# Bar Recursion in Classical Realisability: Dependent Choice and Continuum Hypothesis

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**Abstract**

This paper is about the bar recursion operator in the context of classical realizability. The pioneering work of Berardi, Bezem, Coquand [1] was enhanced by Berger and Oliva [2]. Then Streicher [12] has shown, by means of their bar recursion operator, that the realizability models of ZF, obtained from usual models of $\lambda$-calculus (Scott domains, coherent spaces, ...), satisfy the axiom of dependent choice. We give a proof of this result, using the tools of classical realizability. Moreover, we show that these realizability models satisfy the well ordering of $\mathbb{R}$ and the continuum hypothesis. These formulas are therefore realized by closed $\lambda_c$-terms. This new result allows to obtain programs from proofs of arithmetical formulas using all these axioms.

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## 1 Introduction

This paper is about the bar recursion operator [11], in the context of classical realizability [8, 9]. It is a sequel to the three papers [1, 2, 12]. We use the definitions and notations of the theory of classical realizability as expounded in [7, 8, 9].

In [1], S. Berardi, M. Bezem and T. Coquand have shown that a form of the bar recursion operator can be used, in a proof-program correspondence, to interpret *the axiom of dependent choice* in proofs of $\Pi^0_2$-formulas of arithmetic. Their work was enhanced and adapted to the theory of domains by U. Berger and P. Oliva in [2]. In [12], T. Streicher has shown, by using the bar recursion operator of [2], that the models of ZF, associated with realizability algebras [7, 9] obtained from usual models of $\lambda$-calculus (Scott domains, coherent spaces, ...), satisfy the axiom of dependent choice. We give here a proof of this result, but for a realizability algebra which is built following the presentation of [1], which we call the BBC-algebra.

In Section 2, we define and study this algebra; we define also the bar recusion operator, which is a closed $\lambda$-term. In Sections 3 and 4, which are very similar, we show that this operator realizes the axiom of countable choice (CC), then the axiom of dependent choix (DC). The proof is a little simpler for CC. In Section 5, we deduce from this result, using results of [10] that, in the model of ZF associated with this realizability algebra, *every real (more generally, every sequence of ordinals) is constructible*. The formulas "$\mathbb{R}$ *is well ordered*" and "*Continuum Hypothesis*" are therefore realized in these models by a closed $\lambda_c$-term (i.e. a $\lambda$-term containing the control instruction cc of Felleisen-Griffin). We show also that *every true formula of analysis is realized by a closed $\lambda_c$-term*. Using these new results, we show how to obtain a program (closed $\lambda_c$-term) from any proof of an arithmetical formula in the theory ZF + "Dependent choice" + "Every real is constructible" (and therefore "Well ordering of $\mathbb{R}$" and "Continuum Hypothesis").


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**Remark.** In [8], we obtain programs from proofs in ZF + "Dependent choice", using another realizability algebra (the *thread model*). We cannot use Continuum Hypothesis in this context but, on the other hand, we are not limited to proofs of arithmetical formulas.

## 2 The BBC realizability algebra

The definition and general properties of a *realizability algebra* are given at the beginning of [7]. It consists of a set of *terms* $\Lambda$, a set of *stacks* $\Pi$ and a set of *processes* $\Lambda \star \Pi$. Every closed $\lambda$-term is interpreted as a term.

In this paper, we consider a particular realizability algebra $\mathcal{B} = (\mathbf{\Lambda}, \mathbf{\Pi}, \perp\!\!\!\perp)$, which we call the *BBC algebra* because it is a reformulation of the programming language of [1]. It is defined as follows:

- The set of *processes* $\mathbf{\Lambda} \star \mathbf{\Pi}$ is $\mathbf{\Lambda} \times \mathbf{\Pi}$.
- The set of *terms* $\mathbf{\Lambda}$ is the smallest set which contains the following *constants of term*: $\mathsf{B}$, $\mathsf{C}$, $\mathsf{I}$, $\mathsf{K}$, $\mathsf{W}$ *(Curry's combinators)*, $\mathsf{cc}$ *(Felleisen-Griffin instruction)*, $\mathsf{A}$ *(abort instruction)*, $\mathsf{p}, \mathsf{q}_0, \dots, \mathsf{q}_N$ *(variables)* where $N$ is a fixed integer; and is such that:

  if $\xi, \eta \in \mathbf{\Lambda}$ then $(\xi)\eta \in \mathbf{\Lambda}$ *(application)* ;

  with each sequence $\xi_i (i \in \mathbb{N})$ of closed elements of $\mathbf{\Lambda}$ (i.e. which contain no variable $\mathsf{p}, \mathsf{q}_0, \dots, \mathsf{q}_N$) is associated, in a one-to-one (and well founded) way, a constant of term denoted by $\bigwedge_i \xi_i$.
  Therefore, each term $\xi \in \mathbf{\Lambda}$ is a finite sequence of constants of term and parentheses.
  $\mathbf{\Lambda}$ is defined by an induction of length $\aleph_1$ and is of cardinality $2^{\aleph_0}$.

**Notations.** The application $(\dots((\xi_1)\xi_2)\dots)\xi_n$ will be often written $(\xi_1)\xi_2 \dots \xi_n$ or even $\xi_1 \xi_2 \dots \xi_n$. The finite sequence $\mathsf{q}_0, \dots, \mathsf{q}_N$ will be often written $\vec{\mathsf{q}}$.

- The set of *stacks* $\mathbf{\Pi}$ is defined as follows: a stack $\pi$ is a finite sequence $t_0 \centerdot \dots \centerdot t_{n-1} \centerdot \pi_0$ with $t_0, \dots, t_{n-1} \in \mathbf{\Lambda}$; it is terminated by the symbol $\pi_0$ which represents the *empty stack.*

For each stack $\pi$, the *continuation* $\mathsf{k}_\pi$ is a term which is defined by recurrence: $\mathsf{k}_{\pi_0} = \mathsf{A}$; $\mathsf{k}_{t \centerdot \pi} = \ell_t \, \mathsf{k}_\pi$, with $\ell_t = ((\mathsf{C})(\mathsf{B})\mathsf{CB})t$ or $\lambda k \lambda x(k)(x)t$.

