input into $\textbf{Back}_m$ and a valid $\psi$ word $g$ such that checking whether $t^{f(0)} \pi \in H_k \ell^s$ for some $s \in \mathbb{Z}$ is equivalent to checking whether $t^{\theta^0} a_{m-1} \in H_k \ell^s$ for some $s \in \mathbb{Z}$. If $t^{f(0)}$ does not pass through an initial $a_m$ of $\pi$ in one of three ways, Proposition 4.10 in [12] says that $t^{f(0)} \pi \notin H_k \ell^8$ for all $s \in \mathbb{Z}$. Putting together the algorithm $\textbf{Front}_m$ with $\textbf{Back}_m$ and implicitly $\textbf{Push}_{m-1}$, we can construct the algorithm $\textbf{Push}_m$. That way, given a word $v$ on $\{a_1, \ldots, a_m\}$ and a $\psi$-word $f$, we have a polynomial time algorithm to determine whether or not $t^{f(0)} v \in H_k \ell^8$ and if so, to give a $\psi$-word $g$ such that $g(0) = s$. We can then use $\textbf{Psi}$ to determine whether $g$ represents zero, and so whether $t^{f(0)} v$ represents an element of $H_k$. $\square$
A Reference to the technical details

The technical details are set out in full in *Taming the hydra: the word problem and extreme integer compression* [12], which is available from the arXiv repository at http://arxiv.org/abs/1509.02557.

References


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