Axiomatizations for Propositional and Modal Team Logic

Martin Lück

Leibniz Universität Hannover, Germany
lueck@thi.uni-hannover.de

Abstract

A framework is developed that extends Hilbert-style proof systems for propositional and modal logics to comprehend their team-based counterparts. The method is applied to classical propositional logic and the modal logic K. Complete axiomatizations for their team-based extensions, propositional team logic PTL and modal team logic MTL, are presented.

1998 ACM Subject Classification F.4.1 Mathematical Logic

Keywords and phrases team logic, propositional team logic, modal team logic, proof system, axiomatization

Digital Object Identifier 10.4230/LIPIcs.CSL.2016.33

1 Introduction

Propositional and modal logics, while their history goes back to ancient philosophers, have assumed an outstanding role in the age of modern computer science, with plentiful applications in software verification, modeling, artificial intelligence, and protocol design. An important property of a logical framework is *completeness*, i.e., that the act of mechanical reasoning can effectively be done by a computer.

A recent extension of classical logics is the generalization to *team semantics*, i.e., formulas are evaluated on whole sets of assignments. So-called *team based logics* allow a more sophisticated expression of facts that regard multiple states of a system simultaneously as well as their internal relationship towards each other. The concept of team logic originated from the idea of quantifier dependence and independence. The question was simple and is long-known in linguistics: How can the statement

For every x there is $y(x)$, and for every u there is $v(u)$ such that P(x,y,u,v)

be formalized? The fact that v should only depend on u cannot be expressed with first-order quantifiers. Some suggestions were the *independence-friendly logic* $\mathcal{IF}$ by Hintikka and Sandu [7] or the *dependence logic* $\mathcal{D}$ by Väänänen [15]. Hodges found that a compositional semantics of $\mathcal{IF}$ can be formulated with the concept of teams [8], which was adapted by Väänänen [14, 15] together with an *atom of dependence*, written $=(x, y)$ or $\mathsf{dep}(x, y)$.

Beside Väänänen's dependence atom a variety of atomic formulas solely for the reasoning in teams were introduced. Galliani and others found a connection to database theory; they defined common constraints like *independence* $\perp$, *inclusion* $\subseteq$ and *exclusion* $|$ in the framework of team semantics [2, 4]. Beside first-order logic, all these atoms were also adapted for modal logic $\mathcal{ML}$ [14] and recently propositional logic $\mathcal{PL}$ [16].

As for any logic system, the question of axiomatizability arose. After all, team logics enable reasoning about sets of valuations, and predicate logic with set quantifiers ($\mathcal{SO}$) is not axiomatizable. An important connection to team logic was found in the sense that dependence

25th EACSL Annual Conference on Computer Science Logic (CSL 2016).
Editors: Jean-Marc Talbot and Laurent Regnier; Article No. 33; pp. 33:1–33:18
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

logic $\mathcal{D}$ is as powerful as existential second-order logic $\mathcal{SO}(\exists)$ [15], and that its extension $\mathcal{TL}$ (where a semantical negation $\sim$ is provided) is even equivalent to full second-order logic $\mathcal{SO}$ [10]; therefore both are non-axiomatizable. Later Kontinen and Väänänen showed that there is a partial axiomatization in the sense that $\mathcal{FO}$ consequences of $\mathcal{D}$ formulas are derivable [11]. For many weaker team logics the question of axiomatizability is open. Exceptions are certain fragments of propositional and modal team logic. They were axiomatized by Sano and Virtema [13] and Yang [16], but these solutions rely on the absence of Boolean negation.

Contribution

In this paper complete axiomatizations of $\mathcal{PTL}$ and $\mathcal{MTL}$ are given, the full propositional and modal team logics. A crucial step in the completeness proof is the fact that $\mathcal{PTL}$ is not more expressive than a (team semantical) Boolean combination of classical $\mathcal{PL}$ formulas, in symbols $\mathcal{PTL} \equiv \mathcal{B}(\mathcal{PL})$. In the modal case analogously $\mathcal{MTL} \equiv \mathcal{B}(\mathcal{ML})$ holds. For a similar application to first-order logic see the technical report [12].

The paper is built as follows: After reminding the reader of several foundational definitions (Section 2), complete axiomatizations for the Boolean closures $\mathcal{B}(\mathcal{PL})$ and $\mathcal{B}(\mathcal{ML})$ are presented (Section 3). The collapses from $\mathcal{PTL}$ and $\mathcal{MTL}$ to $\mathcal{B}(\mathcal{PL})$ and $\mathcal{B}(\mathcal{ML})$ are then proven step-wise by axiomatizing the elimination of splitting (Section 4) and modalities (Section 5).

2 Preliminaries

If in the following A is a set, then $\mathfrak{P}(A)$ refers to its power set. The notation $[n]$ will be used for the set $\{1, \ldots, n\}$, assuming $n \in \mathbb{N}$.

We define a *logic* as a triple $\mathcal{L} = (\Phi, \mathfrak{A}, \vDash)$. The component Φ is a countable set consisting of finite words over some alphabet Σ, the so-called *formulas* of $\mathcal{L}$. The set $\mathfrak{A}$ contains possible *valuations* of formulas in Φ, and the binary relation $\vDash$ is the *truth* or *satisfaction relation* between $\mathfrak{A}$ and Φ. To distinguish between different satisfaction relations we sometimes write $\vDash_{\mathcal{L}}$. We use the same symbol for the *entailment relation*, $\varphi \vDash \psi$ meaning that $\forall A \in \mathfrak{A} : A \vDash \varphi$ implies $A \vDash \psi$. These relations are as usual generalized to sets, $A \vDash \Phi$ meaning $\forall \varphi \in \Phi : A \vDash \varphi$, and $\Phi \vDash \psi$ meaning that $\forall A \in \mathfrak{A} : A \vDash \Phi$ implies $A \vDash \psi$.

The reader is assumed to be familiar with the foundations of classical propositional and modal logics. We define classical propositional logic via a countable set $\mathcal{PS} := \{x_1, x_2, \ldots\}$ of atomic propositional statements and the connectives $\rightarrow$ and $\neg$. Truth $\top$ and falsum $\bot$ are defined as $(x_1 \rightarrow x_1)$ and $\neg\top$, respectively. On top of propositional logic, modal logic is defined with the additional unary modality $\Box$ with the standard Kripke semantics.

2.1 Team logics

Let $\mathcal{L}$ be a logic. We introduce two new operators to $\mathcal{L}$: The unary *strong negation* $\sim$ and the binary *material implication* $\multimap$ (under the assumption that they are not symbols in formulas of $\mathcal{L}$). The logic $\mathcal{B}(\mathcal{L})$ is the *Boolean closure* of $\mathcal{L}$ and is defined by the following grammar, where α stands for any $\mathcal{L}$-formula: $\varphi ::= \alpha \mid {\sim}\varphi \mid (\varphi \multimap \varphi)$. Note that in particular any atom of the logic $\mathcal{L}$ is an atom of $\mathcal{B}(\mathcal{L})$, but connecting $\mathcal{L}$-formulas with $\sim$ or $\multimap$ always yields formulas not in $\mathcal{L}$. We further use the symbol $⫫$ (*strong falsum*), $⫫ := {\sim}(\psi \multimap \psi)$, and the abbreviations $(\varphi \owedge \psi) := ({\sim}\varphi \multimap \psi)$, $(\varphi \ovee \psi) := {\sim}(\varphi \multimap {\sim}\psi)$ and $(\varphi \multimapboth \psi) := (\varphi \multimap \psi) \ovee (\psi \multimap \varphi)$. The semantics of $\mathcal{B}(\mathcal{L})$ extend $\mathcal{L}$ by, given some valuation $A \in \mathfrak{A}$ of $\mathcal{L}$, as follows: $A \vDash {\sim}\varphi \Leftrightarrow A \nvDash \varphi$ and $A \vDash \varphi \multimap \psi \Leftrightarrow A \nvDash \varphi$ or $A \vDash \psi$.

The next operator introduced in team logic is the binary operator $\multimap$, called *linear implication*, similar as in Väänänen's first-order team logic $\mathcal{TL}$ [15]. Assume that the logic $\mathcal{L} = (\Phi, \mathfrak{A}, \vDash)$ has a *splitting relation* $\sigma_{\mathcal{L}} \subseteq \mathfrak{A}^3$. If $(A, B, C) \in \sigma_{\mathcal{L}}$ then we say that (B, C) is a *splitting* or *division* of A. The semantics is that $A \vDash \varphi \multimap \psi$ if for all (B, C) with $(A, B, C) \in \sigma_{\mathcal{L}}$ it holds $B \nvDash \varphi$ or $C \vDash \psi$. Abbreviate $\varphi \otimes \psi := {\sim}(\varphi \multimap {\sim}\psi)$. If a logic $\mathcal{L}$ has a splitting relation, then the syntax of $\mathcal{S}(\mathcal{L})$ is the extension of $\mathcal{B}(\mathcal{L})$ by the grammar rule $\varphi ::= (\varphi \multimap \varphi)$.

Propositional team logic $\mathcal{PTL}$ is the logic of $\mathcal{S}(\mathcal{PL})$-formulas. A valuation of $\mathcal{PTL}$ is a *team* T which is a (possibly empty) set of propositional assignments $s\colon \mathcal{PS} \to \{0, 1\}$. If $\varphi \in \mathcal{PL}$ then $T \vDash \varphi$ if $s \vDash \varphi$ in $\mathcal{PL}$ semantics for all $s \in T$. A division of a team T is simply a pair (S, U) such that $S \cup U = T$.

Modal team logic $\mathcal{MTL}$ is the closure of $\mathcal{ML}$ under $\sim, \multimap\!\!\!\!\rightarrow, \multimap$ (as above) and the unary modalities $\Box$ and Δ. Abbreviate $\Diamond := {\sim}\Delta{\sim}$. In contrast to $\mathcal{ML}$, valuations are not pointed Kripke structures $(\mathcal{K}, w)$ but have the form $(\mathcal{K}, T)$, where $T \subseteq W$ is called a *team*. For $\varphi \in \mathcal{ML}$ it holds $(\mathcal{K}, T) \vDash \varphi$ if $(\mathcal{K}, w) \vDash \varphi$ in $\mathcal{ML}$ semantics for all $w \in T$. A division of $(\mathcal{K}, T)$ is a pair $((\mathcal{K}, S), (\mathcal{K}, U))$ such that $S \cup U = T$.