Thus, if the stack $\pi$ is $t_0 \centerdot \dots \centerdot t_{n-1} \centerdot \pi_0$, we have:

$$\mathsf{k}_\pi = (\ell_{t_0}) \dots (\ell_{t_{n-1}})\mathsf{A} or \lambda x(\mathsf{A})(x)t_0 \dots t_{n-1} \,.$$

The *integer* $\underline{n}$ is defined as follows:

$$\underline{0} = (\mathsf{K})\mathsf{I} or \lambda x \lambda y \, y; \; \underline{n+1} = (\sigma)\underline{n} with \sigma = (\mathsf{BW})(\mathsf{C})(\mathsf{B})\mathsf{BB} or \lambda n \lambda f \lambda x(f)(n)fx \,.$$

The relation of *execution* $\succ$ is the least preorder on $\mathbf{\Lambda} \star \mathbf{\Pi}$ defined by the following rules (with $\xi, \eta, \zeta \in \mathbf{\Lambda}, \pi \in \mathbf{\Pi}$ and $n \in \mathbb{N}$):

1. $(\xi)\eta \star \pi \succ \xi \star \eta \centerdot \pi$; *(push)*
2. $\mathsf{B} \star \xi \centerdot \eta \centerdot \zeta \centerdot \pi \succ \xi \star (\eta)\zeta \centerdot \pi$; *(apply)*
3. $\mathsf{C} \star \xi \centerdot \eta \centerdot \zeta \centerdot \pi \succ \xi \star \zeta \centerdot \eta \centerdot \pi$; *(switch)*
4. $\mathsf{I} \star \xi \centerdot \pi \succ \xi \star \pi$; *(no operation)*
5. $\mathsf{K} \star \xi \centerdot \eta \centerdot \pi \succ \xi \star \pi$; *(delete)*
6. $\mathsf{W} \star \xi \centerdot \eta \centerdot \pi \succ \xi \star \eta \centerdot \eta \centerdot \pi$; *(copy)*
7. $\mathsf{cc} \star \xi \centerdot \pi \succ \xi \star \mathsf{k}_\pi \centerdot \pi$; *(save the stack)*
8. $\mathsf{A} \star \xi \centerdot \pi \succ \xi \star \pi_0$; *(abort)* or *(delete the stack)*
9. $\bigwedge_i \xi_i \star \underline{n} \centerdot \pi \succ \xi_n \star \pi$; *(oracle)*

There is no execution rule for $\mathsf{p}$ and $\mathsf{q}_i$ which are, therefore, *stop instructions*; $\mathsf{p}$ plays a special role because of the definition of $\perp\!\!\!\perp$ below.

When $\xi, \eta \in \mathbf{\Lambda}$, we set $\xi \succ \eta$ iff $(\forall \pi \in \mathbf{\Pi})(\xi \star \pi \succ \eta \star \pi)$.

- *Execution of processes.* For every process $\xi \star \pi$, at most one among the rules 1 to 9 applies. By iterating these rules, we obtain the *reduction* or the *execution* of the process $\xi \star \pi$. This execution stops if and only if the stack is insufficient (rules 2 to 8) or does not begin with an integer (rule 9) or else if the process has the form $\mathsf{p} \star \varpi$ or $\mathsf{q}_i \star \varpi$.
- *Definition of* $\perp\!\!\!\perp$. We set $\perp\!\!\!\perp = \{\xi \star \pi \in \mathbf{\Lambda} \star \mathbf{\Pi}\ ;\ (\exists \varpi \in \mathbf{\Pi})(\xi \star \pi \succ \mathsf{p} \star \varpi)\}$.
- *Proof-like terms.* Let $\mathsf{PL}_0$ be the countable set of terms built with the constants $\mathsf{B}$, $\mathsf{C}$, $\mathsf{I}$, $\mathsf{K}$, $\mathsf{W}$, $\mathsf{cc}$ and the application. It is the smallest possible set of *proof-like terms*. We shall also consider the set $\mathsf{PL}$ of *closed terms* (i.e. with no occurrence of $\mathsf{p}, \vec{\mathsf{q}}$) which is much bigger: it contains $\mathsf{A}$ and the oracles $\bigwedge_i \xi_i$ and is therefore of cardinality $2^{\aleph_0}$.

▶ **Lemma 1.** *$\mathcal{B}$ is a coherent realizability algebra.*

**Proof.** *$\mathcal{B}$ is a realizability algebra*: It remains to check that $\mathsf{k}_\pi \star \xi \,.\, \varpi \succ \xi \star \pi$, which is done by recurrence on $\pi$:

- if $\pi = \pi_0$, it is rule 8;
- if $\pi = t \,.\, \rho$ we have $\mathsf{k}_\pi \star \xi \,.\, \varpi = \mathsf{k}_{t\,.\,\rho} \star \xi \,.\, \varpi = \ell_t \mathsf{k}_\rho \star \xi \,.\, \varpi \succ (\mathsf{k}_\rho)(\xi)t \star \varpi \succ \mathsf{k}_\rho \star \xi t \,.\, \varpi \succ \xi t \star \rho$ (recurrence hypothesis) $\succ \xi \star t \,.\, \rho$.

*$\mathcal{B}$ is coherent*: If $\theta \in \mathsf{PL}$ then $\theta \star \pi_0 \notin \perp\!\!\!\perp$; indeed, $\mathsf{p}$ does not appear during the execution of $\theta \star \pi_0$. ◀

## Models and functionals

A coherent realizability algebra is useful in order to give *truth values* to formulas of ZF. In fact, we use a theory called $\mathrm{ZF}_\varepsilon$ [8] which is a conservative extension of ZF. This theory has an additional strong membership relation symbol $\varepsilon$ *which is not extensional*.

For each closed formula $F$ of $\mathrm{ZF}_\varepsilon$, we define *two truth values*, denoted $\|F\|$ and $|F|$, with $\|F\| \subset \mathbf{\Pi}$ and $|F| \subset \mathbf{\Lambda}$, with the relation $\xi \in |F| \Leftrightarrow (\forall \pi \in \|F\|)(\xi \star \pi \in \perp\!\!\!\perp)$. The relation $\xi \in |F|$ is also written $\xi \Vdash F$ and reads *"the term $\xi$ realizes the formula $F$".* All the necessary definitions are given in [7, 8, 9].

The following Lemma 2 is a useful property of the BBC realizability algebra $\mathcal{B}$.

