AC Dependency Pairs Revisited*

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Abstract

Rewriting modulo AC, i.e., associativity and/or commutativity of certain symbols, is among the most frequently used extensions of term rewriting by equational theories. In this paper we present a generalization of the dependency pair framework for termination analysis to rewriting modulo AC. It subsumes existing variants of AC dependency pairs, admits standard dependency graph analyses, and in particular enjoys the minimality property in the standard sense. As a direct benefit, important termination techniques are easily extended; we describe usable rules and the subterm criterion for AC termination, which properly generalize the non-AC versions.

We also perform these extensions within IsaFoR – the Isabelle formalization of rewriting – and thereby provide the first formalization of AC dependency pairs. Consequently, our certifier CeTA now supports checking proofs of AC termination.

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1 Introduction

The *dependency pair (DP) method* of Arts and Giesl [2] and its successor, the *DP framework* [7], have become the de facto standard in termination proving for term rewrite systems, providing the foundation of state-of-the-art termination provers.

Various authors extended the DP method/framework for proving termination *modulo AC*, i.e., certain symbols are assumed to be associative and/or commutative. Each variant has its own strengths and weaknesses: Kusakari and Toyama [12] introduced a version that results in less dependency pairs than others, but requires a special treatment of AC symbols. Marché and Urbain [13] introduced a version that considers *flattened* terms and results in more dependency pairs than the Kusakari-Toyama version. However, their original version was later pointed out to be incorrect [14]. As a specialization of their more general *equational DP method*, Giesl and Kapur [6] mentioned a version that does not require flattening or a special treatment of AC symbols, so that many techniques for the standard DP framework can be incorporated without major modifications. However, Alarcón et al. [1] pointed out that the minimality property, which is needed for some important techniques such as *usable rules* and the *subterm criterion* [9], does not carry over to the AC case.

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In this paper, we introduce yet another AC-DP framework. Our approach enjoys the strengths of both the Kusakari-Toyama approach and the Giesl-Kapur approach: the number of generated dependency pairs is small and most techniques for the standard DP framework can be integrated without major modifications. Moreover, our approach ensures the minimality property in the standard sense, and thus both the usable rules technique and the subterm criterion become available in our framework.

The key idea of our AC-DP framework is to consider AC axioms just as rewrite rules and to take the dependency pairs of these rules. This choice allows us to reuse most of the reasoning from the standard DP framework, including the notion of minimality.

The results of this paper are formalized using the Isabelle proof assistant [15]. The formalization, consisting of 7069 lines of Isabelle code, is incorporated into our *Isabelle formalization of rewriting* IsaFoR [23], and as it provides a formalized correctness proof of our certifier CeTA, we can now formally validate AC termination proofs. Through experiments CeTA revealed a long-lurking bug in AProVE [5] (in the computation of AC usable rules), which is now fixed. IsaFoR, CeTA, and the experiments are accessible from

`http://cl-informatik.uibk.ac.at/software/ceta/experiments/ac-dp`

The paper is organized as follows. After preliminaries in Section 2, we describe our AC-DP framework in Section 3. In Section 4 we show how to characterize AC termination in our AC-DP framework. We present AC variants of usable rules and the subterm criterion in Sections 5 and 6, respectively. The practical relevance of our work is discussed in Section 7, whereas Section 8 contains a theoretical comparison to related work.

2 Preliminaries

We assume familiarity with term rewriting and only recall some notations required in the remainder; for details on term rewriting, we refer to textbooks [3, 21].

Given a set $\mathcal{F}$ of function symbols with associated arities and a set $\mathcal{V}$ of variables such that $\mathcal{F} \cap \mathcal{V} = \emptyset$, a *term* is either a variable $x \in \mathcal{V}$ or of the form $f(s_1, \ldots, s_n)$, where n is the arity of $f \in \mathcal{F}$ and the arguments $s_1, \ldots, s_n$ are terms. We often abbreviate a list of terms $s_1, \ldots, s_n$ by $\vec{s}_n$. The *root symbol* of a term $s = f(\vec{s}_n)$ is f and denoted by $\mathsf{root}(s)$.

A *substitution* σ assigns each variable x a term $x\sigma$. For a term s, $s\sigma$ denotes the term obtained by replacing every variable x by $x\sigma$ in s. A *position* in a term s is represented by a sequence of natural numbers as usual. The *subterm* of a term s at position p is denoted by $s|_p$, and the term obtained by replacing the subterm by a term t is denoted by $s[t]_p$.

A *rewrite rule* is a pair of terms, written $l \to r$, such that $l \notin \mathcal{V}$ and variables appearing in r also appear in l. A *term rewrite system (TRS)* is a set $\mathcal{R}$ of rewrite rules. There is an *$\mathcal{R}$-rewrite step* from s to t at position p, written $s \xrightarrow[\mathcal{R}]{p} t$, iff there exist a rule $l \to r \in \mathcal{R}$ and a substitution σ such that $s|_p = l\sigma$ and $t = s[r\sigma]_p$. We write $s \xrightarrow[\mathcal{R}]{} t$ if p is irrelevant.

The *associativity* and *commutativity* of a binary symbol $+ \in \mathcal{F}$ are respectively specified by the following axioms, where we use infix notation for $+$:

$$x + (y + z) \approx (x + y) + z \qquad (\mathsf{A}) \qquad\qquad x + y \approx y + x \qquad (\mathsf{C})$$

We assume $\mathcal{E}$ to be a set of associativity and/or commutativity axioms, which we call an *AC theory*. We write $\mathcal{F}_\mathsf{A}$ and $\mathcal{F}_\mathsf{C}$ for the set of symbols which are associative and commutative in $\mathcal{E}$, respectively, and write $\mathcal{F}_\mathsf{AC}$ for $\mathcal{F}_\mathsf{A} \cap \mathcal{F}_\mathsf{C}$. The equivalence relation induced by $\mathcal{E}$ is denoted by $\underset{\mathcal{E}}{\approx}$.

For a relation R, we denote its transitive closure by R^+ and its reflexive transitive closure by R^*. Given another relation S, we denote the relation $S^* \cdot R \cdot S^*$ by R/S.

For a TRS $\mathcal{R}$ and an AC theory $\mathcal{E}$, $\mathcal{R}$-rewriting modulo $\mathcal{E}$ considers the relation $\xrightarrow[\mathcal{R}/\mathcal{E}]{}$, which is defined as $\xrightarrow[\mathcal{R}]{}/\underset{\mathcal{E}}{\approx}$. A term s is *$\mathcal{R}/\mathcal{E}$-nonterminating* iff there exists an infinite sequence $s \xrightarrow[\mathcal{R}/\mathcal{E}]{} s_1 \xrightarrow[\mathcal{R}/\mathcal{E}]{} s_2 \xrightarrow[\mathcal{R}/\mathcal{E}]{} \cdots$, and *$\mathcal{R}/\mathcal{E}$-terminating* otherwise. An $\mathcal{R}/\mathcal{E}$-nonterminating term is *minimal* iff all its proper subterms are $\mathcal{R}/\mathcal{E}$-terminating.

If every term is $\mathcal{R}/\mathcal{E}$-terminating, then we say $\mathcal{R}$ is *terminating modulo* $\mathcal{E}$, or $\mathcal{R}/\mathcal{E}$ is terminating. This terminology carries over to the non-AC case: we say a term is $\mathcal{R}$-terminating, $\mathcal{R}$ is terminating, etc., when the same holds for $\mathcal{R}/\emptyset$.

We extensively use the following notion in subsequent proofs.

▶ **Definition 1** (Well-Founded Order Pair). A *quasi-order pair* on a set A is a pair $\langle \succsim, \succ \rangle$ consisting of a quasi-order $\succsim$ and a transitive relation $\succ$ on A, satisfying $\succ/\succsim \ \subseteq \ \succ$. If moreover $\succ$ is well-founded, then we call $\langle \succsim, \succ \rangle$ a *well-founded order pair*.

It is well known that a lexicographic composition of well-founded order pairs again forms a well-founded order pair. Moreover, a well-founded order pair can be extended to a well-founded order pair over multisets [4, 22]. Below $\uplus$ denotes the multiset union.

▶ **Definition 2.** The *multiset extension* of a quasi-order pair $\langle \succsim, \succ \rangle$ on A is the quasi-order pair $\langle \succsim^{\mathsf{mul}}, \succ^{\mathsf{mul}} \rangle$ on multisets over A which is defined as follows: $X \succsim^{\mathsf{mul}} Y$ iff X and Y are of the form $X = \{x_1, \ldots, x_n\} \uplus X'$ and $Y = \{y_1, \ldots, y_n\} \uplus Y'$ such that $\forall i \in \{1, \ldots, n\}.\ x_i \succsim y_i$, and $\forall y \in Y'.\ \exists x \in X'.\ x \succ y$. We have $X \succ^{\mathsf{mul}} Y$ if it also holds that $X' \neq \emptyset$.

3 An AC-Dependency Pair Framework

Before introducing our AC-DP framework, we first recall the basics of dependency pairs. We say a symbol $f \in \mathcal{F}$ is *defined* in $\mathcal{R}$ if there is a rewrite rule $f(\vec{s}_n) \to r \in \mathcal{R}$, and denote the set of defined symbols in $\mathcal{R}$ by $\mathcal{D}_\mathcal{R}$. We assume a fresh *marked* symbol $f^\sharp$ for every $f \in \mathcal{D}_\mathcal{R}$, and write $s^\sharp$ to denote the term $f^\sharp(\vec{s}_n)$ for $s = f(\vec{s}_n)$.

▶ **Definition 3.** A *dependency pair* of a TRS $\mathcal{R}$ is a rule $l^\sharp \to r|_p^\sharp$ such that $l \to r \in \mathcal{R}$ and $\mathsf{root}(r|_p) \in \mathcal{D}_\mathcal{R}$. The set of all dependency pairs of $\mathcal{R}$ is denoted by $\mathsf{DP}(\mathcal{R})$.