If (W, R, V) is a Kripke structure, then we define the *image* $R[T]$ of a team $T \subseteq W$ as $\{\, w \in W \mid \exists v \in T : vRw \,\}$ and the *pre-image* $R^{-1}[T]$ as $\{\, w \in W \mid \exists v \in T : wRv \,\}$. A *successor team* T' of T is a team such that $T' \subseteq R[T]$ and $T \subseteq R^{-1}[T']$. The semantics of $\Box$ and Δ is $(K, T) \vDash \Box\varphi$ if $(K, R[T]) \vDash \varphi$, and $(K, T) \vDash \Delta\varphi$ if for all successor teams T' of T it holds $(K, T') \vDash \varphi$.

In the following we drop parentheses according to the usual precedence rules; further we assume $\to$, $\multimap\!\!\!\!\rightarrow$ and $\multimap$ as right-associative and $\wedge, ⊘, \vee, ⦸, \otimes$ as left-associative.

Also we reserve the letters $\alpha, \beta, \gamma, \ldots$ for classical $\mathcal{PL}$, $\mathcal{ML}$ formulas; and we use $\varphi, \psi, \vartheta, \ldots$ for general $\mathcal{PTL}$ and $\mathcal{MTL}$ formulas.

2.2 Proof systems

Proof systems or *calculi* are connected to the so-called *Entscheidungsproblem*, the problem of algorithmically deciding if a given formula φ of a logic $\mathcal{L}$ is valid. Formally we define a proof system as a triple $\Omega = (\Xi, \Psi, I)$ where Ξ is a set of formulas, $\Psi \subseteq \Xi$ is a set of *axioms*, and $I \subseteq \mathfrak{P}(\Xi) \times \Xi$ is a set of *inference rules*. Ξ, Ψ and I are all countable and decidable.

An *Ω-proof* $\mathcal{P}$ from a given set of premises $\Phi \subseteq \Xi$ is a finite sequence $\mathcal{P} = (P_1, \ldots, P_n)$ of finite sets $P_i \subseteq \Xi$ such that $\xi \in P_i$ implies $\xi \in P_{i-1} \cup \Psi \cup \Phi$ or $I(P'_{i-1}, \xi)$ for some $P'_{i-1} \subseteq P_{i-1}$. We say that $\mathcal{P}$ *proves* or *derives* a formula φ from Φ if $\varphi \in P_n$ and $\mathcal{P}$ is an Ω-proof from Φ. We write $\Phi \vdash_\Omega \varphi$ if there is some Ω-proof that proves φ from Φ. If Ω is clear then we just write $\Phi \vdash \varphi$. If two formulas φ and φ' prove each other, i.e., $\{\varphi\} \vdash \varphi'$ and $\{\varphi'\} \vdash \varphi$, then we write $\varphi \dashv\vdash \varphi'$. For sets write $\Phi \dashv\vdash \Phi'$ if for every $\varphi \in \Phi$ it holds $\Phi' \vdash \varphi$, and for every $\varphi' \in \Phi'$ it holds $\Phi \vdash \varphi'$.

A calculus Ω is *sound* for a logic $\mathcal{L}$ if for $\Phi \subseteq \mathcal{L}$, $\varphi \in \mathcal{L}$ it holds that $\Phi \vdash_\Omega \varphi$ implies $\Phi \vDash_{\mathcal{L}} \varphi$, and it is *complete* if conversely $\Phi \vDash_{\mathcal{L}} \varphi$ implies $\Phi \vdash_\Omega \varphi$. We say Ω' is *stronger* than Ω, in symbols $\Omega' \succeq \Omega$, if $\Phi \vdash_\Omega \varphi$ implies $\Phi \vdash_{\Omega'} \varphi$. Clearly if Ω' is sound, then Ω is sound, and if Ω is complete, then Ω' is complete. The union of two systems Ω, Ω' is defined as component-wise union and just written $\Omega\Omega'$.

The proof systems presented in this article are based on classical Hilbert-style axiomatizations of propositional and modal logic, as depicted in Figure 1. The propositional system H^0 consists of the axiom schemas (A1)–(A3) and the inference rule (E$\to$) (*modus ponens*). The modal logic K, the weakest normal modal logic, is obtained by the system $\mathsf{H}^\Box$, which

(A1)	$\alpha \to (\beta \to \alpha)$
(A2)	$(\alpha \to (\beta \to \gamma)) \to (\alpha \to \beta) \to (\alpha \to \gamma)$
(A3)	$(\neg\alpha \to \neg\beta) \to (\beta \to \alpha)$
(K)	$\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$
(E$\to$)	$\dfrac{\alpha \qquad \alpha \to \beta}{\beta}$
(Nec)	$\dfrac{\alpha}{\Box\alpha}$ (α theorem)

Figure 1 Hilbert-style axiomatizations of $\mathcal{PL}$ and $\mathcal{ML}$.

consists of the axiom schemas (A1)–(A3) and the (K) axiom schema as well as the inference rules (E$\to$) and (Nec) (*necessitation*).

In Figure 1, *theorem* means that α was derived without assumptions. Indeed the deduction $\alpha \vdash \Box\alpha$ would not be valid otherwise.

We defined $\mathcal{PL}$ and $\mathcal{ML}$ in team semantics to be *flat*, i.e., to have the *flatness property*: A formula is satisfied by a team T in team semantics exactly when all of T's members satisfy it in classical semantics. In the following we emphasize this by referring to flat logics as $\mathcal{F}$. From the flatness property we can prove that it is unnecessary to distinguish between a classical and a team-semantical entailment relation.

▶ **Proposition 2.1.** *Let $\mathcal{F} \in \{\mathcal{PL}, \mathcal{ML}\}$, $\Gamma \subseteq \mathcal{F}, \alpha \in \mathcal{F}$. Then $\Gamma \vDash \alpha$ holds in team semantics if and only if it holds in classical semantics.*

Proof. We prove the $\mathcal{PL}$ case. "$\Rightarrow$" follows since assignments are just singleton teams. For "$\Leftarrow$" let $T \vDash \Gamma$. Then $\forall s \in T : s \vDash \Gamma$, hence $\forall s \in T : s \vDash \alpha$. By flatness again $T \vDash \alpha$. ◀

▶ **Corollary 2.2.** *In team semantics the calculi* H^0 *and* $\mathsf{H}^\Box$ *are sound and complete for $\mathcal{PL}$ and $\mathcal{ML}$.*

Other straightforward consequences of flatness are the following, where $\mathcal{F} \in \{\mathcal{PL}, \mathcal{ML}\}$:

▶ **Proposition 2.3** (Downward closure)**.** *If $A \vDash \alpha$ for a formula $\alpha \in \mathcal{F}$, then $A_1 \vDash \alpha$ and $A_2 \vDash \alpha$ for all divisions (A_1, A_2) of A.*

▶ **Proposition 2.4** (Union closure)**.** *If $A_1 \vDash \alpha$ and $A_2 \vDash \alpha$ for a formula $\alpha \in \mathcal{F}$, then $A \vDash \alpha$ for all A which have a division into (A_1, A_2).*

▶ **Proposition 2.5** (Flatness of $\otimes$)**.** *For all $\alpha, \beta \in \mathcal{F}$ it holds $A \vDash \alpha \vee \beta$ if and only if $A \vDash \alpha \otimes \beta$.*

Note that propositional and modal team logics are defined in literature using only literals like $p, \neg p$ as atoms. The classical operators $\neg, \vee$ and $\wedge$ (plus $\Box, \Diamond$ in the modal case) are then the primitive connectives (see e.g. Väänänen, Sano and Virtema [13, 14, 15]), where $\vee$ is written $\otimes$ here. With this approach, every team-logical formula with the flatness property is already syntactically identical with the corresponding classical formula.

The rationale behind deviating from this notation is twofold. First, embedding the classical logics with their semantics as its own "layer" in team logic allows to comfortably build onto their proof systems. Second, Hilbert-style proof systems contain introduction rules of the form $\alpha \vdash \Box\alpha$, where α is a tautology. We will show similar introduction rules for all team-logical operators, and such introduction rules for the existential form of the operators, i.e., $\otimes$ and $\Diamond$ instead of $\multimap$ and Δ, are simply unsound.

(L1)	$\varphi \multimap (\psi \multimap \varphi)$
(L2)	$(\varphi \multimap (\psi \multimap \vartheta)) \multimap (\varphi \multimap \psi) \multimap (\varphi \multimap \vartheta)$
(L3)	$(\sim\varphi \multimap \sim\psi) \multimap (\psi \multimap \varphi)$
(L4)	$(\alpha \to \beta) \multimap (\alpha \multimap \beta)$
(E$\multimap$)	$\dfrac{\varphi \qquad \varphi \multimap \psi}{\psi}$

Figure 2 Hilbert-style axiomatization L of $\mathcal{B}(\mathcal{F})$.

3 Axioms of the Boolean closure

We begin the development of a proof system for team logic with the operators $\multimap$ and $\sim$. They are purely truth-functional; hence we can only reason about Boolean combinations of classical formulas. The presented calculus for this is the system L (for *lifted propositional axioms*) shown in Figure 2. The system L corresponds to the usual propositional axioms, with exception of (L4) which relates the propositional and the material implication. In this section it is shown how any complete proof system for a logic $\mathcal{F}$ can be augmented with L to obtain a complete system for $\mathcal{B}(\mathcal{F})$.

For the team-logical material implication $\multimap$ a new *modus ponens* inference rule is introduced. While the systems H^0 and $\mathsf{H}^\Box$ can only be applied to classical formulas $\alpha, \beta, \gamma, \ldots$, i.e., where no team-logical operators occur, the axioms and rules in L are permitted for general team-logical formulas $\varphi, \psi, \vartheta, \ldots$.

The proof of completeness of L is based on a generalized deduction theorem. The thought behind this strategy is that the deduction theorem implies *Lindenbaum's lemma* which allows the construction of a maximal consistent set, the usual method for completeness proofs of propositional axioms. We begin with identifying a family of proof systems which guarantee a deduction theorem, extending the ideas of Hakli and Negri [5].

▶ **Definition 3.1.** Let $\Omega = (\Xi, \Psi, I)$ be a calculus. Say that a rule $(\{\xi_1, \ldots, \xi_k\}, \psi) \in I$ has *weakening* if $\{\, \varphi \multimap \xi_i \mid i \in [k] \,\} \vdash \varphi \multimap \psi$ for all $\varphi \in \Xi$.