▶ **Lemma 2.** *For all formulas $A, B$ of $ZF_\varepsilon$, and all terms $\xi \in \mathbf{\Lambda}$, we have:*

$$\xi \Vdash A \to B \text{ iff } (\forall \eta \in \mathbf{\Lambda})(\eta \Vdash A \Rightarrow \xi\eta \Vdash B)\,.$$

**Proof.** Indeed, by the general definition of $\Vdash$, we have:

$$(\xi \Vdash A \to B) \Leftrightarrow (\forall \eta \Vdash A)(\forall \pi \in \|B\|)(\xi \star \eta \,.\, \pi \in \perp\!\!\!\perp)\,.$$

Now, by the above definition of $\perp\!\!\!\perp$, it is clear that $(\xi \star \eta \,.\, \pi \in \perp\!\!\!\perp) \Leftrightarrow (\xi\eta \star \pi \in \perp\!\!\!\perp)$ from which the result follows. ◀

Classical realizability is an extension of forcing. As in forcing, we start with an ordinary model $\mathcal{M}$ of ZFC or even ZF + V = L (the Gödel constructibility axiom) which we call the *ground model*, and we build a *realizability model* $\mathcal{N}$ which satisfies $\mathrm{ZF}_\varepsilon$ in the following sense: $\mathcal{M}$ and $\mathcal{N}$ have the same domain, but neither the same language, nor the same truth values. The language of $\mathcal{N}$ has the additional binary symbol $\varepsilon$ of *strong membership*. The truth values of $\mathcal{N}$ are not $0, 1$ as for $\mathcal{M}$, but are taken in $\mathcal{P}(\mathbf{\Pi})$ endowed with a suitable structure of Boolean algebra [7, 9]. We say that $\mathcal{N}$ *satisfies a formula* $F$ iff there is a proof-like term $\theta$ which realizes $F$ or equivalently, if the truth value $\|F\|$ of $F$ is the unit of the Boolean algebra $\mathcal{P}(\mathbf{\Pi})$.

A *functional* on the ground model $\mathcal{M}$ is a formula $F(\vec{x},y)$ of ZF with parameters in $\mathcal{M}$, such that $\mathcal{M} \models \forall\vec{x}\,\exists!y\,F(\vec{x},y)$. Denoting such a functional by $f$, we write $y = f(\vec{x})$ for $F(\vec{x},y)$.

Since $\mathcal{M}$ and $\mathcal{N}$ have the same domain, all the functionals defined on $\mathcal{M}$ are also defined on $\mathcal{N}$ and *they satisfy the same equations* and even the same *Horn formulas* i.e. formulas of the form $\forall\vec{x}(f_1(\vec{x}) = g_1(\vec{x}), \ldots, f_n(\vec{x}) = g_n(\vec{x}) \to f(\vec{x}) = g(\vec{x}))$.

A particularly useful binary functional on $\mathcal{M}$ (and thus also on $\mathcal{N}$) is the *application*, denoted by app, which is defined as follows: $\mathrm{app}(f,x) = \{y\ ;\ (x,y) \in f\}$.

We shall often write $f[x]$ for $\mathrm{app}(f,x)$. This allows to consider each set in $\mathcal{M}$ (and in $\mathcal{N}$) as a unary functional.

▶ Remark. We can define a set $f$ in $\mathcal{M}$ by giving $f[x]$ for every $x$, provided that there exists a set $X$ such that $f[x] = \emptyset$ for all $x \notin X$: take $f = \bigcup_{x\in X}\{x\}\times f[x]$.

In the ground model $\mathcal{M}$, every function is defined in this way but in general, *this is false in* $\mathcal{N}$.

## Quantifiers restricted to $\mathbb{N}$

In [9], we defined the quantifier $\forall x^{\mathrm{int}}$, by setting:

$$\|\forall x^{\mathrm{int}} F[x]\| = \bigcup_{n\in\mathbb{N}} \|\{\underline{n}\} \to F[n]\}\| = \{\underline{n}\,.\,\pi\ ;\ n \in \mathbb{N}, \pi \in \|F[n]\|\}\,,$$

so that we have:

$$\xi \Vdash \forall x^{\mathrm{int}} F[x] \ \Leftrightarrow\ \xi\underline{n} \Vdash F[n] \text{ for all } n \in \mathbb{N}\,;$$

and it is shown that it is a correct definition of the restricted quantifier to $\mathbb{N}$.

Indeed the equivalence $\forall x^{\mathrm{int}} F[x] \leftrightarrow \forall x(\mathrm{int}[x] \to F[x])$ is realized by a closed $\lambda$-term independent of $F$, called a *storage operator*.

The formula $\mathrm{int}[x]$ is any formula of ZF which says that $x$ is an integer.

▶ **Theorem 3.** *If we take PL for the set of proof-like terms, and if the ground model $\mathcal{M}$ is transitive and countable, then there exists a countable realizability model $\mathcal{N}$ which has only standard integers, i.e. which is an $\omega$-model.*

**Proof.** Let $\mathcal{T}$ be the theory formed with closed formulas, with parameters in $\mathcal{M}$, which are realized by a proof-like term. $\mathcal{T}$ is $\omega$-complete: indeed, if $\theta_n \in \mathsf{PL}$ and $\theta_n \Vdash F[n]$ for $n \in \mathbb{N}$, let us set $c = \bigwedge_i \theta_i$. Then $c\underline{n} \Vdash F[n]$ for all $n \in \mathbb{N}$ and therefore $c \Vdash \forall n^{\mathrm{int}} F[n]$, i.e. $\forall n^{\mathrm{int}} F[n] \in \mathcal{T}$. It follows that $\mathcal{T}$ has a countable $\omega$-model. ◀

▶ **Proposition 4.** *Let $f : \mathbb{N} \to 2$ and $\theta \in PL$, $\theta \Vdash \exists n^{int}(f(n) = 1)$. Then $\theta\star\mathsf{p}\,.\,\pi_0 \succ \mathsf{p}\star\underline{n}\,.\,\varpi$ with $f(n) = 1$.*

**Proof.** There exists $\tau \in \mathbf{\Lambda}$ such that $\tau\underline{n} \succ \mathsf{p}$ if $f(n) = 1$ and $\tau\underline{n} \succ \mathsf{q}_0$ if $f(n) = 0$: set $\tau = \lambda x(\bigwedge_i \xi_i)x\,\mathsf{p}\mathsf{q}_0$ with $\xi_n = \mathsf{K}$ if $f(n) = 1$ and $\xi_n = \mathsf{KI}$ if $f(n) = 0$.