The DP framework works on *DP problems*, which are just pairs of TRSs.

▶ **Theorem 4** ([2]). *A TRS $\mathcal{R}$ is terminating iff the DP problem $\langle \mathsf{DP}(\mathcal{R}), \mathcal{R} \rangle$ is* finite, *i.e., there is no infinite chain $s_0 \xrightarrow[\mathsf{DP}(\mathcal{R})]{\epsilon} t_0 \xrightarrow[\mathcal{R}]{}^* s_1 \xrightarrow[\mathsf{DP}(\mathcal{R})]{\epsilon} t_1 \xrightarrow[\mathcal{R}]{}^* \cdots$ where every t_i is $\mathcal{R}$-terminating.*

Our key idea towards an AC-DP framework is to represent an AC theory by a (nonterminating) TRS. Commutativity is trivial: axiom (C) can be seen as the rewrite rule $x+y \to y+x$. For associativity, i.e., axiom (A), the following two rules obviously suffice:

$$x + (y + z) \to (x + y) + z \qquad (\mathsf{A}_1) \qquad\qquad (x + y) + z \to x + (y + z) \qquad (\mathsf{A}_2)$$

The benefit of this approach is that now we can take the dependency pairs $\mathsf{DP}(\mathcal{E})$ of AC axioms. Commutativity induces the following dependency pair

$$x +^\sharp y \to y +^\sharp x \qquad (\mathsf{C}^\sharp)$$

and associativity yields the following dependency pairs:

$$x +^\sharp (y + z) \to (x + y) +^\sharp z \qquad (x + y) +^\sharp z \to x +^\sharp (y + z) \qquad (\mathsf{A}^\sharp)$$

$$x +^\sharp (y + z) \to x +^\sharp y \qquad (x + y) +^\sharp z \to y +^\sharp z \qquad (\mathsf{a}^\sharp)$$

Note that there are other representations; e.g., if $+$ is both associative and commutative, then one of (A_1) and (A_2) may be dropped, since it is a consequence of the other and (C). In the formalization, we only demand that an AC theory $\mathcal{E}$ is represented by a TRS $\mathcal{E}'$ such that $\xrightarrow[\mathcal{E}']{}^{*} = \underset{\mathcal{E}}{\approx}$. In the rest of the paper, we identify $\mathcal{E}'$ with $\mathcal{E}$.

▶ **Definition 5.** An *AC-DP problem* is a quadruple of TRSs $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, and $\mathcal{E}$, denoted by $\langle \mathcal{P}/\mathcal{Q}, \mathcal{R}/\mathcal{E}\rangle$. A $\langle \mathcal{P}/\mathcal{Q}, \mathcal{R}/\mathcal{E}\rangle$*-chain* is a finite or infinite sequence of the form

$$s_0 \xrightarrow[\mathcal{P}\cup\mathcal{Q}]{\epsilon} t_0 \xrightarrow[\mathcal{R}\cup\mathcal{E}]{}^{*} s_1 \xrightarrow[\mathcal{P}\cup\mathcal{Q}]{\epsilon} t_1 \xrightarrow[\mathcal{R}\cup\mathcal{E}]{}^{*} \cdots$$

and is called *minimal* if every t_i is $\mathcal{R}/\mathcal{E}$-terminating. An AC-DP problem is said to be *finite* if it admits no minimal chain containing infinitely many $\mathcal{P}$-steps.

We often write $\xrightarrow[\mathcal{P}\cup\mathcal{Q}]{}$ instead of $\xrightarrow[\mathcal{P}\cup\mathcal{Q}]{\epsilon}$ if it is clear from the context that $\mathcal{P}$ and $\mathcal{Q}$ are the first two components of an AC-DP problem.

Our AC-DP problems are quite similar to *relative DP problems* [19, Definition 1], which form the basis of the formalized DP framework in IsaFoR; ignoring minimality, an AC-DP problem $\langle \mathcal{P}/\mathcal{Q}, \mathcal{R}/\mathcal{E}\rangle$ corresponds to the relative DP problem $\langle \mathcal{P}, \mathcal{Q}, \emptyset, \mathcal{R}\cup\mathcal{E}\rangle$. Hence all techniques which do not rely on minimality, are immediately applicable.

The most important technique in proving the finiteness of DP problems is the *reduction pair processor* [2, 7]: a reduction pair $\langle \succsim, \succ\rangle$ is a well-founded order pair on terms such that $\succsim$ and $\succ$ are closed under substitutions and $\succsim$ is closed under contexts.

▶ **Theorem 6.** *Let $\langle \succsim, \succ\rangle$ be a reduction pair such that $\mathcal{P}\cup\mathcal{Q}\cup\mathcal{R}\cup\mathcal{E} \subseteq \succsim$. Then, $\langle \mathcal{P}/\mathcal{Q}, \mathcal{R}/\mathcal{E}\rangle$ is finite iff $\langle \mathcal{P}'/\mathcal{Q}', \mathcal{R}'/\mathcal{E}\rangle$ is, where $\mathcal{P}' = \mathcal{P}\setminus\succ$, $\mathcal{Q}' = \mathcal{Q}\setminus\succ$, and $\mathcal{R}' = \mathcal{R}$ or $\mathcal{R}' = \mathcal{R}\setminus\succ$ if $\succ$ is closed under contexts.*

The *dependency graph processor* [8, 7] is also easily adapted.

▶ **Theorem 7.** *Let $\mathcal{G}$ be an (estimated) dependency graph, whose set of nodes is $\mathcal{P}\cup\mathcal{Q}$ and there is an arc from $s \to t$ to $u \to v$ whenever there exist substitutions σ and τ such that $t\sigma \xrightarrow[\mathcal{R}\cup\mathcal{E}]{}^{*} u\tau$. Then $\langle \mathcal{P}/\mathcal{Q}, \mathcal{R}/\mathcal{E}\rangle$ is finite iff every AC-DP problem $\langle \mathcal{P}\cap\mathcal{C}_i/\mathcal{Q}\cap\mathcal{C}_i, \mathcal{R}/\mathcal{E}\rangle$ is, where $\mathcal{C}_1, \dots, \mathcal{C}_n$ are the strongly connected components (SCCs) of $\mathcal{G}$.*

More work has to be done if minimality is involved, since minimality in a relative DP problem $\langle \mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{E}\rangle$ considers termination w.r.t. $\mathcal{R}\cup\mathcal{E}$, whereas we consider termination w.r.t. $\mathcal{R}/\mathcal{E}$ (cf. Sections 5 and 6). But before we can apply any technique, we first have to construct an initial AC-DP problem whose finiteness corresponds to AC termination.

4 From AC Termination to AC-DP Problems

In this section we reduce the termination of a TRS $\mathcal{R}$ modulo an AC theory $\mathcal{E}$ to the finiteness of certain AC-DP problems.

For the standard DP method, i.e., $\mathcal{E} = \mathcal{Q} = \emptyset$, Theorem 4 means that the finiteness of the DP problem is equivalent to the termination of $\mathcal{R}$. In the case of AC termination, however, it is well known that the argument is not directly applicable.

▶ **Example 8** (cf. [12]). Consider the TRS $\mathcal{R} = \{\, x + x \to \mathsf{a} + \mathsf{b} \,\}$ with $+ \in \mathcal{F}_{\mathcal{A}}$. The DP problem $\langle \mathsf{DP}(\mathcal{R})/\mathsf{DP}(\mathcal{E}), \mathcal{R}/\mathcal{E}\rangle$ is finite; it is easy to see that $\mathsf{DP}(\mathcal{R}) = \{\, x +^{\sharp} x \to \mathsf{a} +^{\sharp} \mathsf{b} \,\}$ cannot constitute an infinite chain. However, $\mathcal{R}/\mathcal{E}$ is nonterminating, as illustrated by

$$(\mathsf{a}+\mathsf{a})+\mathsf{b} \xrightarrow[\mathcal{R}]{} (\mathsf{a}+\mathsf{b})+\mathsf{b} \underset{\mathcal{E}}{\approx} \mathsf{a}+(\mathsf{b}+\mathsf{b}) \xrightarrow[\mathcal{R}]{} \mathsf{a}+(\mathsf{a}+\mathsf{b}) \underset{\mathcal{E}}{\approx} \cdots$$

Nevertheless, since we take dependency pairs of $\mathcal{E}$, we can reuse the following key lemma from the standard DP method. To be more precise, we will instantiate the lemma for both $\mathcal{R}$ and $\mathcal{E}$ in the later development.

▶ **Lemma 9.** *Let $\mathcal{R}$ be a TRS. If $s \xrightarrow[\mathcal{R}]{\epsilon} t$ for a minimal nonterminating term s and nonterminating t, then there is a minimal nonterminating subterm t' of t with $s^\sharp \xrightarrow[\mathsf{DP}(\mathcal{R})]{\epsilon} t'^\sharp$.*

The following lemma is a variant of [12, Lemma 3.16], but differs in the following way: While [12, Lemma 3.16] requires the so-called *head subterm relation* in a chain to ensure minimality, we employ the rules in $\mathsf{DP}(\mathcal{E})$, especially those of form $(\mathsf{a}^\sharp)$, for this purpose.

▶ **Lemma 10.** *If s is a minimal nonterminating term, then there exist a minimal nonterminating term v and a minimal chain $s^\sharp \xrightarrow[\mathcal{E}\cup\mathsf{DP}(\mathcal{E})]{}^{*} \cdot \xrightarrow[\mathcal{R}\cup\mathsf{DP}(\mathcal{R})]{} v^\sharp$.*

The lemma is proved by induction w.r.t. the following well-founded order pair. In the formalization we prove its well-foundedness from the fact that $\mathcal{E}$-equivalent terms have the same size. The same result could be deduced also from a result by Jouannaud and Kirchner [10].