In other words, every derivation using an inference rule can also be proven under arbitrary assumptions. Say that a calculus Ω has weakening if all inference rules have weakening.

▶ **Lemma 3.2.** *If $\Omega \succeq$ L and Ω has weakening then it has the deduction theorem: $\Phi \vdash (\varphi \multimap \psi)$ if and only if $\Phi \cup \{\varphi\} \vdash \psi$.*

Proof. The direction from left to right is clear as L has (E$\multimap$). From right to left we do an induction over the length n of a shortest proof of ψ. If $\psi \in \Phi$, $\psi = \varphi$, or if ψ is an axiom, then by (L1) and (E$\multimap$) $\Phi \vdash (\varphi \multimap \psi)$. For $n = 1$ these are the only cases. Let $n > 1$. Then ψ could be obtained by application of some inference rule $(\{\xi_1, \ldots, \xi_k\}, \psi)$. $\xi_1, \ldots, \xi_k$ all have a proof of length $\leq n - 1$ from $\Phi \cup \{\varphi\}$, so by induction hypothesis $\Phi \vdash \varphi \multimap \xi_i$ for $i \in [k]$. By weakening $\Phi \vdash \varphi \multimap \psi$. ◀

▶ **Lemma 3.3** (Deduction theorem of L)**.** *If $\Omega \succeq$ L and all inference rules of Ω except (E$\to$) and (E$\multimap$) yield only theorems (i.e., formulas provable without assumptions), then Ω has the deduction theorem.*

Proof. By the preceding lemma we can instead show that Ω has weakening. If a formula ψ produced by an inference rule is always a theorem, i.e., provable without assumptions, then by (L1) we can trivially prove $\xi \multimap \psi$ for any ξ. Consider (E$\multimap$). From the assumptions $\xi \multimap (\varphi \multimap \psi)$ and $\xi \multimap \varphi$ just derive $\xi \multimap \psi$ by (L2). For (E$\rightarrow$), i.e., $(\{\alpha, \alpha \rightarrow \beta\}, \beta)$, weakening is shown as follows: $\theta := \xi \multimap (\alpha \rightarrow \beta) \multimap (\alpha \multimap \beta)$ is a theorem due to (L4) and (L1). Apply (L2) twice on the formulas $\xi \multimap (\alpha \rightarrow \beta)$, θ and $\xi \multimap \alpha$ to obtain $\xi \multimap \beta$. Hence all rules have weakening. ◀

3.1 Completeness of the Boolean closure

The typical textbook proof of completeness of propositional or first-order logic uses Lindenbaum's lemma to construct a maximal consistent set. For this we need the notion of inconsistency.

▶ **Definition 3.4.** Let $\Omega = (\Xi, \Psi, I)$ be a proof system. A set Φ is *Ω-inconsistent* (or just *inconsistent*) if $\Phi \vdash \Xi$. Φ is *Ω-consistent* (or just *consistent*) if it is not Ω-inconsistent.

▶ **Lemma 3.5.** *Let $\Omega = (\Xi, \Psi, I), \Omega \succeq \mathsf{L}$. The following statements are equivalent:*
1. *$\Phi \vdash \varphi$ and $\Phi \vdash {\sim}\varphi$ for some φ,*
2. *Φ is inconsistent,*
3. *$\Phi \vdash \perp\!\!\!\perp$.*

Proof. For 1. $\Rightarrow$ 2. we have to show $\Phi \vdash \xi$ for all $\xi \in \Xi$. First $\Phi \vdash ({\sim}\xi \multimap {\sim}\varphi)$ follows from $\Phi \vdash {\sim}\varphi$, (L1) and (E$\multimap$); by (L3) and (E$\multimap$) then follows $\Phi \vdash (\varphi \multimap \xi)$, and again by (E$\multimap$) then $\Phi \vdash \xi$. 3. is a special case of 2. For 3. $\Rightarrow$ 1. it suffices to derive $(\psi \multimap \psi)$ by a textbook proof, since $\perp\!\!\!\perp := {\sim}(\psi \multimap \psi)$. ◀

▶ **Lemma 3.6** (Relative consistency)**.** *Let $\Omega \succeq \mathsf{L}$ have weakening and let Φ be consistent. Then $\Phi \nvdash \varphi$ implies that $\Phi \cup \{{\sim}\varphi\}$ is consistent, and $\Phi \vdash \varphi$ implies that $\Phi \cup \{\varphi\}$ is consistent.*

Proof. If $\Phi \nvdash \varphi$ and $\Phi \cup \{{\sim}\varphi\}$ was inconsistent, then $\Phi \cup \{{\sim}\varphi\} \vdash {\sim}\psi$ for any axiom ψ and thus by Lemma 3.2 $\Phi \vdash ({\sim}\varphi \multimap {\sim}\psi)$. By (L3) then $\Phi \vdash \psi, \psi \multimap \varphi$, so by (E$\multimap$) $\Phi \vdash \varphi$, contradiction.

If $\Phi \vdash \varphi$ and $\Phi \cup \{\varphi\}$ was inconsistent, then again $\Phi \vdash \varphi, \varphi \multimap \perp\!\!\!\perp$: contradiction to consistency of Φ and Lemma 3.5. ◀

▶ **Definition 3.7.** If $\Omega = (\Xi, \Psi, I)$ then $\Phi \subseteq \Xi$ is *maximal consistent* if it is consistent and contains ξ or ${\sim}\xi$ for every $\xi \in \Xi$.

▶ **Lemma 3.8** (Lindenbaum's Lemma)**.** *Let $\Omega = (\Xi, \Psi, I), \Omega \succeq \mathsf{L}$. If Ω has weakening, then every consistent set $\Phi \subseteq \Xi$ has a maximal consistent superset $\Phi^* \subseteq \Xi$.*

Proof. Straightforward by enumerating all formulas and applying Lemma 3.6: For every formula ξ, add either ξ or ${\sim}\xi$. See the appendix for details. ◀

The application of Lindenbaum's lemma is usually as follows: If a set Φ is maximal consistent, then there is a model fulfilling all its atomic formulas. By the maximality of Φ then one can inductively claim that also all Boolean combinations of atomic formulas in Φ are automatically fulfilled as well. The main work here is required for the induction basis — the model satisfying the atomic formulas. In our context, an "atom" is in fact any formula of the underlying classical logic, in this case $\mathcal{PL}$ or $\mathcal{ML}$. This additional complexity requires the next property as an additional step to completeness.

▶ **Definition 3.9.** Let $\mathcal{F}$ be a logic. $\mathcal{F}$ admits *counter-model merging* if it has the following property for arbitrary sets $\Gamma, \Delta \subseteq \mathcal{F}$: If for every $\delta \in \Delta$ there is a valuation falsifying δ and satisfying Γ, then there is a valuation falsifying all formulas in Δ and satisfying Γ.

▶ **Lemma 3.10.** *$\mathcal{PL}$ and $\mathcal{ML}$, under team semantics, admit counter-model merging.*

Proof. We prove only the $\mathcal{ML}$ case as $\mathcal{PL}$ works similar. Let $\Gamma, \Delta \subseteq \mathcal{ML}$. Assume for each $\delta \in \Delta$ a Kripke structure $(\mathcal{K}_\delta, T_\delta)$ that falsifies δ and satisfies Γ. Define $\mathfrak{K}$ as the disjoint union (see Goranko and Otto [3]) of all Kripke structures $\mathcal{K}_\delta$. Then $(\mathfrak{K}, T_\delta) \vDash \Gamma$, $(\mathfrak{K}, T_\delta) \nvDash \delta$ as $\mathcal{ML}$ is invariant under disjoint union of structures [3] and due to flatness of $\mathcal{ML}$. Define the team $\mathcal{T} := \bigcup_{\delta \in \Delta} T_\delta$. As $\mathcal{ML}$ is union closed (Proposition 2.4), $(\mathfrak{K}, \mathcal{T})$ satisfies Γ, and as it is downwards closed (Proposition 2.3), it falsifies each $\delta \in \Delta$. ◀

▶ **Definition 3.11.** A calculus Ω is *refutation complete* for $\mathcal{L}$ if for every unsatisfiable $\Phi \subseteq \mathcal{L}$ there is a φ s. t. $\Phi \vdash \varphi, \sim\varphi$.

Write $\sim\mathcal{F}$ for the fragment of $\mathcal{B}(\mathcal{F})$ restricted to the formulas $\{ \sim\varphi \mid \varphi \in \mathcal{F} \}$.

▶ **Lemma 3.12.** *If $\mathcal{F}$ has counter-model merging and Ω is complete for $\mathcal{F}$, then Ω is refutation complete for $\mathcal{F} \cup \sim\mathcal{F}$.*

Proof. Let a set $\Phi \subseteq \mathcal{F} \cup \sim\mathcal{F}$ be unsatisfiable. Abbreviate $\Gamma := \Phi \cap \mathcal{F}$ and $\Delta := \Phi \cap \sim\mathcal{F}$. It is not the case that $\Gamma \cup \{\sim\delta\}$ is satisfiable for every $\sim\delta \in \Delta$, because then Φ would be satisfiable by counter-model merging. Hence for some $\sim\delta \in \Delta$ the set $\Gamma \cup \{\sim\delta\}$ is unsatisfiable, i.e., $\Gamma \vDash \delta$. But then $\Gamma \vdash \delta$ due to the completeness of Ω for $\mathcal{F}$, so $\Phi \vdash \delta, \sim\delta$. ◀

Let us emphasize again the difference to classical logics, say, $\mathcal{PL}$: $\mathcal{PL}$ has $\mathcal{PS}$ as its atoms, and the analogously defined fragment $\mathcal{PS} \cup \neg\mathcal{PS}$ of $\mathcal{PL}$ is trivially "refutation complete": A set $\Gamma \subseteq \mathcal{PS} \cup \neg\mathcal{PS}$ is contradictory if and only if contains $p, \neg p$ for some proposition p, so there is nothing to do for a proof system. This is different for team logics.

After the atoms are handled correctly by the proof system (by refutation completeness of $\mathcal{F} \cup \sim\mathcal{F}$), the induction step goes just for classical logic, and then results in completeness of $\mathcal{B}(\mathcal{F})$.

▶ **Lemma 3.13.** *If $\Omega \succeq \mathsf{L}$ is refutation complete for $\mathcal{F} \cup \sim\mathcal{F}$ and has the deduction theorem, then Ω is refutation complete for $\mathcal{B}(\mathcal{F})$.*

Proof. We must show that every unsatisfiable $\Phi \subseteq \mathcal{B}(\mathcal{F})$ allows deriving φ and $\sim\varphi$ for some φ, or, equivalently due to Lemma 3.5, that it is inconsistent. We prove for contraposition that every consistent $\Phi \subseteq \mathcal{B}(\mathcal{F})$ has a model.