Then we have $\tau \Vdash \forall n^{\mathrm{int}}(f(n) \neq 1)$ and therefore $\theta\tau \Vdash \bot$. We necessarily have: $\theta\star\tau\,.\,\pi_0 \succ \tau\star\underline{n}\,.\,\pi$ for some $n$; furthermore, we have $\tau\underline{n} \succ \mathsf{p}$, otherwise we should have $\tau\underline{n} \succ \mathsf{q}_0$, and thus $\theta\star\tau\,.\,\pi_0 \notin \perp\!\!\!\perp$. Therefore $f(n) = 1$. ◀

▶ Remark. This shows that, from any proof-like term which realizes a given $\Sigma_1^0$ arithmetical formula, we obtain a program which computes an integer satisfying this formula. Such a realizer is given by any proof of this formula by means of axioms which have themselves such realizers.

The theory of classical realizability gives realizers for the axioms of ZF. We show below that the bar recursion operator realizes the axiom of dependent choice. Finally, in Section 5, we get (rather complicated) proof-like realizers for the axioms "$\mathbb{R}$ is well ordered" and "Continuum hypothesis".

## Execution of processes

**Notation.** If $\pi = t_0 \mathbin{.} \ldots \mathbin{.} t_{n-1} \mathbin{.} \pi_0$, we shall write $\pi \mathbin{.} t$ for $t_0 \mathbin{.} \ldots \mathbin{.} t_{n-1} \mathbin{.} t \mathbin{.} \pi_0$. Thus, we obtain $\mathsf{k}_{\pi \,.\, t}$ by replacing, in $\mathsf{k}_\pi$, the last occurrence of $\mathsf{A}$ by $\ell_t\mathsf{A}$.

▶ **Lemma 5.** *If $\xi \star \pi \in \perp\!\!\!\perp$, then $\xi' \star \pi' \in \perp\!\!\!\perp$ and $\xi' \star \pi' \mathbin{.} t \in \perp\!\!\!\perp$, where $\xi' \star \pi'$ is obtained by replacing, in $\xi \star \pi$, some occurrences of $\mathsf{A}$ by $(\ell_u)\mathsf{A} = \mathsf{k}_{u \,.\, \pi_0}$ and some occurrences of the variables $\mathsf{q}_0, \ldots, \mathsf{q}_N$ by $t_0, \ldots, t_N$; $t_0, \ldots, t_N, t, u$ are arbitrary terms.*

▶ Remark. In particular, it follows that $\xi \star \pi_0 \in \perp\!\!\!\perp \Rightarrow \xi \Vdash \perp$.

**Proof.** Proof by recurrence on the length of the execution of $\xi \star \pi \in \perp\!\!\!\perp$ by means of rules 1 to 9. We consider the last used rule. There are two non trivial cases:

- Rule 7 (execution of $\mathsf{cc}$); we must show $\mathsf{cc} \star \xi' \mathbin{.} \pi' \mathbin{.} t \in \perp\!\!\!\perp$.
  We apply the recurrence hypothesis to $\xi \star \mathsf{k}_\pi \mathbin{.} \pi$, in which we replace:
  - $\pi_0$ by $t \mathbin{.} \pi_0$ (thus $\pi$ becomes $\pi \mathbin{.} t$);
  - the last occurrence of $\mathsf{A}$ in $\mathsf{k}_\pi = (\ell_{t_0}) \ldots (\ell_{t_{n-1}})\mathsf{A}$ by $(\ell_t)\mathsf{A}$ (thus $\mathsf{k}_\pi$ becomes $\mathsf{k}_{\pi \,.\, t}$).

  Then, we make the substitutions in $\xi, \pi$, which gives $\xi' \star \mathsf{k}_{\pi' \,.\, t} \mathbin{.} \pi' \mathbin{.} t$.
- Rule 8 (execution of $\mathsf{A}$); we must show $(\ell_u)\mathsf{A} \star \xi' \mathbin{.} \pi' \mathbin{.} t \in \perp\!\!\!\perp$.
  We apply the recurrence hypothesis to $\xi \star \pi_0$, which gives $\xi' \star u \mathbin{.} \pi_0 \in \perp\!\!\!\perp$, thus $\xi' u \star \pi_0 \in \perp\!\!\!\perp$ and therefore $\mathsf{A} \star \xi' u \mathbin{.} \pi' \mathbin{.} t \in \perp\!\!\!\perp$ (rule 8); finally, we obtain $(\ell_u)\mathsf{A} \star \xi' \mathbin{.} \pi' \mathbin{.} t \in \perp\!\!\!\perp$. ◀

In each process $\xi \star \pi \in \perp\!\!\!\perp$, we define an occurrence of $\mathsf{p}$, which is called *efficient*, by recurrence on the length of its reduction. If $\xi = \mathsf{p}$, it is this very occurrence. Otherwise, we consider the first rule used in the reduction, and the definition is clear; for example, if it is rule 7, and if the efficient occurrence in $\xi \star \mathsf{k}_\pi \mathbin{.} \pi$ is in $\mathsf{k}_\pi$ or in $\pi$, then we take the corresponding occurrence in $\mathsf{cc} \star \xi \mathbin{.} \pi$.

▶ **Lemma 6.** *If $\xi \star \pi \in \perp\!\!\!\perp$, then:*

- *$\xi' \star \pi' \in \perp\!\!\!\perp$, where $\xi' \star \pi'$ is obtained by substituting arbitrary terms for the non efficient occurrences of $\mathsf{p}$.*
- *$\xi' \star \pi' \notin \perp\!\!\!\perp$ and indeed $\xi' \star \pi' \succ \mathsf{q}_0 \star \varpi$, where $\xi' \star \pi'$ is obtained by substituting $\mathsf{q}_0$ for the efficient occurrence of $\mathsf{p}$, and arbitrary terms for the non efficient occurrences of $\mathsf{p}$.*

**Proof.** The proof is immediate, by recurrence on the length of the reduction of $\xi \star \pi$ by means of rules 1 to 9: consider the last used rule. ◀

▶ **Corollary 7.** *If $\xi \Vdash \top, \perp \to \perp$ and $\xi \Vdash \perp, \top \to \perp$, then $\xi \Vdash \top, \top \to \perp$, and thus:*

$$\lambda x(x)II \Vdash \neg\forall x^{\beth 2}(x \neq 0, x \neq 1 \to \perp)$$

*and*

$$W \Vdash \forall x^{\beth 2}(\forall y^{\beth 2}(y \neq 0, y \neq x \to y \not\leq x), x \neq 0 \to \perp)\,.$$

▶ Remark. These two formulas express respectively that the Boolean algebra $\daleth 2$, which is defined in [9, 10], is non trivial and even atomless. Intuitively, $\daleth 2$ represents the *type of Booleans* of the realizability model $\mathcal{N}$. It is called the *characteristic Boolean algebra* of $\mathcal{N}$.