▶ **Definition 11** (Proof Ordering 1)**.** We denote the weak and strict subterm relation by $\trianglerighteq$ and $\triangleright$. The relations $\trianglerighteq_{\mathcal{E}}$ and $\triangleright_{\mathcal{E}}$ are defined as $\left(\triangleright \cup \underset{\mathcal{E}}{\approx}\right)^*$ and $\left(\triangleright / \underset{\mathcal{E}}{\approx}\right)^+$, respectively.

Proof of Lemma 10. Consider a minimal nonterminating term s. Then we have a sequence $s \xrightarrow[\mathcal{E}]{}^{n} t \xrightarrow[\mathcal{R}]{} u$ for nonterminating u. We prove the claim by well-founded induction on $\langle s, n\rangle$ w.r.t. the lexicographic composition of $\triangleright_{\mathcal{E}}$ and $>$ on natural numbers.

- Suppose $s = t$. If $t \xrightarrow[\mathcal{R}]{\epsilon} u$, then by Lemma 9 we obtain a minimal nonterminating term v and a minimal chain $s^\sharp = t^\sharp \xrightarrow[\mathsf{DP}(\mathcal{R})]{} v^\sharp$. Otherwise u must also be minimal and thus by taking $v = u$, we have a minimal chain $s^\sharp = t^\sharp \xrightarrow[\mathcal{R}]{} v^\sharp$.
- Suppose $s \xrightarrow[\mathcal{E}]{} s' \xrightarrow[\mathcal{E}]{}^{n-1} t$. Obviously $s^\sharp \xrightarrow[\mathsf{DP}(\mathcal{E})\cup\mathcal{E}]{} s'^\sharp$. Note also that $s \trianglerighteq_{\mathcal{E}} s'$; hence, if s' is minimal, then the induction hypothesis for $\langle s', n-1\rangle$ yields a minimal nonterminating term v and a minimal chain:

  $$s^\sharp \xrightarrow[\mathcal{E}\cup\mathsf{DP}(\mathcal{E})]{} s'^\sharp \xrightarrow[\mathcal{E}\cup\mathsf{DP}(\mathcal{E})]{}^{*} \cdot \xrightarrow[\mathcal{R}\cup\mathsf{DP}(\mathcal{R})]{} v^\sharp$$

  Now suppose s' is not minimal, which is only possible if $s \xrightarrow[\mathcal{E}]{\epsilon} s'$. By Lemma 9, we obtain a minimal nonterminating subterm s'' of s' such that $s^\sharp \xrightarrow[\mathsf{DP}(\mathcal{E})]{} s''^\sharp$. Since s' is not yet minimal, we know $s' \triangleright s''$, and hence $s \triangleright_{\mathcal{E}} s''$. Since s'' is nonterminating, we have a derivation $s'' \xrightarrow[\mathcal{E}]{}^{m} t' \xrightarrow[\mathcal{R}]{} u'$ for some nonterminating t' and u'. To this sequence, we apply the induction hypothesis for $\langle s'', m\rangle$ and obtain a minimal chain

  $$s^\sharp \xrightarrow[\mathsf{DP}(\mathcal{E})]{} s''^\sharp \xrightarrow[\mathcal{E}\cup\mathsf{DP}(\mathcal{E})]{}^{*} \cdot \xrightarrow[\mathcal{R}\cup\mathsf{DP}(\mathcal{R})]{} v^\sharp$$

  with a minimal nonterminating term v. ◀

A repeated application of Lemma 10 converts any infinite $\mathcal{R}/\mathcal{E}$-rewrite sequence into an infinite minimal $\langle \mathsf{DP}(\mathcal{R})/\mathsf{DP}(\mathcal{E}), \mathcal{R}/\mathcal{E}\rangle$-chain. However, this AC-DP problem can be *finite*; the resulting chain may have infinitely many $\mathcal{R}$-steps connected by only $\mathsf{DP}(\mathcal{E})$-steps.

We avoid this case via the finiteness of another AC-DP problem. To this end we employ the notion of *AC-extended rewriting* [16].

▶ **Definition 12.** Let x and y be arbitrary fresh variables. The set $\mathcal{R}_\mathcal{E}$ of *extended rules* is the TRS that consists of the following rule for each $l \to r \in \mathcal{R}$ with $\mathsf{root}(l) = + \in \mathcal{F}_\mathsf{AC}$

$$l + x \to r + x$$

and the following rules for each $l \to r \in \mathcal{R}$ with $\mathsf{root}(l) = + \in \mathcal{F}_\mathsf{A} \setminus \mathcal{F}_\mathsf{C}$.

$$x + l \to x + r \qquad\qquad l + x \to r + x \qquad\qquad x + l + y \to x + r + y$$

We write $\mathcal{R}^\sharp_\mathcal{E}$ for the TRS $\{\, l^\sharp \to r^\sharp \mid l \to r \in \mathcal{R}_\mathcal{E} \,\}$. Note that in this definition, $\mathcal{R}_\mathcal{E}$ does not necessarily contain the rules in $\mathcal{R}$.

Now we state the main theorem of this section.

▶ **Theorem 13.** *A TRS $\mathcal{R}$ is terminating modulo an AC theory $\mathcal{E}$ iff both AC-DP problems $\langle \mathsf{DP}(\mathcal{R})/\mathcal{Q}, \mathcal{R}/\mathcal{E}\rangle$ and $\langle \mathcal{R}^\sharp_\mathcal{E}/\mathcal{Q}, \mathcal{R}/\mathcal{E}\rangle$ are finite, where $\mathcal{Q} = \mathsf{DP}(\mathcal{E})$.*[1]

Before proving the theorem we prepare several notions.

▶ **Definition 14.** A *top position* (also called a *head position* [12]) in an unmarked term $s = f(\vec{s}_n)$ is the root position ϵ if $f \notin \mathcal{F}_\mathsf{A}$, or a position p such that $\mathsf{root}(s|_q) = f$ for every prefix q of p, if $f \in \mathcal{F}_\mathsf{A}$. The top positions in a marked term $s^\sharp$ are those in s. We denote a rewrite step $s \xrightarrow[\mathcal{R}]{p} t$ as $s \xrightarrow[\mathcal{R}]{\mathsf{top}} t$ if p is a top position in s, and as $s \xrightarrow[\mathcal{R}]{>\mathsf{top}} t$ otherwise.

▶ **Lemma 15.** *If $s \xrightarrow[\mathcal{R}]{\mathsf{top}} t$, then $s \xrightarrow[\mathcal{R}]{\epsilon} t$ or $s \approx_\mathcal{E} \cdot \xrightarrow[\mathcal{R}_\mathcal{E}]{\epsilon} \cdot \approx_\mathcal{E} t$.*

As a measure for the inductive proof of the main theorem, we employ *top-flattening* [17]: the top-flattening of a term s w.r.t. $f \in \mathcal{F}_\mathsf{A}$ is the multiset $\triangledown_f(s)$ defined inductively by

$$\triangledown_f(s) = \begin{cases} \{s\} & \text{if } \mathsf{root}(s) \neq f \\ \triangledown_f(s_1) \uplus \triangledown_f(s_2) & \text{if } s = f(s_1, s_2) \end{cases}$$

For a term $s = f(\vec{s}_n)$, we define $\triangledown(s)$ as $\triangledown_f(s)$ if $f \in \mathcal{F}_\mathsf{A}$, and as $\{s_1, \ldots, s_n\}$ otherwise.

The top-flattenings are ordered by the multiset extension of the following quasi-order pair:

▶ **Definition 16** (Proof Ordering 2)**.** We define the relations $\trianglerighteq_{\mathcal{R}/\mathcal{E}}$ and $\rhd_{\mathcal{R}/\mathcal{E}}$ as $\trianglerighteq_{\mathcal{R}/\mathcal{E}} := (\xrightarrow[\mathcal{R}]{} \cup \rhd \cup \approx_\mathcal{E})^*$ and $\rhd_{\mathcal{R}/\mathcal{E}} := (\approx_\mathcal{E} \cdot \xrightarrow[\mathcal{R}]{} \cdot \trianglerighteq \cdot \approx_\mathcal{E})^+$.

The above relations form a well-founded order pair on $\mathcal{R}/\mathcal{E}$-terminating terms, which is easily shown using the fact that the subterm relation preserves termination. It is also not difficult to prove the following lemma.

▶ **Lemma 17.** *The relations $\trianglerighteq_{\mathcal{R}/\mathcal{E}}$ and $\rhd_{\mathcal{R}/\mathcal{E}}$ satisfy the following properties:*

1. *If $s^\sharp \xrightarrow[\mathcal{R}]{>\mathsf{top}} t^\sharp$, then $\triangledown(s) \rhd^{\mathsf{mul}}_{\mathcal{R}/\mathcal{E}} \triangledown(t)$.*
2. *If $s^\sharp \xrightarrow[\mathcal{E}]{} t^\sharp$, then $\triangledown(s) \trianglerighteq^{\mathsf{mul}}_{\mathcal{R}/\mathcal{E}} \triangledown(t)$.*

We further require another well-founded order pair for the inductive proof of the main theorem. We write $s \xrightarrow[\mathcal{P}]{\mathsf{min}} t$ when $s \xrightarrow[\mathcal{P}]{\epsilon} t$ *and* t is $\mathcal{R}/\mathcal{E}$-terminating.

[1] Our formalization shows that $\mathsf{DP}(\mathcal{E})$ can be generated w.r.t. only the defined symbols of $\mathcal{R}$. To ease readability, we do not introduce the definition of such restricted dependency pairs.