If Φ is consistent, then it has a maximal consistent superset Φ^* by Lemma 3.8. Certainly $\Phi^* \cap (\mathcal{F} \cup \sim\mathcal{F})$ is consistent as well, and by refutation completeness it has a model $\mathcal{A}$. We show that $\psi \in \Phi^* \Leftrightarrow \mathcal{A} \vDash \psi$ for all $\psi \in \mathcal{B}(\mathcal{F})$ (then Φ^* and in particular Φ is satisfiable).

The rest of the proof will be an induction over the length of ψ. Let $\psi \in \mathcal{F}$. If $\psi \in \Phi^*$, then $\mathcal{A} \vDash \psi$ by definition of $\mathcal{A}$. If $\psi \notin \Phi^*$, then $\sim\psi \in \Phi^*$ due to the maximality of Φ^*, so $\sim\psi \in \Phi^* \cap \sim\mathcal{F}$, and again $\mathcal{A} \vDash \sim\psi$ by the definition of $\mathcal{A}$, hence $\mathcal{A} \nvDash \psi$ by definition of $\sim$.

The induction step $\psi = \sim\vartheta$ is clear due to the consistency and maximality of Φ^*.

So let $\psi = \psi_1 \multimap \psi_2$. Assume $\psi \in \Phi^*$. Then either $\psi_1 \notin \Phi^*$ or $\sim\psi_2 \notin \Phi^*$, otherwise by modus ponens Φ^* is inconsistent. But then either $\mathcal{A} \nvDash \psi_1$ or $\mathcal{A} \vDash \psi_2$ by induction hypothesis, hence $\mathcal{A} \vDash \psi_1 \multimap \psi_2$. If $\psi \notin \Phi^*$, then $\sim\psi \in \Phi^*$. If now $\mathcal{A} \vDash \psi_2$, then $\psi_2 \in \Phi^*$ by induction hypothesis. From ψ_2 we can derive ψ via (L1). If $\mathcal{A} \vDash \sim\psi_1$, then $\sim\psi_1 \in \Phi^*$. From $\sim\psi_1$ we can infer $\sim\psi_2 \multimap \sim\psi_1$ again with (L1) and by contraposition (L3) we obtain the conditional ψ. But in both cases Φ would then be inconsistent, so $\mathcal{A} \vDash \psi_1$ and $\mathcal{A} \nvDash \psi_2$, hence $\mathcal{A} \nvDash \psi_1 \multimap \psi_2$. ◀

▶ **Theorem 3.14** (Completeness of L). *If $\Omega \succeq \mathsf{L}$ is refutation complete for $\mathcal{F} \cup \sim\mathcal{F}$ and has the deduction theorem, then it is complete for $\mathcal{B}(\mathcal{F})$.*

$(\alpha \otimes \beta) \leftrightarrow (\alpha \vee \beta)$	(F$\otimes$)	Flatness 1.
$\alpha \rightarrow (\varphi \multimap \alpha)$	(F$\multimap$)	Flatness 2.
$\varphi \rightarrow (\varphi \multimap \psi) \rightarrow (\vartheta \multimap \psi)$	(Lax)	Splitting is lax.
$(\varphi \multimap \psi \multimap \vartheta) \rightarrow (\psi \multimap \varphi \multimap \vartheta)$	(Ex$\multimap$)	Exchange of hypotheses.
$(\varphi \multimap \sim\psi) \rightarrow (\psi \multimap \sim\varphi)$	(C$\multimap$)	Contraposition.
$(\varphi \multimap (\psi \rightarrow \vartheta)) \rightarrow (\varphi \multimap \psi) \rightarrow (\varphi \multimap \vartheta)$	(Dis$\multimap$)	Distribution axiom
$\dfrac{\varphi}{\psi \multimap \varphi}$ (φ theorem)	(Nec$\multimap$)	"Necessitation"

Figure 3 The splitting axioms S.

Proof. Let $\Phi \subseteq \mathcal{B}(\mathcal{F})$ and $\varphi \in \mathcal{B}(\mathcal{F})$. We have to show that from $\Phi \vDash \varphi$ it follows $\Phi \vdash \varphi$. Assume for contraposition that $\Phi \nvdash \varphi$. Then Φ is consistent by definition, and due to Lemma 3.6 so is $\Phi \cup \{\sim\varphi\}$ as well. By Lemma 3.13 Ω is refutation complete for $\mathcal{B}(\mathcal{F})$. Hence the consistent set $\Phi \cup \{\sim\varphi\}$ must be satisfiable which implies $\Phi \nvDash \varphi$. ◀

▶ **Corollary 3.15.** $\mathsf{H}^0\mathsf{L}$ *is complete for* $\mathcal{B}(\mathcal{PL})$*.* $\mathsf{H}^\square\mathsf{L}$ *is complete for* $\mathcal{B}(\mathcal{ML})$*.*

Proof. Follows from Corollary 2.2, Lemma 3.3, 3.10 and 3.12 and Theorem 3.14. ◀

Independently it can be shown that the axioms L can already derive all important Boolean tautologies, like De Morgan's laws, commutative, distributive laws and associative laws [12].

4 The axioms of splitting

In the previous sections we considered classical logics in the setting of team semantics, and their closure under the Boolean operators of team logic. With these operators we can express in essence three facts: The existence of certain members in the team, the absence of other members in the team, and further Boolean combinations thereof.

An essential addition to team semantics is the previously introduced splitting disjunction $\otimes$, or sometimes *splitjunction* or *tensor*. The expression $\varphi \otimes \psi$ can be seen as a per-member decision for either φ, ψ, or, as in the classical disjunction, both. This is called *lax semantics*. In the *strict semantics* the two subteams of the division may not overlap; so the strict $\otimes$ is better seen as a member-wise "exclusive or". In this work we will only consider the lax semantics as defined in Section 2.

The non-truth-functional nature of splitting disjunction is an obstacle to axiomatizability; our strategy here is to consider it as a special type of (axiomatizable) modality.

It was shown by Yang [16] that $\mathcal{PTL}$ formulas are equivalent to $\sim$-free formulas except that the atom of non-emptiness ($\text{NE} := \sim\bot$) occurs. A little informally we can call this fragment $\mathcal{S}^+(\mathcal{PL} \cup \text{NE})$ here. Her argumentation for this equivalence is however model-theoretic and not syntactical. With the system S in Figure 3 we get a similar result for $\mathcal{B}(\mathcal{PL})$, but in a purely syntactical way:

▶ **Theorem 4.1.** *Every* $\mathcal{PTL}$ *formula is provably equivalent to a* $\mathcal{B}(\mathcal{PL})$ *formula.*

This result will be proven in this section. The completeness is then just a consequence of the completeness of $\mathcal{B}(\mathcal{PL})$, i.e., Corollary 3.15.

Note that the matter is not so easy for strict splitting semantics. For instance the formula $\text{NE} \oslash \sim(\text{NE} \otimes \text{NE})$ is true in strict semantics if and only if the team contains exactly one element.

▶ **Proposition 4.2.** *There is no finite set $\Phi \subseteq \mathcal{B}(\mathcal{PL})$ that is equivalent to* $\textsc{ne} \oslash \sim(\textsc{ne} \otimes \textsc{ne})$ *in strict semantics.*

Proof. Assume for the sake of contradiction that there was some finite $\Phi \subseteq \mathcal{B}(\mathcal{PL})$ as above. W.l.o.g. the variable x does not occur in Φ. Let $T \vDash \Phi$, then $T = \{s\}$ for some assignment s. It can be easily shown by induction over the length of formulas that $\{s_0^x, s_1^x\} \vDash \Phi$, where $s_c^x(x) = c$ and $s_c^x(y) = s(y)$ for $x \neq y$. ◀

One remark about the naming of the "necessitation" rule in Figure 3. This rule is similar to the rule used in modal logic. In the context of teams, a subteam can as well be seen as a type of "other world". We can, as typical for modal logics, derive no knowledge about ψ in a subteam from knowledge about ψ in the current team. Instead we can see team logics as logics with countable many modalities of the form "$\varphi \multimap$" and introduce corresponding necessitation and distribution rules.

▶ **Proposition 4.3.** *The proof system* $\mathsf{H}^0\mathsf{LS}$ *is sound for* $\mathcal{PTL}$.

Proof. The soundness of H^0 is clear as instances of its axioms may only be $\mathcal{PL}$ formulas and H^0 is sound for $\mathcal{PL}$. The soundness of L is clear as well. So consider the axioms and rules introduced in S: The soundness of $(\mathsf{F}\multimap)$ and $(\mathsf{F}\otimes)$ is due to downward closure (Proposition 2.3) and flatness (Proposition 2.5). (Lax) follows from the definition of splitting, and $(\mathsf{Ex}\multimap)$ and $(\mathsf{C}\multimap)$ can easily be proven by contradiction. The necessitation rule $(\mathsf{Nec}\multimap)$ and the distribution axiom $(\mathsf{Dis}\multimap)$ are as in modal logic, just with the pseudo-modality $\psi \multimap$. $(\mathsf{Nec}\multimap)$ is applied to φ only if φ is a theorem, i.e., $\vdash \varphi$. Its soundness follows therefore straightforwardly by induction over the proof length. ◀

4.1 Completeness of propositional team logic

The proof of Theorem 4.1, the collapse of propositional team logic to the Boolean closure, will be built on several lemmas and the following meta-rules.

▶ **Lemma 4.4.** *If a proof system Ω has the deduction theorem, then it admits the following meta-rules:*

- *If $\Omega \succeq \mathsf{L}$: Reductio ad absurdum* (RAA)*: $\Phi \cup \{\ \varphi\} \vdash \psi, \sim\psi \ \Rightarrow\ \Phi \vdash \sim\varphi$ and*
  $\Phi \cup \{\sim\varphi\} \vdash \psi, \sim\psi \ \Rightarrow\ \Phi \vdash \ \varphi$.
- *If $\Omega \succeq \mathsf{LS}$: Modus ponens in $\multimap$* $(\mathsf{MP}\multimap)$*: $\vdash \varphi \twoheadrightarrow \psi,\ \Phi \vdash \vartheta \multimap \varphi \ \Rightarrow\ \Phi \vdash \vartheta \multimap \psi$.*
- *If $\Omega \succeq \mathsf{LS}$: Modus ponens in $\otimes$* $(\mathsf{MP}\otimes)$*: $\vdash \varphi \twoheadrightarrow \psi,\ \Phi \vdash \vartheta \otimes \varphi \ \Rightarrow\ \Phi \vdash \vartheta \otimes \psi$.*

Proof. Only a few applications of the deduction theorem and the axioms are required, see the appendix for details. ◀

The next definition is required since our strategy of $\multimap$-elimination starts at the innermost subformulas. Equivalent subformulas can always be substituted in compositional semantics, but we still have to show that $\mathsf{H}^0\mathsf{LS}$ proves these substitutions as well.