**Proof.** We apply Lemma 6 to $\xi \star \mathsf{p} \,.\, \mathsf{p} \,.\, \pi_0$. We have $\xi \star \mathsf{q}_0 \,.\, \mathsf{p} \,.\, \pi_0 \in \perp\!\!\!\perp$ and $\xi \star \mathsf{p} \,.\, \mathsf{q}_0 \,.\, \pi_0 \in \perp\!\!\!\perp$, which shows that the efficient occurrence of $\mathsf{p}$ is in $\xi$. Therefore $\xi \star t \,.\, u \,.\, \pi_0 \in \perp\!\!\!\perp$ for every $t, u \in \mathbf{\Lambda}$, again by Lemma 6.

The last two assertions follow from the fact that:

$$\|\forall x^{\daleth 2}(x \neq 0, x \neq 1 \to \bot)\| = \|\top, \bot \to \bot\| \cup \|\bot, \top \to \bot\|$$

and therefore:

$$|\forall x^{\daleth 2}(x \neq 0, x \neq 1 \to \bot)| = |\top, \top \to \bot| \,.$$ ◀

▶ **Theorem 8.** *For every sequence $\xi_i \in \mathbf{\Lambda}$ $(i \in \mathbb{N})$, there exists $\phi \in \mathbf{\Lambda}$ such that:*

- *$\phi \underline{i} \succ \xi_i$ for every $i \in \mathbb{N}$;*
- *for every $U \in \mathbf{\Lambda}$ such that $U\phi \Vdash \bot$, there exists $k \in \mathbb{N}$ such that $U\psi \Vdash \bot$ for every $\psi \in \mathbf{\Lambda}$ such that $\psi \underline{i} \succ \xi_i$ for every $i < k$.*

▶ Remark. Theorem 8 will be used in order to show properties of the bar recursion operator. In fact, the following weaker formulation is sufficient:

For every sequence $\xi_i \in \mathbf{\Lambda}$ $(i \in \mathbb{N})$ and every $U \in \mathbf{\Lambda}$ such that:

$$(\forall k \in \mathbb{N})(\exists \psi \in \mathbf{\Lambda})\{U\psi \not\Vdash \bot, (\forall i < k)(\psi \underline{i} \succ \xi_i)\}$$

there exists $\phi \in \mathbf{\Lambda}$ such that $U\phi \not\Vdash \bot$ and $(\forall i \in \mathbb{N})(\phi \underline{i} \succ \xi_i)$.

In the particular case of forcing, this is exactly the *decreasing chain condition*: every decreasing sequence of (non false) conditions has a lower bound (which is non false).

**Proof.** We set $\eta_i = \lambda \mathsf{p} \lambda \vec{\mathsf{q}}\, \xi_i$; thus, we have $\eta_i \in \mathsf{PL}$ and $\eta_i \mathsf{p} \vec{\mathsf{q}} \succ \xi_i$. Let $\eta = \bigwedge_i \eta_i$ and $\phi = \lambda x(\eta)x\,\mathsf{p}\vec{\mathsf{q}}$. Thus, we have $\eta \in \mathsf{PL}$ and $\phi \underline{i} \succ \xi_i$. We may assume that $\eta$ does not appear in $U$. We have $U\phi \Vdash \bot \Leftrightarrow U \star \phi \,.\, \pi_0 \in \perp\!\!\!\perp$ (Lemma 5). During the execution of the process $U \star \phi \,.\, \pi_0$, the constant $\eta$ arrives in head position a finite number of times, always through $\phi$ (since it is deleted each time it arrives in head position), therefore as follows:

$$\eta \star \underline{i} \,.\, \mathsf{p} \,.\, \vec{\mathsf{q}} \,.\, \pi \succ \xi_i \star \pi.$$

Let $k$ be an integer, greater than all the arguments of $\eta$ during this execution and let $\psi \in \mathbf{\Lambda}$ be such that $\psi \underline{i} \succ \xi_i$ for all $i < k$. Let us set $\tau = \lambda x \lambda \mathsf{p} \lambda \vec{\mathsf{q}}\; \psi x$; thus, we have $\tau \underline{i}\, \mathsf{p}\vec{\mathsf{q}} \succ \psi \underline{i} \succ \xi_i$ for $i < k$. In the process $U \star \phi \,.\, \pi_0$, let us replace the constant $\eta$ by the term $\tau$; we obtain $U \star \psi \,.\, \pi_0$. The execution is the same, and therefore $U \star \psi \,.\, \pi_0 \in \perp\!\!\!\perp$ and $U\psi \Vdash \bot$. ◀

## The bar recursion operator

We define below two *proof-like terms* $\chi$ and $\Psi$ (which are, in fact, closed $\lambda$-terms). In these definitions, the variables $i, k$ represent (intuitively) integers and the variable $f$ represents a function of domain $\mathbb{N}$, with arbitrary values in $\mathbf{\Lambda}$.

- We want a $\lambda$-term $\chi$ such that:

$$\chi \underline{k} f z \underline{i} \succ f \underline{i} \text{ if } i < k; \chi \underline{k} f z \underline{i} \succ z \text{ if } i \geq k \,.$$

Therefore, we set:

$$\chi = \lambda k \lambda f \lambda z \lambda i((i{<}k)(f)i)z$$

where the boolean $(i{<}k)$ is defined by:

$$(i{<}k) = ((kA)\lambda d\,\mathbf{0})(iA)\lambda d\,\mathbf{1}$$

with $\mathbf{0} = \lambda x \lambda y\, y$ or $\mathsf{KI}$, $\mathbf{1} = \lambda x \lambda y\, x$ or $\mathsf{K}$ and $A = \lambda x \lambda y\, yx$ or $\mathsf{CI}$.
The term $\chi \underline{k} f$ is a representation, in $\lambda$-calculus, of the finite sequence $(f\underline{0}, f\underline{1}, \ldots, f\underline{k-1})$.
We want a $\lambda$-term $\Psi$ such that:

$$\Psi g u \underline{k} f \succ (u)(\chi \underline{k} f)(g)\lambda z(\Psi g u \underline{k}^+)(\chi)\underline{k} f z$$

where $k^+ = ((\mathsf{BW})(\mathsf{C})(\mathsf{B})\mathsf{BB})k$ or $\lambda f \lambda x(f)(k)fx$ is the successor of the integer $k$.
Thus, we set:

$$\Psi = \lambda g \lambda u(\mathsf{Y})\lambda h \lambda k \lambda f(u)(\chi k f)(g)\lambda z(hk^+)(\chi)kfz,,$$

where $\mathsf{Y}$ is the Turing fix point operator:

$$\mathsf{Y} = XX \text{ with } X = \lambda x \lambda f(f)(x)xf = (\mathsf{W})(\mathsf{B})(\mathsf{BW})(\mathsf{C})\mathsf{B}\,.$$

The term $\Psi$ will be called the *bar recursion operator*.