▶ **Definition 18** (Proof Ordering 3)**.** Let TRSs $\mathcal{Q}$, $\mathcal{R}$, and $\mathcal{E}$ be fixed. For a TRS $\mathcal{P}$, we define the relations $\succsim_{\mathcal{P}}$ and $\succ_{\mathcal{P}}$ by $\big(\xrightarrow[\mathcal{P}\cup\mathcal{Q}]{\mathsf{min}} \cup \xrightarrow[\mathcal{R}\cup\mathcal{E}]{>\epsilon}\big)^*$ and $\big(\xrightarrow[\mathcal{P}]{\mathsf{min}} / (\xrightarrow[\mathcal{Q}]{\mathsf{min}} \cup \xrightarrow[\mathcal{R}\cup\mathcal{E}]{>\epsilon})\big)^+$, respectively.

It is easy to see that $\langle \succsim_{\mathcal{P}}, \succ_{\mathcal{P}} \rangle$ forms a quasi-order pair. On the other hand, its well-foundedness depends on the finiteness of the underlying AC-DP problem.

▶ **Lemma 19.** *If $\langle \mathcal{P}/\mathcal{Q}, \mathcal{R}/\mathcal{E} \rangle$ is finite, then $\langle \succsim_{\mathcal{P}}, \succ_{\mathcal{P}} \rangle$ is a well-founded order pair on $\mathcal{R}/\mathcal{E}$-terminating terms.*

Proof. We prove the well-foundedness of $\succ_{\mathcal{P}}$ by contradiction: Suppose that it is not well-founded on $\mathcal{R}/\mathcal{E}$-terminating terms. So we have an $\mathcal{R}/\mathcal{E}$-terminating term s that starts an infinite reduction $s \succ_{\mathcal{P}} \cdot \succ_{\mathcal{P}} \cdots$, i.e.,

$$s \,\big(\xrightarrow[\mathcal{Q}]{\mathsf{min}} \cup \xrightarrow[\mathcal{R}\cup\mathcal{E}]{>\epsilon}\big)^* \cdot \xrightarrow[\mathcal{P}]{\mathsf{min}} \cdot \big(\xrightarrow[\mathcal{Q}]{\mathsf{min}} \cup \xrightarrow[\mathcal{R}\cup\mathcal{E}]{>\epsilon}\big)^* \cdot \xrightarrow[\mathcal{P}]{\mathsf{min}} \cdots$$

This sequence is an infinite minimal chain, contradicting the finiteness of $\langle \mathcal{P}/\mathcal{Q}, \mathcal{R}/\mathcal{E} \rangle$. ◀

Now we are ready to prove the main theorem.

Proof of Theorem 13. The "only if" direction is proved as in the standard DP framework, where we additionally use the fact that $s^\sharp \xrightarrow[\mathcal{R}^\sharp_{\mathcal{E}}]{} t^\sharp$ implies $s \xrightarrow[\mathcal{R}]{} t$.

For the "if" direction, suppose that $\langle \mathsf{DP}(\mathcal{R})/\mathcal{Q}, \mathcal{R}/\mathcal{E} \rangle$ and $\langle \mathcal{R}^\sharp_{\mathcal{E}}/\mathcal{Q}, \mathcal{R}/\mathcal{E} \rangle$ are finite. We prove that any minimal $\mathcal{R}/\mathcal{E}$-nonterminating term s derives a contradiction, by induction on $\langle s^\sharp, s^\sharp, \triangledown(s) \rangle$, which is ordered by the lexicographic composition of $\succ_{\mathsf{DP}(\mathcal{R})}$, $\succ_{\mathcal{R}^\sharp_{\mathcal{E}}}$, and $\rhd^{\mathsf{mul}}_{\mathcal{R}/\mathcal{E}}$.

By Lemma 10, we obtain a minimal $\mathcal{R}/\mathcal{E}$-nonterminating term u and a minimal chain $s^\sharp \xrightarrow[\mathcal{E}\cup\mathsf{DP}(\mathcal{E})]{}{}^* \, t^\sharp \xrightarrow[\mathcal{R}\cup\mathsf{DP}(\mathcal{R})]{} u^\sharp$. We distinguish the following cases:

- Suppose $t^\sharp \xrightarrow[\mathsf{DP}(\mathcal{R})]{} u^\sharp$. In this case, we have $s^\sharp \succ_{\mathsf{DP}(\mathcal{R})} u^\sharp$. Hence, the induction hypothesis for u on the first component derives a contradiction.
- Suppose $t^\sharp \xrightarrow[\mathcal{R}]{>\mathsf{top}} u^\sharp$. By definition we have $s^\sharp \succsim_{\mathsf{DP}(\mathcal{R})} u^\sharp$ and $s^\sharp \succsim_{\mathcal{R}^\sharp_{\mathcal{E}}} u^\sharp$. Using Lemma 17, we also have $\triangledown(s) \unrhd^{\mathsf{mul}}_{\mathcal{R}/\mathcal{E}} \triangledown(t) \rhd^{\mathsf{mul}}_{\mathcal{R}/\mathcal{E}} \triangledown(u)$. Hence, the induction hypothesis for u on the third component derives a contradiction.
- Suppose $t^\sharp \xrightarrow[\mathcal{R}]{\mathsf{top}} u^\sharp$. By Lemma 15 we obtain a sequence $t = t_0 \xrightarrow[\mathcal{E}]{} \cdots \xrightarrow[\mathcal{E}]{} t_n \xrightarrow[\mathcal{R}_{\mathcal{E}}]{\epsilon} v \approx_{\mathcal{E}} u$.
  - If every t_i is minimal, then we have

    $$s^\sharp \;\succsim_{\mathsf{DP}(\mathcal{R})}\; t^\sharp \;=\; t_0^\sharp \;\succsim_{\mathsf{DP}(\mathcal{R})}\; t_n^\sharp \;\succsim_{\mathsf{DP}(\mathcal{R})}\; v^\sharp$$

    and similarly, $s^\sharp \succsim_{\mathcal{R}^\sharp_{\mathcal{E}}} t_n^\sharp \succ_{\mathcal{R}^\sharp_{\mathcal{E}}} v^\sharp$. Thus the induction hypothesis for v on the second component yields a contradiction.
  - Otherwise, for some $i < n$, $t_0, \dots, t_i$ are minimal but t_{i+1} is not. Take a minimal nonterminating subterm t' of t_{i+1}. Using Lemma 9, we get a minimal chain

    $$s^\sharp \xrightarrow[\mathcal{E}\cup\mathsf{DP}(\mathcal{E})]{}{}^* \, t^\sharp = t_0^\sharp \xrightarrow[\mathcal{E}\cup\mathsf{DP}(\mathcal{E})]{}{}^* \, t_i^\sharp \xrightarrow[\mathsf{DP}(\mathcal{E})]{} t'^\sharp$$

    and thus, $s^\sharp \succsim_{\mathsf{DP}(\mathcal{R})} t'^\sharp$ and $s^\sharp \succsim_{\mathcal{R}^\sharp_{\mathcal{E}}} t'^\sharp$. By Lemma 17 we have $\triangledown(s) \unrhd^{\mathsf{mul}}_{\mathcal{R}/\mathcal{E}} \triangledown(t) \unrhd^{\mathsf{mul}}_{\mathcal{R}/\mathcal{E}} \triangledown(t_i)$, and due to the shape of AC axioms, $\triangledown(t_i) \supset \triangledown(t')$. Hence, the induction hypothesis for t' on the third component derives a contradiction. ◀

In contrast to the non-AC case, Theorem 13 generates two AC-DP problems, where $\mathsf{DP}(\mathcal{R})$ and $\mathcal{R}^\sharp_{\mathcal{E}}$ are separated. These two sets of pairs can be merged into one problem.

▶ **Corollary 20.** *A TRS $\mathcal{R}$ is terminating modulo an AC theory $\mathcal{E}$ iff the AC-DP problem $\langle (\mathsf{DP}(\mathcal{R}) \cup \mathcal{R}^\sharp_{\mathcal{E}})/\mathsf{DP}(\mathcal{E}), \mathcal{R}/\mathcal{E} \rangle$ is finite.*

Clearly, solving this merged AC-DP problem cannot be easier than tackling the smaller AC-DP problems separately. Nevertheless, it remains open whether practically there is any difference in power between Theorem 13 and Corollary 20.

5 Usable Rules

In this section, we prove that the usable rules technique is applicable within our AC-DP framework. To this end, we generalize the currently most powerful variant of usable rules in the DP framework [18] to the AC case. To be more precise, we adapt the statements and proofs of the existing formalization in such a way that they become applicable both for the DP framework and the AC-DP framework.

Let us first recapitulate the notions of (non-AC) usable rules. The following function tcap is used to estimate the shape of a term after rewriting:

$$\begin{aligned} \mathsf{tcap}_{\mathcal{R}}(x) &= \text{a fresh variable} \\ \mathsf{tcap}_{\mathcal{R}}(f(\vec{s}_n)) &= \begin{cases} \text{a fresh variable} & \text{if } \exists l \to r \in \mathcal{R}.\ l \sim f(\mathsf{tcap}_{\mathcal{R}}(\vec{s}_n)) \\ f(\mathsf{tcap}_{\mathcal{R}}(\vec{s}_n)) & \text{otherwise} \end{cases} \end{aligned}$$

Here, $\mathsf{tcap}_{\mathcal{R}}(\vec{s}_n)$ is an abbreviation for $\mathsf{tcap}_{\mathcal{R}}(s_1), \dots, \mathsf{tcap}_{\mathcal{R}}(s_n)$, and $\sim$ denotes *unifiability*: $s \sim t$ iff there exist substitutions σ and τ such that $s\sigma = t\tau$.