▶ **Definition 4.5.** Let $\mathfrak{f}$ be an n-ary connective. Say that a proof system Ω has *substitution in* $\mathfrak{f}$ if $\varphi_i \dashv\vdash \psi_i$ f. a. $i \in [n]$ implies $\mathfrak{f}(\varphi_1, \ldots, \varphi_n) \dashv\vdash \mathfrak{f}(\psi_1, \ldots, \psi_n)$.

Note that due to symmetry it suffices to prove only $\mathfrak{f}(\varphi_1, \ldots, \varphi_n) \vdash \mathfrak{f}(\psi_1, \ldots, \psi_n)$ to show substitution in $\mathfrak{f}$.

▶ **Lemma 4.6.** *If $\Omega \succeq \mathsf{LS}$ has the deduction theorem, then it has substitution in $\sim$, $\twoheadrightarrow$ and $\multimap$.*

$(\varphi \otimes \psi) \leftrightarrow (\psi \otimes \varphi)$	$(\mathsf{Com}\otimes)$	Commutative law for $\otimes$
$((\varphi \otimes \psi) \otimes \vartheta) \leftrightarrow (\varphi \otimes (\psi \otimes \vartheta))$	$(\mathsf{Ass}\otimes)$	Associative law for $\otimes$
$\alpha \owedge (\varphi \otimes \psi) \leftrightarrow (\alpha \owedge \varphi) \otimes (\alpha \owedge \psi)$	$(\mathsf{D}\owedge\otimes)$	Distributive law for $\otimes$ and $\owedge$
$\varphi \otimes (\psi \ovee \vartheta) \leftrightarrow (\varphi \otimes \psi) \ovee (\varphi \otimes \vartheta)$	$(\mathsf{D}\ovee\otimes)$	Distributive law for $\otimes$ and $\ovee$
$(\varphi \otimes \psi) \owedge (\varphi \multimap \vartheta) \rightarrow (\varphi \otimes (\psi \owedge \vartheta))$	$(\mathsf{Aug}\otimes)$	
$(\mathsf{E}\alpha \otimes \varphi) \rightarrow \mathsf{E}\alpha$	$(\mathsf{Abs}\otimes)$	Absorption law of $\otimes$
$(\alpha \owedge \mathsf{E}\beta) \rightarrow \mathsf{E}(\alpha \wedge \beta)$	(JoinE)	
$(\varphi \otimes (\alpha \owedge \mathsf{E}\beta)) \leftrightarrow (\varphi \otimes \alpha) \owedge \mathsf{E}(\alpha \wedge \beta)$	$(\mathsf{IsolateE})$	

Figure 4 Auxiliary theorems S' of splitting.

Proof. Let $\varphi = \xi_1 \rightarrow \xi_2$, $\xi_1 \dashv\vdash \psi_1$ and $\xi_2 \dashv\vdash \psi_2$. Then $\{\psi_1, \varphi\} \vdash \xi_2$ in L and hence $\{\psi_1, \varphi\} \vdash \psi_2$. By the deduction theorem $\varphi \vdash \psi_1 \rightarrow \psi_2$. Let $\varphi = {\sim}\xi$ and $\xi \dashv\vdash \psi$. Obviously $\{\varphi, \psi\} \vdash \xi, {\sim}\xi$ in L. By Lemma 4.4 (RAA) is derivable, so we obtain $\varphi \vdash {\sim}\psi$. The $\multimap$ case is proven with applications of $(\mathsf{MP}\multimap)$ and $(\mathsf{C}\multimap)$. ◀

If α is a classical formula, in the following write $\mathsf{E}\alpha$ as an abbreviation for ${\sim}\neg\alpha$. The meaning of $\mathsf{E}\alpha$ is intuitively that at least one element in the current team satisfies α (in particular $\mathsf{E}\alpha$ implies NE).

▶ **Lemma 4.7** ([12]). *Let $\Omega \succeq \mathsf{H}^0\mathsf{LS}$ have the deduction theorem. Then Ω proves the theorems S' (see Figure 4).*

We formally describe the translation from $\mathcal{PTL}$ to $\mathcal{B}(\mathcal{PL})$ as $\multimap$-*elimination.* The proof is a step-wise translation with the help of the following lemmas. We implicitly use Lemma 4.6 which permits derivations applied to subformulas inside $\rightarrow$, $\sim$ and $\multimap$, as well as the meta-rules in Lemma 4.4 and the theorems in Lemma 4.7.

▶ **Lemma 4.8.** *If Ω has the deduction theorem and $\Omega \succeq \mathsf{H}^0\mathsf{LS}$, then the following formulas are theorems of Ω:*

$$\bigowedge_{i=1}^{n} \mathsf{E}\beta_i \leftrightarrow \bigotimes_{i=1}^{n} \mathsf{E}\beta_i \tag{1}$$

$$\alpha \owedge \left(\bigowedge_{i=1}^{n} \mathsf{E}\beta_i\right) \leftrightarrow \bigotimes_{i=1}^{n} (\alpha \owedge \mathsf{E}\beta_i) \tag{2}$$

$$\bigotimes_{i=1}^{n} (\alpha_i \owedge \mathsf{E}\beta_i) \leftrightarrow \left(\bigotimes_{i=1}^{n} \alpha_i\right) \owedge \bigowedge_{i=1}^{n} \mathsf{E}(\alpha_i \wedge \beta_i) \tag{3}$$

Proof. The proof of $\rightarrow$ for (1) is by induction over n. $n = 1$ is clear, so let $n > 1$. By induction hypothesis we can assume that $\bigotimes_{i=1}^{n-1} \mathsf{E}\beta_i$ and $\mathsf{E}\beta_n$ are derivable from $\bigowedge_{i=1}^{n} \mathsf{E}\beta_i$. For sake of contradiction take $\bigotimes_{i=1}^{n-1} \mathsf{E}\beta_i \multimap {\sim}\mathsf{E}\beta_n$ as assumption; by (Lax) derive $(\top \multimap {\sim}\mathsf{E}\beta_n)$, by $(\mathsf{C}\multimap)$ then $(\mathsf{E}\beta_n \multimap {\sim}\top)$ and by (Lax) $(\top \multimap {\sim}\top)$. But certainly $\top \vee \top$ is a theorem of $\mathsf{H}^0\mathsf{LS}$ and hence $\top \otimes \top$ due to $\mathsf{F}\otimes$. This is a contradiction, so (RAA) yields ${\sim}(\bigotimes_{i=1}^{n} \mathsf{E}\beta_i \multimap {\sim}\mathsf{E}\beta_n)$, i.e., $\bigotimes_{i=1}^{n} \mathsf{E}\beta_i$. The theorems $(\mathsf{Abs}\otimes)$, $(\mathsf{Ass}\otimes)$ and $(\mathsf{Com}\otimes)$ are used to derive each conjunct for $\leftarrow$ (1).

For (2) first apply (1) to substitute $\bigowedge_{i=1}^{n} \mathsf{E}\beta_i$, then distribute α with repeated application of $(\mathsf{D}\owedge\otimes)$, $(\mathsf{Ass}\otimes)$ and $(\mathsf{Com}\otimes)$. The reverse direction is possible as both steps are symmetric.

Consider (3). From $\bigotimes_{i=1}^{n}(\alpha_i \owedge \mathsf{E}\beta_i)$ we obtain $\bigotimes_{i=1}^{n}\alpha_i$ by (Ass$\otimes$), (Com$\otimes$) and (MP$\otimes$) as $(\alpha_i \owedge \mathsf{E}\beta_i) \rightarrowtriangle \alpha_i$ for all $i \in [n]$. Apply (JoinE) to also derive $\bigotimes_{i=1}^{n}\mathsf{E}(\alpha_i \wedge \beta_i)$ the same way, and by (1) then $\bigowedge_{i=1}^{n}\mathsf{E}(\alpha_i \wedge \beta_i)$.

For the other implication we repeatedly apply the theorem (IsolateE), i.e., $(\varphi \otimes \alpha) \owedge \mathsf{E}(\alpha \wedge \beta) \rightarrowtriangle \varphi \otimes (\alpha \owedge \mathsf{E}\beta)$ as follows: Assume that the formula has the following form after k applications.

$$\left(\bigotimes_{i=1}^{k}(\alpha_i \owedge \mathsf{E}\beta_i) \otimes \bigotimes_{i=k+1}^{n}\alpha_i\right) \owedge \bigowedge_{i=k+1}^{n}\mathsf{E}(\alpha_i \wedge \beta_i).$$

For $k = 0$ this is indeed the case. With commutative and associative laws we isolate a single subformula on each side:

$$\left[\left(\bigotimes_{i=1}^{k}(\alpha_i \owedge \mathsf{E}\beta_i) \otimes \bigotimes_{i=k+2}^{n}\alpha_i\right) \otimes \alpha_{k+1}\right] \owedge \mathsf{E}(\alpha_{k+1} \wedge \beta_{k+1}) \owedge \bigowedge_{i=k+2}^{n}\mathsf{E}(\alpha_i \wedge \beta_i)$$

Then we apply the theorem on the left two conjuncts, resulting in

$$\left[\left(\bigotimes_{i=1}^{k}(\alpha_i \owedge \mathsf{E}\beta_i) \otimes \bigotimes_{i=k+2}^{n}\alpha_i\right) \otimes (\alpha_{k+1} \owedge \mathsf{E}\beta_{k+1})\right] \owedge \bigowedge_{i=k+2}^{n}\mathsf{E}(\alpha_i \wedge \beta_i)$$

and again with commutative and associative laws in

$$\left(\bigotimes_{i=1}^{k+1}(\alpha_i \owedge \mathsf{E}\beta_i) \otimes \bigotimes_{i=k+2}^{n}\alpha_i\right) \owedge \bigowedge_{i=k+2}^{n}\mathsf{E}(\alpha_i \wedge \beta_i)$$

so that we arrive at the same form again and repeat the steps. ◀

▶ **Lemma 4.9** (Flatness Properties). *If Ω has the deduction theorem and $\Omega \succeq \mathsf{H}^0\mathsf{LS}$, then the following formulas are theorems of Ω:*

$$\bigotimes_{i=1}^{n}\alpha_i \leftrightarrowtriangle \bigvee_{i=1}^{n}\alpha_i \qquad (4) \qquad\qquad \bigowedge_{i=1}^{n}\alpha_i \leftrightarrowtriangle \bigwedge_{i=1}^{n}\alpha_i \qquad (5)$$

Proof. (4): By induction over n. The case $n = 1$ is trivial. For $n > 1$ let $\varphi := \bigotimes_{i=1}^{n-1}\alpha_i$ and $\gamma := \bigvee_{i=1}^{n-1}\alpha_i$ be given as assumptions. By induction hypothesis $\varphi \dashv\vdash \gamma$. Then $\varphi \otimes \alpha_n \dashv\vdash \gamma \otimes \alpha_n$ by (Com$\otimes$) and (MP$\otimes$), and by (F$\otimes$) we obtain $\varphi \otimes \alpha_n \dashv\vdash \gamma \otimes \alpha_n \dashv\vdash \gamma \vee \alpha_n$ and hence (4).