## 3 Realizing countable choice

The *axiom of countable choice* is the following formula:

$$\text{(CC)} \qquad \forall n \exists x\, F[n,x] \to \exists f \forall n^{\text{int}} F[n, f[n]]$$

where $F[n,x]$ is an arbitrary formula of $\mathrm{ZF}_\varepsilon$(see [8]), with parameters and two free variables. The notation $f[n]$ stands for $\mathrm{app}(f,n)$ (the functional app has been defined above).

▶ Remark. This is a strong form of countable choice which shows that, in the realizability model $\mathcal{N}$, every countable sequence has the form $n \mapsto f[n]$ for some $f$. This will be used in Section 5.

▶ **Theorem 9.** $\lambda g \lambda u(\Psi)gu\,\underline{0}\,\underline{0} \Vdash \mathrm{CC}$.

**Proof.** The axiom of countable choice is therefore realized in the model of ZF associated with the BBC realizability algebra (in fact, it is sufficient that the realizability algebra satisfies the property formulated in the remark following Theorem 8). We write the axiom of countable choice as follows:

$$\text{(CC)} \qquad \forall n \neg \forall x \neg F[n,x], \forall f \neg \forall n^{\text{int}} F[n, f[n]] \to \bot\,.$$

Let $G, U \in \mathbf{\Lambda}$ be such that $G \Vdash \forall n \neg \forall x \neg F[n,x]$ and $U \Vdash \forall f \neg \forall n^{\text{int}} F[n, f[n]]$. We set $H = \Psi G U$ and we have to show that $H\underline{0}\,\underline{0} \Vdash \bot$. In fact, we shall show that $H\underline{0}\xi \Vdash \bot$ for every $\xi \in \mathbf{\Lambda}$.

▶ **Lemma 10.** *Let $k \in \mathbb{N}$ and $\phi \in \mathbf{\Lambda}$ be such that $(\forall i < k)\exists a_i(\phi \underline{i} \Vdash F[i, a_i])$. If $H\underline{k}\,\phi \nVdash \bot$, then there exist a set $a_k$ and a term $\zeta_{k,\,\phi} \in \mathbf{\Lambda}$ such that:*

$$\zeta_{k,\,\phi} \Vdash F[k, a_k] \quad \textit{and} \quad (H\underline{k}^+)(\chi)\underline{k}\,\phi\,\zeta_{k,\,\phi} \nVdash \bot\,.$$

**Proof.** Define $\eta_{k,\,\phi} = \lambda z(H\underline{k}^+)(\chi)\underline{k}\,\phi z$, so that $H\underline{k}\,\phi \succ (U)(\chi\underline{k}\,\phi)(G)\eta_{k,\,\phi}$.

If $\eta_{k,\,\phi} \Vdash \forall x \neg F[k,x]$ then, by hypothesis on $G$, we have $G\eta_{k,\,\phi} \Vdash \bot$. Let us check that:

$$(\chi\underline{k}\,\phi)(G)\eta_{k,\,\phi} \Vdash \forall n^{\mathrm{int}} F[n, f_k[n]]$$

where $f_k$ is defined by: $f_k[i] = a_i$ if $i < k$ (i.e. $i \in k$); $f_k[i] = \emptyset$ if $i \notin k$.

Indeed, if we set $\phi' = (\chi\underline{k}\phi)(G)\eta_{k,\,\phi}$, we have:

$$\phi'\underline{i} \succ \phi\underline{i} \Vdash F[i, a_i] \text{ for } i < k \text{ and } \phi'\underline{i} \succ (G)\eta_{k,\,\phi} \Vdash \bot$$

for $i \geq k$, and therefore $\phi'\underline{i} \Vdash F[i, \emptyset]$. By hypothesis on $U$, it follows that $(U)(\chi\underline{k}\,\phi)(G)\eta_{k,\,\phi} \Vdash \bot$, in other words $H\underline{k}\,\phi \Vdash \bot$. Thus, we have shown that, if $H\underline{k}\,\phi \nVdash \bot$, then $\eta_{k,\,\phi} \nVdash \forall x \neg F[k,x]$, which gives immediately the desired result. ◀

Let $\phi_0 \in \mathbf{\Lambda}$ be such that $H\underline{0}\phi_0 \nVdash \bot$. By means of Lemma 10, we define $\phi_{k+1} \in \mathbf{\Lambda}$ and $a_k$ recursively on $k$, by setting $\phi_{k+1} = \chi\underline{k}\,\phi_k\,\zeta_{k,\,\phi_k}$. By definition of $\chi$, we have $\phi_{k+1}\underline{i} \succ \zeta_{k,\phi_k}$ for $i \geq k$. Then, we show easily, by recurrence on $k$:

$$\phi_{k+1}\underline{i} \succ \phi_{i+1}\underline{i} \succ \zeta_{i,\phi_i} \Vdash F[i, a_i] \text{ for } i \leq k; H\underline{k}\phi_k \nVdash \bot\,.$$