Let $\mathcal{R}$ and $\mathcal{U}$ be TRSs. An *argument filter* π assigns each function symbol a subset of its argument positions, where $i \notin \pi(f)$ indicates that the argument s_i of any term $f(\vec{s}_n)$ will be ignored for the usable rules. The formula $\mathsf{urClosed}^{\pi}_{\mathcal{U},\mathcal{R}}(\mathcal{S})$ is defined as follows:

$$\begin{aligned} \mathsf{urClosed}^{\pi}_{\mathcal{U},\mathcal{R}}(x) &= \mathsf{true} \\ \mathsf{urClosed}^{\pi}_{\mathcal{U},\mathcal{R}}(f(\vec{s}_n)) &= \bigwedge_{i \in \pi(f)} \mathsf{urClosed}^{\pi}_{\mathcal{U},\mathcal{R}}(s_i) \wedge \bigwedge_{l \to r \in \mathcal{R}} (l \sim f(\mathsf{tcap}_{\mathcal{R}}(\vec{s}_n)) \Longrightarrow l \to r \in \mathcal{U}) \\ \mathsf{urClosed}^{\pi}_{\mathcal{U},\mathcal{R}}(\mathcal{S}) &= \bigwedge_{l \to r \in \mathcal{S}} \mathsf{urClosed}^{\pi}_{\mathcal{U},\mathcal{R}}(r) \end{aligned}$$

Rewrite rules which might be invoked from the right-hand sides of dependency pairs should be considered as usable rules. This is captured formally in the following definition.

▶ **Definition 21** (from [18]). We say $\mathcal{U}$ is a set of *usable rules* for a DP problem $\langle \mathcal{P}, \mathcal{R} \rangle$ w.r.t. an argument filter π iff the following formula is satisfied:

$$\mathsf{urClosed}^{\pi}_{\mathcal{U},\mathcal{R}}(\mathcal{P}) \wedge \mathsf{urClosed}^{\pi}_{\mathcal{U},\mathcal{R}}(\mathcal{U})$$

The close connection between our AC-DP framework and the standard DP framework admits reusing the above notions without major effort; we simply instantiate $\mathcal{R}$ to $\mathcal{R} \cup \mathcal{E}$ and $\mathcal{P}$ to $\mathcal{P} \cup \mathcal{Q}$. A more precise alternative might integrate *$\mathcal{E}$-unification* into tcap; however, we refrain from this possibility as IsaFoR has no $\mathcal{E}$-unification algorithm formalized yet.

▶ **Definition 22** (Usable Rules for AC). We say $\mathcal{U}$ is a set of usable rules for an AC-DP problem $\langle \mathcal{P}/\mathcal{Q}, \mathcal{R}/\mathcal{E} \rangle$ w.r.t. an argument filter π iff the following formula is satisfied:

$$\mathsf{urClosed}^{\pi}_{\mathcal{U},\mathcal{R}\cup\mathcal{E}}(\mathcal{P} \cup \mathcal{Q}) \wedge \mathsf{urClosed}^{\pi}_{\mathcal{U},\mathcal{R}\cup\mathcal{E}}(\mathcal{U})$$

The following theorem generalizes the corresponding theorem in the DP framework [18, Theorem 4.6]. Instead of requiring a weak decrease for all rules in $\mathcal{R} \cup \mathcal{E}$, one just has to look at the usable rules $\mathcal{U}$. Moreover, under certain conditions one can even delete all non-usable rules from $\mathcal{R}$. In the rest of this section, we assume a fresh infix function symbol $\circ$, and denote the TRS $\{\, x \circ y \to x,\ x \circ y \to y \,\}$ by $\mathcal{C}_e$.

▶ **Theorem 23.** *Let $\mathcal{R}$ be a finite TRS, π an argument filter, $\mathcal{U}$ a set of usable rules for $\langle\mathcal{P}/\mathcal{Q},\mathcal{R}/\mathcal{E}\rangle$ w.r.t. π, and $\langle\gtrsim,\succ\rangle$ a reduction pair such that*

- *$\mathcal{P}\cup\mathcal{Q}\cup\mathcal{U}\cup\mathcal{C}_e\subseteq\gtrsim$, and*
- *$\gtrsim$ is π-compatible; i.e., $f(s_1,\dots,s_i,\dots,s_n)\gtrsim f(s_1,\dots,s_i',\dots,s_n)$ whenever $i\notin\pi(f)$.*

Then $\langle\mathcal{P}/\mathcal{Q},\mathcal{R}/\mathcal{E}\rangle$ is finite if $\langle\mathcal{P}'/\mathcal{Q}',\mathcal{R}'/\mathcal{E}\rangle$ is, where $\mathcal{P}'=\mathcal{P}\setminus\succ$, $\mathcal{Q}'=\mathcal{Q}\setminus\succ$, and $\mathcal{R}'=\mathcal{R}$ or $\mathcal{R}'=(\mathcal{R}\cap\mathcal{U})\setminus\succ$ if $\mathcal{C}_e\subseteq\succ$ and $\succ$ is closed under contexts.

▶ **Example 24.** To illustrate the application of Theorem 23, consider an AC-DP problem with $\mathcal{F}_\mathsf{A}=\{+,\times\}$, $\mathcal{F}_\mathsf{C}=\{+,\times,\mathsf{eq}\}$, $\mathcal{P}$ consisting of

$$\mathsf{f}^\sharp(\mathsf{s}(x),y,z)\to\mathsf{eq}^\sharp(x,y)\qquad\qquad \mathsf{f}^\sharp(\mathsf{s}(x),y,z)\to\mathsf{f}^\sharp(\mathsf{p}(\mathsf{s}(x)),x+y,x\times z)$$

and $\mathcal{R}$ consisting of a standard rules for addition $+$ and multiplication $\times$, some rules for f, and the following two rules for p.

$$\mathsf{p}(\mathsf{s}(x))\to x \qquad (1) \qquad\qquad \mathsf{p}(0)\to 0 \qquad (2)$$

If one applies the theorem with $\pi(\mathsf{f}^\sharp)=\{1,2\}$, then only (1) and the rules for $+$ in $\mathcal{R}\cup\mathcal{E}$ have to be marked as usable. One can ignore

- rules in $\mathcal{R}\cup\mathcal{E}$ for $\times$, since $\times$ occurs only in the third argument of $\mathsf{f}^\sharp$;
- the other rule (2) for p, since its left-hand side does not unify with $\mathsf{p}(\mathsf{s}(x))$; and
- rules in $\mathcal{R}\cup\mathcal{E}$ for f and eq, since only the marked versions of these symbols occur.

It is also necessary to take into account the pairs in $\mathcal{Q}$ for determining usable rules.

▶ **Example 25.** Consider the AC-DP problem $\langle\mathcal{P}/\mathcal{Q},\mathcal{R}/\mathcal{E}\rangle$ where $\mathcal{E}=\{(\mathsf{A}_1)\}$ and

$$\mathcal{P}=\{\mathsf{f}(x)+^\sharp\mathsf{c}\to\mathsf{a}+^\sharp x\}\quad \mathcal{Q}=\{x+^\sharp(y+z)\to(x+y)+^\sharp z\}\quad \mathcal{R}=\{\mathsf{a}+\mathsf{b}\to\mathsf{f}(\mathsf{b}+\mathsf{c})\}$$

This AC-DP problem is not finite, as illustrated by the following chain:

$$\mathsf{f}(\mathsf{b}+\mathsf{c})+^\sharp\mathsf{c}\xrightarrow[\mathcal{P}]{}\mathsf{a}+^\sharp(\mathsf{b}+\mathsf{c})\xrightarrow[\mathcal{Q}]{}(\mathsf{a}+\mathsf{b})+^\sharp\mathsf{c}\xrightarrow[\mathcal{R}]{}\mathsf{f}(\mathsf{b}+\mathsf{c})+^\sharp\mathsf{c}\xrightarrow[\mathcal{P}]{}\cdots$$

Now assume one ignores usable rules from $\mathcal{Q}$ – then no rule is usable, and one could wrongly deduce finiteness, e.g., with a polynomial interpretation $\mathcal{A}$ such that

$$x+^\sharp_\mathcal{A}y=x+_\mathcal{A}y=x+y\qquad\qquad \mathsf{f}_\mathcal{A}(x)=x\qquad\qquad \mathsf{c}_\mathcal{A}>\mathsf{a}_\mathcal{A}$$

In the non-AC case, a key to proving Theorem 23 is a transformation $\mathcal{I}$, that converts any minimal $\langle\mathcal{P},\mathcal{R}\rangle$-chain into a $\langle\mathcal{P},\mathcal{U}\cup\mathcal{C}_e\rangle$-chain by applying $\mathcal{I}$ to all terms in the chain. Here, minimality is essential since $\mathcal{I}$ is applicable only to $\mathcal{R}$-terminating terms.

In the AC-DP framework, we cannot directly reuse $\mathcal{I}$, since minimality ensures only $\mathcal{R}/\mathcal{E}$-termination, but not $\mathcal{R}\cup\mathcal{E}$-termination. Thus we base our soundness proof on the following transformation $\overline{\cdot}$. Below, we write $\overline{\vec{s}_n}$ to denote the list $\overline{s_1},\dots,\overline{s_n}$.

▶ **Definition 26.** Consider a finite TRS $\mathcal{R}$, an AC theory $\mathcal{E}$, and a set $\mathcal{U}$ of usable rules. For an $\mathcal{R}/\mathcal{E}$-terminating term s, we define $\overline{s}$ as follows: $\overline{x}=x$ and for $s=f(\vec{s}_n)$,

$$\overline{s}=\begin{cases}\mathsf{list}\big(\{g(\overline{\vec{t}_n})\mid s\underset{\mathcal{E}}{\approxeq}g(\vec{t}_n)\}\cup\{\overline{t}\mid s\xrightarrow[\mathcal{R}/\mathcal{E}]{}t\}\big) & \text{if }\exists l\to r\in(\mathcal{R}\cup\mathcal{E})\setminus\mathcal{U}.\ l\sim f(\mathsf{tcap}_{\mathcal{R}\cup\mathcal{E}}(\vec{s}_n))\\ f(\overline{\vec{s}_n}) & \text{otherwise}\end{cases}$$

Here, $\mathsf{list}(\{s_1,\dots,s_n\})$ denotes the term $s_1\circ\cdots\circ s_n$, where the elements are sorted w.r.t. an arbitrary but fixed total order on terms.