(5): Again by induction, so let $\varphi := \bigowedge_{i=1}^{n-1}\alpha_i$ and $\gamma := \bigwedge_{i=1}^{n-1}\alpha_i$ be given such that $\varphi \dashv\vdash \gamma$. Then $\varphi \owedge \alpha_n \dashv\vdash \gamma \owedge \alpha_n$ in L. To obtain $\varphi \owedge \alpha_n \dashv\vdash \gamma \owedge \alpha_n \dashv\vdash \gamma \wedge \alpha_n$ we prove the general theorem $\alpha \wedge \beta \dashv\vdash \alpha \owedge \beta$ for classical α, β. Clearly $\alpha \wedge \beta$ proves α and β in H^0 and thus $\alpha \owedge \beta$ in $\mathsf{H}^0\mathsf{L}$. To prove $\alpha \wedge \beta$ from $\alpha \owedge \beta$ we use (RAA), i.e., assume $\alpha \owedge \beta$ and $\sim(\alpha \wedge \beta)$. It holds $(\alpha \wedge \beta) \dashv\vdash \neg(\alpha \to \neg\beta)$ in H^0, so we derive $\mathsf{E}(\alpha \to \neg\beta)$ from $\sim(\alpha \wedge \beta)$. Via L and the theorem (JoinE) we obtain in two steps first $\mathsf{E}(\alpha \wedge (\alpha \to \neg\beta))$ and then $\mathsf{E}(\beta \wedge \alpha \wedge (\alpha \to \neg\beta))$ which turns into $\mathsf{E}\bot$ in $\mathsf{H}^0\mathsf{L}$. But $\mathsf{E}\bot = \sim\neg\bot$ proves $\perp\!\!\!\perp$, contradiction. ◀

With the above lemmas we are ready to state the full translation from $\mathcal{PTL}$ to $\mathcal{B}(\mathcal{PL})$.

▶ **Definition 4.10.** Let $\mathcal{F}$ be a logic. Let $\Omega = (\Xi, \Psi, I)$ be a proof system. Let $\mathfrak{f}$ be a connective of arity n. We say that $\mathcal{B}(\mathcal{F})$ has $\mathfrak{f}$-*elimination in Ω* if the following holds for all formulas $\xi_1, \ldots, \xi_n \in \Xi$: If for all $i \in [n]$ there is $\xi_i' \in \mathcal{B}(\mathcal{F})$ s.t. $\xi_i \dashv\vdash \xi_i'$, then also $\mathfrak{f}(\xi_1, \ldots, \xi_n) \dashv\vdash \varphi$ for some $\varphi \in \mathcal{B}(\mathcal{F})$.

In other words: If $\xi_1, \ldots, \xi_n$ have provably equivalent $\mathcal{B}(\mathcal{F})$ formulas, then so has $\mathfrak{f}(\xi_1, \ldots, \xi_n)$.

▶ **Lemma 4.11** ($\multimap$-elimination)**.** *Let $\mathcal{F}$ be a logic closed under $\neg, \vee, \wedge$. Let Ω be a proof system with the deduction theorem s. t. $\Omega \succeq \mathsf{H}^0\mathsf{LS}$. Then $\mathcal{B}(\mathcal{F})$ has $\multimap$-elimination in Ω.*

Proof. Let $\varphi = \psi \multimap \vartheta$. Let $\psi', \vartheta' \in \mathcal{B}(\mathcal{F})$ such that $\psi \dashv\vdash \psi'$ and $\vartheta \dashv\vdash \vartheta'$. It holds $\vartheta' \dashv\vdash {\sim}{\sim}\vartheta'$ in L. Lemma 4.6 applies to Ω, so by substitution in $\multimap$ we can translate $\varphi := \psi \multimap \vartheta$ to $\psi' \multimap {\sim}{\sim}\vartheta'$, and therefore in L to ${\sim}{\sim}(\psi' \multimap {\sim}{\sim}\vartheta') = {\sim}(\psi' \otimes {\sim}\vartheta')$.

In the system L we can apply De Morgan's laws and distributive laws on both ψ' and ${\sim}\vartheta'$. We can therefore derive two formulas ψ'', ϑ'' in disjunctive normal form (DNF) over $\owedge, \ovee$; and obtain with substitution in $\sim$ and $\multimap$ the following equivalent form of φ, where all $\alpha, \beta, \gamma, \delta, \ldots \in \mathcal{F}$:

$$\sim\left[\bigovee_{i=1}^{n}\left(\bigowedge_{j=1}^{o_i}\alpha_{i,j} \owedge \bigowedge_{j=1}^{m_i}\mathsf{E}\beta_{i,j}\right) \otimes \bigovee_{i=1}^{n'}\left(\bigowedge_{j=1}^{o'_i}\alpha'_{i,j} \owedge \bigowedge_{j=1}^{m'_i}\mathsf{E}\beta'_{i,j}\right)\right]$$

That the negative literals can be represented with E prefix is due to H^0 proving $\neg\neg$ introduction, thus ${\sim}\beta \dashv\vdash {\sim}\neg\neg\beta = \mathsf{E}\neg\beta$ for any β. Apply the following derivation in $\mathsf{H}^0\mathsf{LS}$:

$$\begin{aligned}
&\text{Lemma 4.9 (5)} \dashv\vdash && \sim\left[\bigovee_{i=1}^{n}\left(\alpha_i \owedge \bigowedge_{j=1}^{m_i}\mathsf{E}\beta_{i,j}\right) \otimes \bigovee_{i=1}^{n'}\left(\alpha'_i \owedge \bigowedge_{j=1}^{m'_i}\mathsf{E}\beta'_{i,j}\right)\right] \quad (\alpha_i, \alpha'_i \in \mathcal{F})\\
&\begin{matrix}(\mathrm{D}\ovee\otimes)\\(\mathrm{Com}\otimes)\\(\mathrm{Ass}\otimes)\end{matrix} \dashv\vdash && \sim\bigovee_{\substack{1\le i\le n\\1\le i'\le n'}}\left(\alpha_i \owedge \bigowedge_{j=1}^{m_i}\mathsf{E}\beta_{i,j}\right) \otimes \left(\alpha'_{i'} \owedge \bigowedge_{j=1}^{m'_{i'}}\mathsf{E}\beta'_{i',j}\right)\\
&\text{Lemma 4.8 (2)} \dashv\vdash && \sim\bigovee_{\substack{1\le i\le n\\1\le i'\le n'}}\left(\bigotimes_{j=1}^{m_i}\left(\alpha_i \owedge \mathsf{E}\beta_{i,j}\right) \otimes \bigotimes_{j=1}^{m'_{i'}}\left(\alpha'_{i'} \owedge \mathsf{E}\beta'_{i',j}\right)\right)\\
&\text{(Renaming)} \dashv\vdash && \sim\bigovee_{i=1}^{\ell}\bigotimes_{j=1}^{k_i}\left(\gamma_{i,j} \owedge \mathsf{E}\delta_{i,j}\right)\\
&\text{Lemma 4.8 (3)} \dashv\vdash && \sim\bigovee_{i=1}^{\ell}\left(\bigotimes_{j=1}^{k_i}\gamma_{i,j} \owedge \bigowedge_{j=1}^{k_i}\mathsf{E}\left(\gamma_{i,j} \wedge \delta_{i,j}\right)\right)\\
&\text{Lemma 4.9 (4)} \dashv\vdash && \sim\bigovee_{i=1}^{\ell}\left(\bigvee_{j=1}^{k_i}\gamma_{i,j} \owedge \bigowedge_{j=1}^{k_i}\mathsf{E}\left(\gamma_{i,j} \wedge \delta_{i,j}\right)\right) \quad =: \varphi' \in \mathcal{B}(\mathcal{F}).
\end{aligned}$$

◀

▶ **Theorem 4.12.** *Every $\mathcal{PTL}$ formula is provably equivalent to a $\mathcal{B}(\mathcal{PL})$ formula in* $\mathsf{H}^0\mathsf{LS}$*.*

Proof. Let $\varphi \in \mathcal{PTL}$. We show derivability of an equivalent $\varphi' \in \mathcal{B}(\mathcal{PL})$ by induction over $|\varphi|$. So assume $\varphi \notin \mathcal{B}(\mathcal{PL})$. If $\varphi = \psi \rightarrowtriangle \vartheta$ or $\varphi = {\sim}\psi$ then we have only to apply the induction hypothesis to ψ and ϑ and substitute in the sense of Lemma 4.6. The remaining case is $\varphi = \psi \multimap \vartheta$ for which we apply the previous lemma. ◀

▶ **Lemma 4.13.** *Let $\mathcal{F}, \mathcal{L}$ be logics, $\mathcal{B}(\mathcal{F}) \subseteq \mathcal{L}$, s. t. every $\mathcal{L}$ formula is provably equivalent to a $\mathcal{B}(\mathcal{F})$ formula in Ω. If Ω is complete for $\mathcal{B}(\mathcal{F})$ and sound for $\mathcal{L}$, then Ω is complete for $\mathcal{L}$.*

Proof. Assume $\Phi \subseteq \mathcal{L}, \varphi \in \mathcal{L}$. For completeness we have to show that $\Phi \vDash \varphi$ implies $\Phi \vdash \varphi$ in Ω. By assumption every $\mathcal{L}$ formula is provably equivalent to a $\mathcal{B}(\mathcal{F})$ formula, so $\Phi \dashv\vdash \Phi'$ for some set $\Phi' \subseteq \mathcal{B}(\mathcal{F})$. Similar $\varphi \dashv\vdash \varphi'$ for some $\varphi' \in \mathcal{B}(\mathcal{F})$.