Therefore, we can define:

a function $f$ of domain $\mathbb{N}$ such that $f[i] = a_i$ for every $i \in \mathbb{N}$;

and, by Theorem 8, a term $\phi \in \mathbf{\Lambda}$ such that $\phi\underline{k} \succ \zeta_{k,\,\phi_k}$ for all $k \in \mathbb{N}$. Therefore, we have $\phi\underline{i} \Vdash F[i, f[i]]$ for every $i \in \mathbb{N}$, that is to say $\phi \Vdash \forall n^{\mathrm{int}} F[n, f[n]]$. By hypothesis on $U$, it follows that $U\phi \Vdash \bot$. Therefore, by Theorem 8, applied to the sequence $\xi_i = \zeta_{i,\,\phi_i}$, there exists an integer $k$ such that $U\psi \Vdash \bot$, for every term $\psi \in \mathbf{\Lambda}$ such that $\psi\underline{i} \succ \zeta_{i,\,\phi_i}$ for $i < k$. hus, in particular, we have $(U)(\chi\underline{k}\,\phi_k)\xi \Vdash \bot$ for every $\xi \in \mathbf{\Lambda}$. Now, by definition of $H$, we have $H\underline{k}\,\phi_k \succ (U)(\chi\underline{k}\,\phi_k)\xi$ with $\xi = (G)\lambda z(H\underline{k}^+)(\chi)\underline{k}\,\phi_k z$, and therefore $H\underline{k}\,\phi_k \Vdash \bot$, that is a contradiction. Thus, we have shown that $H\underline{0}\phi_0 \Vdash \bot$ for every $\phi_0 \in \mathbf{\Lambda}$. ◀

## 4 Realizing dependent choice

The *axiom of dependent choice* is the following formula:

$$\text{(DC)} \qquad \forall x \exists y\, F[x,y] \to \exists f \forall n^{\mathrm{int}} F[f[n], f[n+1]]$$

where $F[x,y]$ is an arbitrary formula of $\mathrm{ZF}_\varepsilon$, with parameters and two free variables. The notation $f[n]$ stands for $\mathrm{app}(f,n)$ as defined above.

▶ **Theorem 11.** $\lambda g \lambda u(\Psi)gu\,\underline{0}\,\underline{0} \Vdash \mathrm{DC}$.

▶ Remark. The axiom of dependent choice is therefore realized in the model of ZF associated with the BBC realizability algebra (or, more generally, with any realizability algebra satisfying the property formulated in the remark after Theorem 8).

**Proof.** The proof of Theorem 11 is almost the same as Theorem 9. We write the *axiom of dependent choice* as follows:

$$\text{(DC)} \qquad \forall x\, \neg\forall y\, \neg F[x,y],\ \forall f \neg\forall n^{\mathrm{int}} F[f[n], f[n+1]] \to \bot.$$

Let $G, U \in \mathbf{\Lambda}$ be such that $G \Vdash \forall x\, \neg\forall y\, \neg F[x,y]$ and $U \Vdash \forall f \neg\forall n^{\mathrm{int}} F[f[n], f[n+1]]$. We set $H = \Psi GU$ and we have to show that $H\underline{0}\,\underline{0} \Vdash \bot$. In fact, we shall show that $H\underline{0}\xi \Vdash \bot$ for every $\xi \in \mathbf{\Lambda}$.

▶ **Lemma 12.** *Let $a_0, \dots, a_k$ be a finite sequence in $\mathcal{M}$ and $\phi \in \mathbf{\Lambda}$ be such that $(\forall i < k)(\phi\underline{i} \Vdash F[a_i, a_{i+1}])$. If $H\underline{k}\,\phi \not\Vdash \bot$, then there exist $\zeta \in \mathbf{\Lambda}$ and $a_{k+1}$ in $\mathcal{M}$ such that:*

$$\zeta \Vdash F[a_k, a_{k+1}] and (H\underline{k}^+)(\chi)\underline{k}\,\phi\,\zeta \not\Vdash \bot\,.$$

**Proof.** Define $\eta_{k,\,\phi} = \lambda z(H\underline{k}^+)(\chi)\underline{k}\,\phi z$, so that $H\underline{k}\,\phi \succ (U)(\chi\underline{k}\,\phi)(G)\eta_{k,\,\phi}$. If $\eta_{k,\,\phi} \Vdash \forall y\,\neg F[a_k, y]$ then, by hypothesis on $G$, we have $G\eta_{k,\,\phi} \Vdash \bot$. We check that:

$$(\chi\underline{k}\,\phi)(G)\eta_{k,\,\phi} \Vdash \forall n^{\mathrm{int}} F[f_k[n], f_k[n+1]]$$

where $f_k$ is defined by $f_k[i] = a_i$ for $i \le k$ (i.e. $i \in k+1$); $f_k[i] = \emptyset$ for $i \notin k+1$. Indeed, if we set $\phi' = (\chi k\phi)(G)\eta_{k,\,\phi}$, we have: $\phi'\underline{i} \succ \phi\underline{i} \Vdash F[a_i, a_{i+1}]$ for $i < k$ and $\phi'\underline{i} \succ (G)\eta_{k,\,\phi} \Vdash \bot$ for $i \ge k$. Therefore, we have $\phi'\underline{i} \Vdash F[f_k[i], f_k[i+1]]$ for every $i \in \mathbb{N}$. By hypothesis on $U$, it follows that $(U)(\chi\underline{k}\,\phi)(G)\eta_{k,\,\phi} \Vdash \bot$, that is $H\underline{k}\,\phi \Vdash \bot$. Thus, we have shown that, if $H\underline{k}\,\phi \not\Vdash \bot$, then $\eta_{k,\,\phi} \not\Vdash \forall y\,\neg F[a_k, y]$, which gives immediately the desired result. ◀

Let $\phi_0 \in \mathbf{\Lambda}$ be such that $H\underline{0}\phi_0 \not\Vdash \bot$ and let $a_0 = \emptyset$. Using Lemma 12, we define $\phi_{k+1} \in \mathbf{\Lambda}$ and $a_{k+1}$ in $\mathcal{M}$ recursively on $k$, by setting $\phi_{k+1} = \chi\underline{k}\,\phi_k\,\zeta_{k,\,\phi_k}$, where $\zeta_{k,\,\phi_k}$ is given by Lemma 12, where we set $\phi = \phi_k$. By definition of $\chi$, we have $\phi_{k+1}\underline{i} \succ \zeta_{k,\phi_k}$ for $i \ge k$.

Then, we show easily, by recurrence on $k$:

$$\phi_{k+1}\underline{i} \succ \phi_{i+1}\underline{i} \succ \zeta_{i,\phi_i} \Vdash F[a_i, a_{i+1}] \text{ for } i \le k; H\underline{k}\phi_k \not\Vdash \bot.$$

Therefore, we can define:

a function $f$ of domain $\mathbb{N}$ such that $f[i] = a_i$ for every $i \in \mathbb{N}$;

and, by means of Theorem 8, a term $\phi \in \mathbf{\Lambda}$ such that $\phi\underline{k} \succ \zeta_{k,\,\phi_k}$ for every $k \in \mathbb{N}$. Thus, we have $\phi\underline{i} \Vdash F[f[i], f[i+1]]$ for every $i \in \mathbb{N}$, that is to say $\phi \Vdash \forall n^{\mathrm{int}} F[f[n], f[n+1]]$.