It is easy to see that $\overline{s}$ is defined for every $\mathcal{R}/\mathcal{E}$-terminating s, using an inductive argument and the fact that there are only finitely many $\mathcal{E}$-equivalent terms.

Most of the existing proofs using transformation $\mathcal{I}$ can be immediately generalized to the new transformation; various properties [18, Lemma 4.9] are satisfied after straightforward modifications, e.g., replacing $\mathcal{R}$-termination by $\mathcal{R}/\mathcal{E}$-termination.

Nevertheless, in order to prove Theorem 23 we have to add some new properties which are required for simulating $\mathcal{E}$-equivalence. In the following crucial property we rely upon the fact that $\mathcal{E}$ is an AC theory; in many other places it was sufficient to know that $\mathcal{E}$ is symmetric and has only finite equivalence classes. We omit presenting other details of the proof of Theorem 23.

▶ **Lemma 27.** *Suppose that $\mathcal{E}$ is an AC theory and $\mathsf{urClosed}^{\pi}_{\mathcal{U},\mathcal{R}\cup\mathcal{E}}(\mathcal{U})$ is satisfied. If $f(\vec{t}_n)$ is $\mathcal{R}/\mathcal{E}$-terminating, $f(\vec{t}_n) \underset{\mathcal{E}}{\approx} s$, and $\overline{f(\vec{t}_n)} \neq f(\overline{\vec{t}_n})$, then $\overline{f(\vec{t}_n)} = \overline{s}$.*

Proof. By the preconditions s must be of the form $f(\vec{s}_n)$, and due to the symmetry of $\underset{\mathcal{E}}{\approx}$, $f(\vec{s}_n) \underset{\mathcal{E}}{\approx} f(\vec{t}_n)$. Furthermore, there must be a rule $l \to r \in (\mathcal{R} \cup \mathcal{E}) \setminus \mathcal{U}$ such that $l \sim f(\mathsf{tcap}_{\mathcal{R}\cup\mathcal{E}}(\vec{t}_n))$. In particular, $\mathsf{root}(l)$ must be f.

The challenge is showing $\overline{f(\vec{s}_n)} \neq f(\overline{\vec{s}_n})$, i.e., finding a non-usable rule where the left-hand side unifies with $f(\mathsf{tcap}_{\mathcal{R}\cup\mathcal{E}}(\vec{s}_n))$. Having proved this fact, the claim immediately follows by the transitivity of $\underset{\mathcal{E}}{\approx}$ and the definition of $\overline{\cdot}$.

To determine such a non-usable rule we consider the following two cases.

- If $s_i \underset{\mathcal{E}}{\approx} t_i$ for all $1 \le i \le n$, then $f(\mathsf{tcap}_{\mathcal{R}\cup\mathcal{E}}(\vec{s}_n)) = f(\mathsf{tcap}_{\mathcal{R}\cup\mathcal{E}}(\vec{t}_n))$. Hence, the above $l \to r$ is a desired non-usable rule.
- Otherwise, we have a root $\mathcal{E}$-step, i.e., for some AC rule $u \to v \in \mathcal{E}$, $f(\vec{s}_n) \underset{\mathcal{E}}{\approx} f(\vec{w}_n) = u\sigma$ and $v\sigma \underset{\mathcal{E}}{\approx} f(\vec{t}_n)$. The properties of tcap guarantee $u \sim f(\mathsf{tcap}_{\mathcal{R}\cup\mathcal{E}}(\vec{s}_n))$. By the shape of the AC rules and $\mathsf{urClosed}^{\pi}_{\mathcal{U},\mathcal{R}\cup\mathcal{E}}(\mathcal{U})$, we conclude $u \to v \notin \mathcal{U}$ from the fact that $l \to r \notin \mathcal{U}$ and $\mathsf{root}(l) = f$. Hence $u \to v \in (\mathcal{R}\cup\mathcal{E})\setminus\mathcal{U}$, and in combination with $u \sim f(\mathsf{tcap}_{\mathcal{R}\cup\mathcal{E}}(\vec{s}_n))$ we have found the desired non-usable rule. ◀

6 Subterm Criterion

The subterm criterion [9] is an efficient technique for proving termination in the standard DP framework. It is based on the notion of *simple projections*, i.e., a mapping π that assigns to each marked symbol $f^\sharp$ one of its argument positions $\pi(f^\sharp)$. Simple projections are extended to terms as follows: $\pi(f^\sharp(s_1,\dots,s_n)) = s_i$ for $i = \pi(f^\sharp)$. For a relation $\sqsupset$ on terms, we denote $\pi(s) \sqsupset \pi(t)$ by $s \sqsupset^\pi t$.

▶ **Theorem 28** ([9])**.** *Let $\langle \mathcal{P}, \mathcal{R} \rangle$ be a (standard) DP problem, and π a simple projection such that $\mathcal{P} \subseteq \trianglerighteq^\pi$. Then, $\langle \mathcal{P}, \mathcal{R} \rangle$ is finite iff $\langle \mathcal{P} \setminus \triangleright^\pi, \mathcal{R} \rangle$ is.*

This technique however is not directly applicable if AC symbols appear. For commutative symbols, neither of the argument positions can be projected.

▶ **Example 29.** For the TRS $\{\, \mathsf{s}(x) + \mathsf{p}(y) \to x + y \,\}$ with $+ \in \mathcal{F}_{\mathsf{C}}$ we construct the following pair for the TRS and in addition the pair $(\mathsf{C}^\sharp)$ for the equations.

$$\mathsf{s}(x) +^\sharp \mathsf{p}(y) \to x +^\sharp y \tag{3}$$

We would like to delete (3) by the subterm criterion. However, projecting either argument of $+^\sharp$, $(\mathsf{C}^\sharp)$ cannot be oriented by $\trianglerighteq^\pi$. On the other hand, one cannot ignore $(\mathsf{C}^\sharp)$; e.g., consider removing a pair $\mathsf{s}(x) +^\sharp x \to x +^\sharp \mathsf{s}(x)$ via $\pi(+^\sharp) = 1$.

This motivates us to generalize the range of π to *multisets of arguments*. For instance, in Example 29 one can choose $\pi(+^\sharp) = \{1, 2\}$. Then $(3) \in \rhd^\pi$ and $(\mathsf{C}^\sharp) \subseteq \unrhd^\pi$.

Associativity rules also require a careful treatment. As in the commutative case, ignoring $(\mathsf{A}^\sharp)$ is unsound, but $(\mathsf{A}^\sharp) \subseteq \unrhd^\pi$ is impossible.

▶ **Example 30** (Example 29 continued)**.** If we assume that $+ \in \mathcal{F}_{\mathsf{AC}}$, the AC-DP problem will contain dependency pairs from $(\mathsf{A}^\sharp)$ and $(\mathsf{a}^\sharp)$. For $\pi(+^\sharp) = \{1, 2\}$, the projected left- and right-hand sides of $(\mathsf{A}^\sharp)$ are $\{x, y + z\}$ and $\{x + y, z\}$, which are incomparable.

This motivates us to also allow projections for unmarked symbols, e.g., $+$ in the above example, and perform the projection recursively.

▶ **Definition 31.** A *multi-projection* π for a set $\mathcal{G}$ of symbols is a mapping that assigns every symbol $f \in \mathcal{G}$ a non-empty multiset $\pi(f)$ of its argument positions. From π we define a mapping from terms to multisets of terms, also denoted by π, as follows:

$$\pi(s) = \begin{cases} \pi(s_{i_1}) \uplus \cdots \uplus \pi(s_{i_m}) & \text{if } s = f(\vec{s}_n),\ f \in \mathcal{G} \text{ and } \pi(f) = \{i_1, \ldots, i_m\} \\ \{s\} & \text{otherwise} \end{cases}$$

For a relation $\sqsupset$ on terms, we write $s \sqsupset^\pi t$ for $\pi(s) \sqsupset^{\mathsf{mul}} \pi(t)$.

Now we can satisfy the constraints in Example 30: $\pi(+^\sharp) = \pi(+) = \{1, 2\}$ will ensure $\{(\mathsf{C}^\sharp), (\mathsf{A}^\sharp)\} \subseteq \unrhd^\pi$ and $\{(3), (\mathsf{a}^\sharp)\} \subseteq \rhd^\pi$. However, allowing projections for a defined symbol – no matter whether it is an AC symbol or not – requires a further side-condition.

▶ **Example 32.** Consider the TRS $\mathcal{R} = \{\, \mathsf{f}(\mathsf{s}(x)) \to \mathsf{f}(\mathsf{g}(x)), \mathsf{g}(x) \to \mathsf{s}(x) \,\}$ and the multi-projection $\pi(\mathsf{f}^\sharp) = \pi(\mathsf{g}^\sharp) = \pi(\mathsf{g}) = \{1\}$. We have $\mathsf{DP}(\mathcal{R}) = \{\, \mathsf{f}^\sharp(\mathsf{s}(x)) \to \mathsf{f}^\sharp(\mathsf{g}(x)) \,\} \subseteq \rhd^\pi$, but the DP problem $\langle \mathsf{DP}(\mathcal{R}), \mathcal{R} \rangle$ is not finite.

Thus in the following theorem, we demand that whenever $\mathcal{G}$ contains a defined symbol f, the rules that define f should be oriented by $\unrhd^\pi$.