By soundness it holds $\Phi \equiv \Phi'$ and $\varphi \equiv \varphi'$, so $\Phi' \vDash \varphi'$ follows. By completeness of Ω for $\mathcal{B}(\mathcal{F})$ we have $\Phi' \vdash \varphi'$, and so $\Phi \vdash \Phi' \vdash \varphi' \vdash \varphi$. The lemma follows as $\vdash$ is transitive. ◀

▶ **Theorem 4.14.** *The proof system* $\mathsf{H}^0\mathsf{LS}$ *is sound and complete for* $\mathcal{PTL}$.

Proof. For soundness see Proposition 4.3. $\mathsf{H}^0\mathsf{L}$ is complete for $\mathcal{B}(\mathcal{PL})$ due to Corollary 3.15. By soundness, Theorem 4.12, and Lemma 4.13 we obtain completeness for $\mathcal{PTL}$. ◀

It follows that axiomatizations of $\mathcal{PTL}$-definable constraints, like dependence, independence and inclusion over Boolean relations, can automatically be found in the presented proof system. For instance the dependency atom $=(x, y)$ ("y is a function of x") can be written as $\top \multimap (=(x) \twoheadrightarrow =(y))$, where $=(p) := p \otimes \neg p$.

▶ **Example 4.15.** We give a proof of Armstrong's axiom of transitivity of functional dependence [1]. The axiom says the following: From $=(x, y)$ and $=(y, z)$ infer $=(x, z)$. A proof sketch follows.

A	$=(x, y)$	
B	$=(y, z)$	
1	$(=(x) \twoheadrightarrow =(y)) \twoheadrightarrow ((=(y) \twoheadrightarrow =(z)) \twoheadrightarrow (=(x) \twoheadrightarrow =(z)))$	L
2	$\top \multimap \big((=(x) \twoheadrightarrow =(y)) \twoheadrightarrow ((=(y) \twoheadrightarrow =(z)) \twoheadrightarrow (=(x) \twoheadrightarrow =(z)))\big)$	Nec$\multimap$ (1)
3	$\big(\top \multimap (=(x) \twoheadrightarrow =(y))\big) \twoheadrightarrow$ $\big(\top \multimap ((=(y) \twoheadrightarrow =(z)) \twoheadrightarrow (=(x) \twoheadrightarrow =(z)))\big)$	Dis$\multimap$ + E$\twoheadrightarrow$ (2)
4	$\top \multimap ((=(y) \twoheadrightarrow =(z)) \twoheadrightarrow (=(x) \twoheadrightarrow =(z))))$	E$\twoheadrightarrow$ (A + 3)
5	$\big(\top \multimap (=(y) \twoheadrightarrow =(z))\big) \twoheadrightarrow \big(\top \multimap (=(x) \twoheadrightarrow =(z))\big)$	Dis$\multimap$ + E$\twoheadrightarrow$ (4)
6	$\top \multimap (=(x) \twoheadrightarrow =(z))$	E$\twoheadrightarrow$ (B + 5)
	$=(x, z)$	

▶ **Example 4.16.** The formula $(\alpha \multimap \beta) \twoheadrightarrow \beta$ is valid for all $\alpha, \beta \in \mathcal{PL}$: α is satisfied by the empty team, and as for every team T there is a division into $\emptyset \cup T$, the team T should satisfy β. We sketch a proof in the system $\mathsf{H}^0\mathsf{LS}$: It holds $\mathsf{H}^0 \vdash \bot \rightarrow \alpha$, thus $\mathsf{H}^0\mathsf{LS} \vdash {\sim}\alpha \twoheadrightarrow {\sim}\bot$. From ${\sim}\alpha \twoheadrightarrow {\sim}\bot$ and $\alpha \multimap \beta$ it follows $\bot \multimap \beta$ by (C$\multimap$) and (MP$\multimap$), hence ${\sim}\beta \multimap {\sim}\bot$. To prove β now we assume ${\sim}\beta$ for (RAA). From ${\sim}\beta$ and ${\sim}\beta \multimap {\sim}\bot$ we obtain $\top \multimap {\sim}\bot$ by (Lax) which contradicts $\top \otimes \bot := {\sim}(\top \multimap {\sim}\bot)$, but $\top \otimes \bot$ follows from $\mathsf{H}^0 \vdash \top \vee \bot$ and (F$\otimes$).

5 Modal team logic

In modal team logic we have team-wide modalities to relate teams of worlds in Kripke structures to each other. As for the splitting operator, we axiomatize the modalities just enough so that we can eliminate them. Kontinen, Müller, Schnoor and Vollmer [9] proved that every $\mathcal{MTL}$ formula is equivalent to an $\mathcal{B}(\mathcal{ML})$ formula. But as with Yang's argument we improve this result by giving a purely syntactical derivation procedure which does not rely on model-theoretic aspects. Together with Corollary 3.15 this yields a complete axiomatization for $\mathcal{MTL}$, settling an open question of Kontinen et al. [9].

▶ **Proposition 5.1.** *The proof system* $\mathsf{H}^\Box\mathsf{LSM}$ *is sound for* $\mathcal{MTL}$.

Proof. For $\mathsf{H}^\Box$ see Corollary 2.2 and for LS see the proof of Proposition 4.3. The axioms and rules of M can be verified from the definition of $\mathcal{MTL}$, their soundness is shown in the appendix. ◀

$\Box{\sim}\varphi \leftrightarrowtriangle {\sim}\Box\varphi$	(Lin$\Box$)	$\Box$ is linear.
$\Diamond\alpha \leftrightarrowtriangle \neg\Box\neg\alpha$	(F$\Diamond$)	Flatness of $\Diamond$.
$\Diamond(\varphi \otimes \psi) \leftrightarrowtriangle \Diamond\varphi \otimes \Diamond\psi$	(D$\Diamond\otimes$)	$\Diamond$ distributes over splitting.
$\Box\alpha \rightarrowtriangle \Delta\alpha$	(E$\Box$)	Successors are subteams of image team.
$\Diamond\varphi \rightarrowtriangle (\Delta\psi \rightarrowtriangle \Box\psi)$	(I$\Box$)	Image is a successor if one exists.
$\Box(\varphi \rightarrowtriangle \psi) \rightarrowtriangle (\Box\varphi \rightarrowtriangle \Box\psi)$	(Dis$\Box$)	Distribution axiom
$\Delta(\varphi \rightarrowtriangle \psi) \rightarrowtriangle (\Delta\varphi \rightarrowtriangle \Delta\psi)$	(DisΔ)	Distribution axiom
$\dfrac{\varphi}{\Box\varphi}$ (φ theorem)	(Nec$\Box$)	Necessitation
$\dfrac{\varphi}{\Delta\varphi}$ (φ theorem)	(NecΔ)	Necessitation

Figure 5 The modal team logic axioms M.

$\Box(\varphi \rightarrowtriangle \psi) \leftrightarrowtriangle (\Box\varphi \rightarrowtriangle \Box\psi)$	(D$\Box\rightarrowtriangle$)	Distributive law for $\Box$ and $\rightarrowtriangle$
$\Diamond(\varphi \ovee \psi) \leftrightarrowtriangle (\Diamond\varphi \ovee \Diamond\psi)$	(D$\Diamond\ovee$)	Distributive law for $\Diamond$ and $\ovee$
$\Diamond(\alpha \owedge \mathsf{E}\beta) \leftrightarrowtriangle \Diamond\alpha \owedge \mathsf{E}\neg\Box\neg(\alpha \wedge \beta)$	($\Diamond$IsolateE)	

Figure 6 Auxiliary theorems M′ for modalities.

▶ **Lemma 5.2.** $\mathsf{H}^{\Box}\mathsf{LSM}$ *has substitution in* $\rightarrowtriangle, \sim, \multimap, \Box$ *and* Δ.

Proof. $\mathsf{H}^{\Box}\mathsf{LSM}$ has the deduction theorem due to Lemma 3.3, so the first three cases follow from Lemma 4.6. The cases $\varphi = \Box\xi$ and $\varphi = \Delta\xi$ are easily shown with (Nec$\Box$), (Dis$\Box$), (NecΔ) and (DisΔ). ◀

▶ **Lemma 5.3** ([12])**.** $\mathsf{H}^{\Box}\mathsf{LSM}$ *proves the theorems* M′ *(see Figure 6).*

▶ **Lemma 5.4.** $\mathcal{B}(\mathcal{ML})$ *has* $\Box$*-elimination in* $\mathsf{H}^{\Box}\mathsf{LM}$.

Proof. Transform the argument of $\Box$ applying Lemma 5.2, the rest follows immediately from the axiom (Lin$\Box$) as well as the distributive law (D$\Box\rightarrowtriangle$): Just push the $\Box$ inwards until it precedes only classical $\mathcal{ML}$ subformulas. ◀

▶ **Lemma 5.5.** $\mathcal{B}(\mathcal{ML})$ *has* Δ*-elimination in* $\mathsf{H}^{\Box}\mathsf{LSM}$.