By hypothesis on $U$, it follows that $U\phi \Vdash \bot$. Therefore, by Theorem 8, applied to the sequence $\xi_i = \zeta_{i,\,\phi_i}$, there exists an integer $k$ such that $U\psi \Vdash \bot$, for every term $\psi \in \mathbf{\Lambda}$ such that $\psi\underline{i} \succ \zeta_{i,\,\phi_i}$ for $i < k$. Thus, in particular, we have $(U)(\chi\underline{k}\,\phi_k)\xi \Vdash \bot$ for every $\xi \in \mathbf{\Lambda}$. But, by definition of $H$, we have $H\underline{k}\,\phi_k \succ (U)(\chi\underline{k}\,\phi_k)\xi$ with $\xi = (G)\lambda z(H\underline{k}^+)(\chi)\underline{k}\,\phi_k z$, and therefore $H\underline{k}\,\phi_k \Vdash \bot$, that is a contradiction.

Thus, we have shown that $H\underline{0}\phi_0 \Vdash \bot$ for every $\phi_0 \in \mathbf{\Lambda}$. ◀

## 5 Well ordering on R and continuum hypothesis

In this section, we use the notations and the results of [9] and [10]. If $F$ is a closed formula of $\mathrm{ZF}_\varepsilon$, the notation $\Vdash F$ means that there exists a proof-like term $\theta \in \mathsf{PL}_0$ (i.e. a closed $\lambda_c$-term) such that $\theta \Vdash F$. In Section 3, we have realized the axiom of countable choice (CC). We replace $F[n,x]$ with $\mathrm{int}(n) \to F[n,x]$ and we add a parameter $\phi$; we obtain:

$$\Vdash \forall\phi \left(\forall n^{\mathrm{int}} \exists x\, F[n,x,\phi] \to \exists f \forall n^{\mathrm{int}} F[n, f[n], \phi]\right)$$

for every formula $F[n,x,\phi]$ of $\mathrm{ZF}_\varepsilon$. In particular, taking $\phi\,\varepsilon\,2^{\mathbb{N}}$ and $F[n,x,\phi] \equiv (x = \phi(n)) \wedge (x = 0 \vee x = 1)$ (i.e. $(n,x)\,\varepsilon\,\phi \wedge (x = 0 \vee x = 1)$), we find:

$$\Vdash (\forall\phi\,\varepsilon\,2^{\mathbb{N}}) \exists f \forall n^{\mathrm{int}} \left((f[n] = \phi(n)) \wedge (f[n] = 0 \vee f[n] = 1)\right)\,.$$

For any set $f$ in the ground model $\mathcal{M}$, let $g = \{x\ ;\ f[x] = 1\}$. We have trivially $\mathsf{I} \Vdash \langle n \in g\rangle = \langle f[n] = 1\rangle$. [1] It follows that: $\Vdash \forall f \exists g \forall n\,((f[n] = 0 \lor f[n] = 1) \to f[n] = \langle n \in g\rangle)$. We have shown that: $\Vdash (\forall \phi\,\varepsilon\,2^{\mathbb{N}})\exists g \forall n^{\mathrm{int}}(\phi(n) = \langle n \in g\rangle)$.

Now, in [10], we have built an ultrafilter $\mathcal{D} : \daleth 2 \to 2$ on the Boolean algebra $\daleth 2$, with the following property: the model $\mathcal{N}$, equipped with the binary relations $\mathcal{D}(\langle x \in y\rangle)$, $\mathcal{D}(\langle x = y\rangle)$, is a model of ZF, denoted $\mathcal{M}_{\mathcal{D}}$, which is an elementary extension of the ground model $\mathcal{M}$. Moreover, $\mathcal{M}_{\mathcal{D}}$ is isomorphic to a transitive submodel of $\mathcal{N}$ (considered as a model of ZF), which contains every ordinal of $\mathcal{N}$. $\mathcal{M}_{\mathcal{D}}$ satisfies the axiom of choice, because we suppose that $\mathcal{M} \models$ ZFC. If we suppose that $\mathcal{M} \models$ V = L, then $\mathcal{M}_{\mathcal{D}}$ is isomorphic to the class $L^{\mathcal{N}}$ of constructible sets of $\mathcal{N}$. For every $\phi : \mathbb{N} \to 2$, we have obviously $\mathcal{D}(\phi(n)) = \phi(n)$. It follows that:

$$\Vdash (\forall \phi\,\varepsilon\,2^{\mathbb{N}})\exists g \forall n^{\mathrm{int}}\,(\phi(n) = \mathcal{D}\langle n \in g\rangle)\,.$$

This shows that the subset of $\mathbb{N}$ defined by $\phi$ is in the model $\mathcal{M}_{\mathcal{D}}$: indeed, it is the element $g$ of this model. We have just shown that $\mathcal{N}$ *and* $\mathcal{M}_{\mathcal{D}}$ *have the same reals*.

Therefore, $\mathbb{R}$ is well ordered in $\mathcal{N}$, and we have: $\Vdash$ ($\mathbb{R}$ *is well ordered*). Moreover, if the ground model $\mathcal{M}$ satisfies V = L, we have: $\Vdash$ (*every real is constructible*). Therefore, the continuum hypothesis is realized.

Since the models $\mathcal{N}$ and $\mathcal{M}_{\mathcal{D}}$ have the same reals, every formula of analysis (closed formula with quantifiers restricted to $\mathbb{N}$ or $\mathbb{R}$) has the same truth value in $\mathcal{M}_{\mathcal{D}}$, $\mathcal{M}$ or $\mathcal{N}$. It follows that:

*For every formula $F$ of analysis, we have $\mathcal{M} \models F$ if and only if $\Vdash F$.*

In particular, we have $\Vdash F$ or $\Vdash \neg F$.

References [7, 8, 9, 10] are available at `http://www.irif.univ-paris-diderot.fr/~krivine/`.

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[1] The notations $\daleth 2$ and $\langle F\rangle$ where $F$ is a closed formula of ZF, with parameters in the realizability model $\mathcal{N}$, are defined in [9, 10]. $\daleth 2$ is called the *characteristic Boolean algebra of* $\mathcal{N}$. We have $\langle F\rangle\,\varepsilon\,\daleth 2$.

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