▶ **Theorem 33.** *Let $\mathcal{E}$ be a size preserving TRS and π a multi-projection over $\mathcal{G}$ such that $\mathcal{P} \cup \mathcal{Q} \subseteq \unrhd^\pi$ and $l \unrhd^\pi r$ for every $l \to r \in \mathcal{R} \cup \mathcal{E}$ with $\mathsf{root}(l) \in \mathcal{G}$. Then $\langle \mathcal{P}/\mathcal{Q}, \mathcal{R}/\mathcal{E} \rangle$ is finite iff $\langle \mathcal{P}'/\mathcal{Q}', \mathcal{R}/\mathcal{E} \rangle$ is, where $\mathcal{P}' = \mathcal{P} \setminus \rhd^\pi$ and $\mathcal{Q}' = \mathcal{Q} \setminus \rhd^\pi$.*

In order to prove the theorem, we need to prove that both $\rhd^\pi$ and $\unrhd^\pi$ are closed under substitutions – a trivial property of $\rhd$ and $\unrhd$.

▶ **Lemma 34.** *If $s \unrhd^\pi t$ then $s\sigma \unrhd^\pi t\sigma$. If $s \rhd^\pi t$ then $s\sigma \rhd^\pi t\sigma$.*

The proof of the lemma is not so trivial, since substitutions may affect the recursive application of π. Moreover, the restriction in Definition 31 that $\pi(f)$ is nonempty for every $f \in \mathcal{G}$ is crucial for $\rhd^\pi$ to be closed under substitutions.

▶ **Example 35.** Let $\pi(+) = \{1, 2\}$. We have $\mathsf{a} + x \rhd^\pi \mathsf{a}$ since $\{\mathsf{a}, x\} \rhd^{\mathsf{mul}} \{\mathsf{a}\}$. Consider substituting x by $\mathsf{f}(\mathsf{a})$, and allowing $\pi(\mathsf{f}) = \emptyset$. Then $\mathsf{a} + \mathsf{f}(\mathsf{a}) \rhd^\pi \mathsf{a}$ does not hold, since $\{\mathsf{a}\} \rhd^{\mathsf{mul}} \{\mathsf{a}\}$ does not hold.

Proof of Theorem 33. Let $\mathcal{S} = \mathcal{P} \setminus \mathcal{P}' \cup \mathcal{Q} \setminus \mathcal{Q}'$. We only prove that there is no minimal chain containing infinitely many $\mathcal{S}$-steps; the remaining reasoning is trivial. So, assume to the contrary that there is a minimal chain with infinitely many $\mathcal{S}$-steps:

$$s_0 \xrightarrow[\mathcal{P}\cup\mathcal{Q}]{} t_0 \xrightarrow[\mathcal{R}\cup\mathcal{E}]{}{}^{*} s_1 \xrightarrow[\mathcal{P}\cup\mathcal{Q}]{} t_1 \xrightarrow[\mathcal{R}\cup\mathcal{E}]{}{}^{*} s_2 \xrightarrow[\mathcal{P}\cup\mathcal{Q}]{} \cdots$$

Table 1 Experiments.

	APRoVE						NaTT					
	certified		full		no ACDP		Thm. 13		Cor. 20		full	
	#	time	#	time	#	time	#	time	#	time	#	time
yes	128	560.3	128	508.5	82	166.3	78	6.2	78	6.0	113	86.3
no	0	–	2	21.2	0	–	0	–	0	–	1	0.5
maybe	14	310.6	12	340.6	63	189.7	67	46.3	67	28.4	31	149.2
timeout	3	1080.0	3	1080.0	0	–	0	–	0	–	0	–

Let $\langle \succsim, \succ \rangle$ be the quasi-order pair defined as $\succsim := \xrightarrow[\mathcal{R}\cup\mathcal{E}]{}^*$ and $\succ := (\rhd / \xrightarrow[\mathcal{R}\cup\mathcal{E}]{})^+$. By the preconditions and Lemma 34 we can turn every step $s_i \xrightarrow[\mathcal{P}\cup\mathcal{Q}]{} t_i$ into $\pi(s_i) \unrhd^{\mathsf{mul}} \pi(t_i)$, and whenever $s_i \xrightarrow[\mathcal{S}]{}^* t_i$ then $\pi(s_i) \rhd^{\mathsf{mul}} \pi(t_i)$. Moreover, from $t_i \xrightarrow[\mathcal{R}\cup\mathcal{E}]{}^* s_{i+1}$ we derive $\pi(t_i) \succsim^{\mathsf{mul}} \pi(s_{i+1})$ by the conditions on the rules of $\mathcal{R} \cup \mathcal{E}$. Hence, by using $\rhd \subseteq \succ$ we obtain

$$\pi(s_0) \succsim^{\mathsf{mul}} \pi(t_0) \succsim^{\mathsf{mul}} \pi(s_1) \succsim^{\mathsf{mul}} \pi(t_1) \succsim^{\mathsf{mul}} \pi(s_2) \succsim^{\mathsf{mul}} \cdots$$

with an infinite number of $\succ^{\mathsf{mul}}$ steps. Thus, some element $u \in \pi(t_0)$ must start an infinite derivation w.r.t. $\succ$. The size preservation of $\mathcal{E}$ implies that this derivation from u must contain infinitely many $\mathcal{R}$-steps; i.e., u is not $\mathcal{R}/\mathcal{E}$ terminating. This contradicts the minimality of the chain, since u is a subterm of t_0, which is $\mathcal{R}/\mathcal{E}$ terminating. ◀

Theorem 33 generalizes the subterm criterion even in the standard case; one may freely include non-defined symbols into $\mathcal{G}$, which was previously allowed only for marked defined symbols. A similar generalization has been proposed for higher-order rewriting [11]. Nevertheless, in case marked associative symbols are involved, Theorem 33 may not work as one expects from a subterm criterion, as shown in the following example.

▶ **Example 36.** Consider the TRS $\mathcal{R} = \{\, \mathsf{s}(x) + y \to \mathsf{s}(x + y) \,\}$ with $+ \in \mathcal{F}_{\mathsf{A}}$. We construct the following dependency pair for $\mathcal{R}$, in addition to ($\mathsf{A}^\sharp$) and ($\mathsf{a}^\sharp$):

$$\mathsf{s}(x) +^\sharp y \to x +^\sharp y \tag{4}$$

One might expect deleting (4) by the subterm criterion; however, ($\mathsf{A}^\sharp$) demands $+ \in \mathcal{G}$ and thus $\mathcal{R} \subseteq \unrhd^\pi$. This is possible only if s is also in $\mathcal{G}$, resulting $(4) \in \unrhd^\pi$ but $(4) \notin \rhd^\pi$. Hence, on this example other techniques have to be applied, such as reduction pairs with usable rules as in Theorem 23.

7 Experiments

We extended the *certification problem format (CPF)* [20] to our AC-DP framework. For proving finiteness of AC-DP problems, all techniques of Sections 3 to 6 are supported. As a benchmark we use 145 AC termination problems from various sources. Further details on the experimental setup is available on the accompanying website.

We adjusted the termination prover AProVE, that already implements an (unpublished) AC-DP framework based on the Giesl-Kapur variant, to comply to our new framework. We also implemented our technique in NaTT [24], where both Theorem 13 and Corollary 20 are available for comparison. Although any reduction pair supported by CeTA can be used also for AC termination, we disabled the *recursive path order* (RPO) and *Knuth-Bendix order*

(KBO) in the termination provers, since CeTA does not support their AC-compatible variants AC-RPO [17] and AC-KBO [25].

We tested three configurations of AProVE: the new “certified” mode, “full” mode that uses all (uncertified) techniques implemented in AProVE, and “no ACDP” mode that is limited to direct techniques that do not require dependency pairs, which is supported also by an earlier version of CeTA that does not include the AC-DP framework.

The result is shown in the left half of Table 1, where runtime is measured in seconds. Our AC-DP framework is clearly more powerful than “no ACDP” mode. Between “full” and “certified” modes we observe no difference in power; for all examples where “full” mode applied AC-RPO, there are alternative proofs via the AC-DP framework and non-linear polynomial interpretations in “certified” mode. In earlier experiments CeTA rejected one proof generated by a previous version of AProVE, where the usable rules computation in the presence of commutative symbols was incorrect. The bug is now corrected – our work indeed improved the reliability of a state-of-the-art termination prover.

The results for NaTT are shown in the right half of Table 1. Since non-linear polynomials are not supported in NaTT, its “full” strategy is significantly better than the certifiable strategies. For this difference, AC-RPO plays a crucial role.

In one example (`RENAMED-BOOL_complete-noand`), NaTT with Theorem 13 took 21.0sec before it eventually gave up, while with Corollary 20 it gave up after only 0.6sec. This however does not indicate that Theorem 13 is weaker or stronger than Corollary 20; the latter configuration quickly failed due to a small SCC that stems from $\mathsf{DP}(\mathcal{E})$, which would be handled later as another AC-DP problem in the former configuration.

8 Related Work

Kusakari and Toyama [12] introduced a term marking $\cdot^{\#}$ that treats AC symbols specially. If the root symbol of s is an AC symbol $+$, then $s^{\#}$ puts marks on every $+$ occurring at a top position. For instance, $\big(\mathsf{a} + (x + \mathsf{s}(y + z))\big)^{\#} = \mathsf{a} +^{\sharp} (x +^{\sharp} \mathsf{s}(y + z))$. Note that rewriting in such a deeply marked term may bring unmarked AC symbols to a top position; consider, e.g., rewriting $\mathsf{a} \to \mathsf{b} + \mathsf{b}$ in the above term. Therefore, a chain must implicitly perform the following rewriting (AC-mark) and, in order to maintain minimality, also (AC-del).

$$(x + y) +^{\sharp} z \leftrightarrow (x +^{\sharp} y) +^{\sharp} z \quad (\textsf{AC-mark}) \qquad (x +^{\sharp} y) +^{\sharp} z \to x +^{\sharp} y \quad (\textsf{AC-del})$$

The rule (AC-del) is similar but not equal to ($\mathsf{a}^{\sharp}$); besides the nested marks, notice the difference in the right-hand sides.

The special behavior of marking causes major difficulties with respect to implementation and formalization. We also tried but did not succeed to prove usable rules for the Kusakari-Toyama formulation. Fortunately, such an effort is actually not necessary, since our formulation subsumes the Kusakari-Toyama version, in the following sense:

▶ **Proposition 37.** *If a TRS is shown to be AC terminating by the AC-DPs of Kusakari-Toyama with some reduction pairs, then our version succeeds with the same reduction pairs.*

Proof. Consider an AC-compatible reduction pair $\langle \succsim, \succ \rangle$ weakly orienting (AC-mark) and (AC-del). Whenever $s^{\#} \mathrel{\underset{(\sim)}{\succsim}} t^{\#}$ we have $s^{\sharp} \mathrel{\underset{(\sim)}{\succsim}} t^{\sharp}$, because of (AC-mark). Also ($\mathsf{a}^{\sharp}$) is at least weakly oriented due to (AC-mark), (AC-del), and the fact that $+^{\sharp}$ is assumed to be associative in their formulation. ◀

The converse does not hold if estimated dependency graph techniques are allowed.

▶ **Example 38.** Consider the TRS $\mathcal{R}$ with the following three rules:

$$(\mathsf{a}+\mathsf{a})+x \to (\mathsf{a}+\mathsf{b})+x \qquad \mathsf{b}+\mathsf{c} \to \mathsf{d}(\mathsf{a}+\mathsf{c}) \qquad \mathsf{c} \to \mathsf{a}+\mathsf{a}$$

where $+ \in \mathcal{F}_{\mathsf{A}}$. The dependency pairs of $\mathcal{R}$ in terms of [12] are

$$(\mathsf{a}+^\sharp \mathsf{a})+^\sharp x \to (\mathsf{a}+^\sharp \mathsf{b})+^\sharp x \quad (5) \qquad \mathsf{b}+^\sharp \mathsf{c} \to \mathsf{a}+^\sharp \mathsf{c} \quad (7) \qquad \mathsf{c}^\sharp \to \mathsf{a}+^\sharp \mathsf{a} \quad (9)$$
$$(\mathsf{a}+^\sharp \mathsf{a})+^\sharp x \to \mathsf{a}+^\sharp \mathsf{b} \quad (6) \qquad \mathsf{b}+^\sharp \mathsf{c} \to \mathsf{c}^\sharp \quad (8)$$

and the extended rules are

$$((\mathsf{a}+^\sharp \mathsf{a})+^\sharp x)+^\sharp z \to ((\mathsf{a}+^\sharp \mathsf{b})+^\sharp x)+^\sharp z \quad (10) \qquad (\mathsf{b}+^\sharp \mathsf{c})+^\sharp z \to \mathsf{d}(\mathsf{a}+\mathsf{c})+^\sharp z \quad (11)$$

A polynomial interpretation with $\mathsf{c}_\mathcal{A} = \mathsf{c}^\sharp_\mathcal{A} = 1$ and $\mathsf{a}_\mathcal{A} = 0$ would easily remove (9), and afterwards (8) does not constitute an SCC. The extended rule (11) would also be easy.

The remaining SCC is $\{(5),(6),(7),(10)\}$, where no AC-reduction pair technique we know can be applied. Note that in the Kusakari-Toyama formulation, the dependency graph has an edge from (5) to (7), and there is clearly one for the other direction.

In our formulation, the corresponding SCC (ignoring the nested marks) is $\{(5),(6),(7),(\mathsf{a}^\sharp),(\mathsf{A}^\sharp)\}$. From here $(\mathsf{a}^\sharp)$ can be removed, e.g., by a polynomial interpretation with $x +_\mathcal{A} y = x + y + 1$. Then (7) does not constitute an SCC, and remaining $\{(5),(6),(\mathsf{A}^\sharp)\}$ is easy. The AC-DP problem for the extended rules is also easy. Hence our formulation proves the termination of $\mathcal{R}/\mathcal{E}$, while the Kusakari-Toyama formulation fails.

Marché and Urbain [13] considered *flattened* terms, which would require some extra formalization work. However, their original version (with marks) was reported to be unsound [14], since it lacks a rule that corresponds to (AC-mark). They also do not impose a counterpart of our $(\mathsf{a}^\sharp)$ or (AC-del) of Kusakari-Toyama; the price for this omission is that the minimality of a chain becomes nontrivial to define [1]. It is indeed unclear, e.g., how to adapt the usable rules technique; cf. Example 25. Apart from that, their formulation takes $\mathsf{DP}(\mathcal{R} \cup \mathcal{R}_\mathcal{E})$, which is in general a superset of $\mathsf{DP}(\mathcal{R})$ and $\mathcal{R}^\sharp_\mathcal{E}$ and often imposes more dependency pairs.

Giesl and Kapur [6] considered a set $\mathcal{E}$ of more general equations than AC, and regarded $\mathsf{DP}(\mathsf{Inst}_\mathcal{E}(\mathcal{R} \cup \mathcal{R}_\mathcal{E}))$ together with $\mathcal{E}^\sharp$, that is, $(\mathsf{C}^\sharp)$ and $(\mathsf{A}^\sharp)$ for AC. They further suggested that for the AC case, taking $\mathsf{Inst}_\mathcal{E}$ is not necessary and hence $\mathsf{DP}(\mathcal{R} \cup \mathcal{R}_\mathcal{E})$ suffices. For this simplification, however, they remarked that the notion of minimality has to be modified [6].

Below we elaborate more on the problem, using an example inspired by Alarcón et al. [1].

▶ **Example 39.** Consider the TRS $\mathcal{R}$ consisting of the following three rules:

$$\mathsf{a}\cdot\mathsf{a} \to \mathsf{a}\cdot\mathsf{b}\cdot\mathsf{c} \qquad \mathsf{b}\cdot\mathsf{b} \to \mathsf{a}\cdot\mathsf{b}\cdot\mathsf{c} \qquad \mathsf{c}\cdot\mathsf{c} \to \mathsf{a}\cdot\mathsf{b}\cdot\mathsf{c}$$

where $\cdot \in \mathcal{F}_{\mathsf{AC}}$. We abbreviate $s \cdot t$ by st. Note that all subterms of abc, the common right-hand side in $\mathcal{R}$, are terminating. Hence, the only dependency pairs in $\mathsf{DP}(\mathcal{R} \cup \mathcal{R}_\mathcal{E})$ that constitute an infinite chain are

$$\mathsf{aa} \cdot^\sharp x \to \mathsf{abc} \cdot^\sharp x \qquad \mathsf{bb} \cdot^\sharp x \to \mathsf{abc} \cdot^\sharp x \qquad \mathsf{cc} \cdot^\sharp x \to \mathsf{abc} \cdot^\sharp x$$

The TRS $\mathcal{R}$ is not terminating modulo $\mathcal{E}$; there are essentially six (modulo $\mathcal{E}$) minimal nonterminating terms, namely aab, aac, abb, bbc, acc, and bcc. Take, e.g., $s = \mathsf{aab}$. Any infinite chain from $s^\sharp$ has the following prefix:

$$\mathsf{aa} \cdot^\sharp \mathsf{b} \to \mathsf{abc} \cdot^\sharp \mathsf{b} \underset{\mathcal{E} \cup \mathcal{E}^\sharp}{\approx} \mathsf{bb} \cdot^\sharp \mathsf{ac} \to \mathsf{abc} \cdot^\sharp \mathsf{ac}$$

From here, there are the following two ways to continue the chain:

- $\mathsf{abc} \cdot^\sharp \mathsf{ac} \underset{\mathcal{E}\cup\mathcal{E}^\sharp}{\approx} \mathsf{aa} \cdot^\sharp \mathsf{bcc} \to \mathsf{abc} \cdot^\sharp \mathsf{bcc} = t^\sharp$, or
- $\mathsf{abc} \cdot^\sharp \mathsf{ac} \underset{\mathcal{E}\cup\mathcal{E}^\sharp}{\approx} \mathsf{cc} \cdot^\sharp \mathsf{aab} \to \mathsf{abc} \cdot^\sharp \mathsf{aab} = u^\sharp$.

Both $t^\sharp$ and $u^\sharp$ contain a nonterminating subterm bcc or aab. Due to symmetry, we cannot have an infinite chain that satisfies the minimality in terms of Giesl and Kapur [6].

Alarcón et al. [1] proposed the notion of A$\vee$C theories, which we simply call AC theories in this paper. In order to define a suitable notion of minimality, they introduced the notion of *stable minimal* terms: any subterm of any AC-equivalent of such a term should be terminating. To maintain stable minimality in a chain, they re-invented (AC-del) without nested marks:

$$(x + y) +^\sharp z \to x +^\sharp y \tag{12}$$

Note that this is still different from our ($\mathsf{a}^\sharp$), and the difference is crucial: (12) is introduced to extract a nonterminating subterm in a chain (an instance of $x + y$) into the main component of the chain (as $x +^\sharp y$). Thus a chain must admit nonterminating subterms, which disallow adapting the proofs of usable rules.[2] Besides, they choose $\mathsf{DP}(\mathcal{R} \cup \mathcal{R}_\mathcal{E})$ and $\mathcal{E}^\sharp$ following Giesl and Kapur. As in Proposition 37, our approach subsumes also this formulation.

9 Conclusion

We have formalized an AC dependency pair framework for proving termination modulo AC axioms. We extended techniques such as dependency graph estimations, reduction pairs with usable rules, and the subterm criterion, for proving AC termination.

A formalization of reduction pairs like AC-RPO and experimental evaluation of the AC subterm criterion is left as future work. Moreover, a formalized algorithm for $\mathcal{E}$-unification would enable more precise estimations of dependency graphs and usable rules.

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