Proof. Let $\Delta\varphi$ be given s. t. $\varphi \dashv\vdash \varphi'$ for $\varphi' \in \mathcal{B}(\mathcal{ML})$. With L and Lemma 5.2 we can prove $\Delta\varphi$ equivalent to ${\sim}{\sim}\Delta{\sim}{\sim}\varphi' = {\sim}\Diamond{\sim}\varphi'$. Again in L we can apply De Morgan's laws and distributive laws on ${\sim}\varphi'$ such that it is provably equivalent to a formula in DNF:

$$\bigovee_{i=1}^{n}\left(\bigowedge_{j=1}^{o_i}\alpha_{i,j} \owedge \bigowedge_{j=1}^{k_i}\mathsf{E}\beta_{i,j}\right)$$

Then $\Delta\varphi$ itself is provably equivalent to:

$$
\begin{aligned}
& \sim \Diamond \bigovee_{i=1}^{n} \left(\bigowedge_{j=1}^{o_i} \alpha_{i,j} \owedge \bigowedge_{j=1}^{k_i} \mathsf{E}\beta_{i,j} \right) \\
\text{Lemma 4.8, 4.9} \dashv\vdash \quad & \sim \Diamond \bigovee_{i=1}^{n} \bigotimes_{j=1}^{k_i} \left(\alpha_i \owedge \mathsf{E}\beta_{i,j} \right) && \text{where } \alpha_i \in \mathcal{ML} \\
(\mathsf{D}\Diamond\owedge), (\mathsf{D}\Diamond\otimes) \dashv\vdash \quad & \sim \bigovee_{i=1}^{n} \bigotimes_{j=1}^{k_i} \Diamond \left(\alpha_i \owedge \mathsf{E}\beta_{i,j} \right) \\
\text{Lemma 5.3} \dashv\vdash \quad & \sim \bigovee_{i=1}^{n} \bigotimes_{j=1}^{k_i} (\Diamond\alpha_i \owedge \mathsf{E}\neg\Box\neg(\alpha_i \wedge \beta_{i,j})) \\
(\mathsf{F}\Diamond) \dashv\vdash \quad & \sim \bigovee_{i=1}^{n} \bigotimes_{j=1}^{k_i} (\neg\Box\neg\alpha_i \owedge \mathsf{E}\neg\Box\neg(\alpha_i \wedge \beta_{i,j})) \\
(\text{Renaming}) \dashv\vdash \quad & \sim \bigovee_{i=1}^{n} \bigotimes_{j=1}^{k_i} (\mu_{i,j} \owedge \mathsf{E}\nu_{i,j}) && \text{where } \mu_{i,j}, \nu_{i,j} \in \mathcal{ML} \\
\text{Lemma 4.8, 4.9} \dashv\vdash \quad & \sim \bigovee_{i=1}^{\ell} \left(\bigvee_{j=1}^{k_i} \mu_{i,j} \owedge \bigowedge_{j=1}^{k_i} \mathsf{E}\left(\mu_{i,j} \wedge \nu_{i,j}\right) \right) \quad \in \mathcal{B}(\mathcal{ML}). \qquad \blacktriangleleft
\end{aligned}
$$

▶ **Theorem 5.6.** *Every $\mathcal{MTL}$ formula is provably equivalent to a $\mathcal{B}(\mathcal{ML})$ formula in* $\mathsf{H}^{\Box}\mathsf{LSM}$.

Proof. Proven by induction as in Theorem 4.12, applying Lemma 4.11, 5.4 and 5.5. ◀

▶ **Theorem 5.7.** *The proof system* $\mathsf{H}^{\Box}\mathsf{LSM}$ *is sound and complete for $\mathcal{MTL}$.*

Proof. For the soundness see Proposition 5.1. The completeness follows from soundness, Corollary 3.15, Lemma 4.13 and Theorem 5.6. ◀

6 Conclusion

The team-semantical extensions of propositional logic $\mathcal{PL}$ and modal logic $\mathcal{ML}$, i.e., $\mathcal{PTL}$ and $\mathcal{MTL}$, have been shown axiomatizable. Their property to collapse to the Boolean closures of their flat base logics, i.e., $\mathcal{B}(\mathcal{PL})$ and $\mathcal{B}(\mathcal{ML})$, allows a completeness proof for the given proof systems.

An important detail there is the use of lax semantics for the operators $\otimes$ and $\Diamond$. It is possible to use strict semantics, i.e., to define team divisions via partitions; and to choose exactly one successor of worlds for $\Diamond$ (see Hella et al. [6]). The semantics of $\otimes$ would then allow to *count* certain elements in the team. The strictness of $\Diamond$ can be chosen accordingly: It distributes over $\otimes$ if and only if both are strict or both are lax.

But even if we recover the distributive laws at this point — counting cannot be expressed in the $\mathcal{B}(\cdot)$ closure (see Proposition 4.2), so there can be no completeness proof based on full operator elimination as in the style of this paper. It is open how complete axiomatizations can be found for team logics strictly stronger than $\mathcal{B}(\cdot)$.

Acknowledgements. I thank Anselm Haak and Juha Kontinen for various comments, hints and for pointing out helpful references. I also thank the anonymous referees for their useful hints and corrections.

References

1 William Ward Armstrong. Dependency structures of data base relationships. In *IFIP Congress*, pages 580–583, 1974.

2 Pietro Galliani. Inclusion and exclusion dependencies in team semantics — On some logics of imperfect information. *Annals of Pure and Applied Logic*, 163(1):68–84, January 2012. doi:10.1016/j.apal.2011.08.005.

3 Valentin Goranko and Martin Otto. Model theory of modal logic. In Johan Van Benthem Patrick Blackburn and Frank Wolter, editors, *Handbook of Modal Logic*, volume 3 of *Studies in Logic and Practical Reasoning*, pages 249–329. Elsevier, 2007. doi:10.1016/S1570-2464(07)80008-5.

4 Erich Grädel and Jouko Väänänen. Dependence and independence. *Studia Logica*, 101(2):399–410, 2013.

5 Raul Hakli and Sara Negri. Does the deduction theorem fail for modal logic? *Synthese*, 187(3):849–867, 2012. doi:10.1007/s11229-011-9905-9.

6 Lauri Hella, Antti Kuusisto, Arne Meier, and Heribert Vollmer. Modal Inclusion Logic: Being Lax is Simpler than Being Strict. In Giuseppe F Italiano, Giovanni Pighizzini, and Donald T. Sannella, editors, *Mathematical Foundations of Computer Science 2015*, volume 9234, pages 281–292. Springer Berlin Heidelberg, Berlin, Heidelberg, 2015.

7 Jaakko Hintikka and Gabriel Sandu. Informational Independence as a Semantical Phenomenon. In *Studies in Logic and the Foundations of Mathematics*, volume 126, pages 571–589. Elsevier, 1989. doi:10.1016/S0049-237X(08)70066-1.

8 Wilfrid Hodges. Compositional semantics for a language of imperfect information. *Logic Journal of the IGPL*, 5(4):539–563, 1997.

9 Juha Kontinen, Julian-Steffen Müller, Henning Schnoor, and Heribert Vollmer. A van benthem theorem for modal team semantics. In *24th EACSL Annual Conference on Computer Science Logic, CSL 2015, September 7-10, 2015, Berlin, Germany*, pages 277–291, 2015. doi:10.4230/LIPIcs.CSL.2015.277.

10 Juha Kontinen and Ville Nurmi. Team Logic and Second-Order Logic. In *Logic, Language, Information and Computation*, volume 5514, pages 230–241. Springer Berlin Heidelberg, Berlin, Heidelberg, 2009.

11 Juha Kontinen and Jouko Väänänen. Axiomatizing first-order consequences in dependence logic. *Annals of Pure and Applied Logic*, 164(11):1101–1117, November 2013. doi:10.1016/j.apal.2013.05.006.

12 Martin Lück. The axioms of team logic. *CoRR*, abs/1510.08786, 2016. URL: http://arxiv.org/abs/1510.08786.

13 Katsuhiko Sano and Jonni Virtema. Axiomatizing Propositional Dependence Logics. In Stephan Kreutzer, editor, *24th EACSL Annual Conference on Computer Science Logic (CSL 2015)*, volume 41 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 292–307, Dagstuhl, Germany, 2015. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. doi:10.4230/LIPIcs.CSL.2015.292.

14 Jouko Väänänen. Modal dependence logic. *New perspectives on games and interaction*, 4:237–254, 2008.

15 Jouko Väänänen. *Dependence logic: a new approach to independence friendly logic*. Number 70 in London Mathematical Society student texts. Cambridge University Press, Cambridge ; New York, 2007.

16 Fan Yang. *On extensions and variants of dependence logic.* PhD thesis, University of Helsinki, 2014. URL: http://www.math.helsinki.fi/logic/people/fan.yang/dissertation_fyang.pdf.

A Appendix

Proof of Lemma 3.8 (Lindenbaum's Lemma). Let $\varPhi$ be consistent, $\varOmega = (\varXi, \varPsi, I)$. We can write the countable set $\varXi$ as $\varXi := \{\xi_1, \xi_2, \dots\}$.

Let $\varPhi_0 := \varPhi$, and for each $i \geq 1$ define $\varPhi_i$ as

$$\varPhi_i := \begin{cases} \varPhi_{i-1} \cup \{\xi_i\} & \text{if } \varPhi_{i-1} \vdash \{\xi_i\}, \\ \varPhi_{i-1} \cup \{\sim\xi_i\} & \text{otherwise.} \end{cases}$$

By Lemma 3.6 the consistency of $\varPhi_{i-1}$ implies that of $\varPhi_i$. Hence by induction all $\varPhi_n$ for $n \geq 0$ are consistent. Let $\varPhi^* := \bigcup_{n\geq 0} \varPhi_n$. $\varPhi^*$ is again consistent, otherwise it could prove $\perp\!\!\!\perp$ already from a finite set $\varPhi_n$ of assumptions, which would be a contradiction to the consistency of all $\varPhi_n$. $\varPhi^*$ is maximal by construction. ◀

Proof of Lemma 4.4. In the first case of (RAA) we have $\varPhi \vdash \varphi \multimap \sim\psi, \varphi \multimap \psi$ by the deduction theorem. L proves the Boolean tautologies $(\varphi \multimap \sim\psi) \multimap (\psi \multimap \sim\varphi)$ and $(\varphi \multimap \sim\varphi) \multimap \sim\varphi$, hence $\varPhi \vdash \sim\varphi$ by (E$\multimap$). The second case is proven with the theorem $\sim\sim\varphi \multimap \varphi$ of L. (MP$\multimap$) is just a shortcut for (Nec$\multimap$), (Dis$\multimap$) and (E$\multimap$). (MP$\otimes$) is proven as follows.

A	$\vdash \varphi \multimap \psi$	MP$\otimes$
B	$\{\vartheta \otimes \varphi\}$	
1	$\vdash \sim\psi \multimap \sim\varphi$	L (A)
2	$\vdash (\vartheta \multimap \sim\psi) \multimap (\vartheta \multimap \sim\varphi)$	Nec$\multimap$, Dis$\multimap$ (1)
3	$\sim(\vartheta \multimap \sim\varphi)$	Def. (B)
4	$\sim(\vartheta \multimap \sim\psi)$	L (2 + 3)
5	$\vartheta \otimes \psi$	Def. (4)
	$\vartheta \otimes \psi$	

◀

Proof of Proposition 5.1 (Soundness of H$^\Box$LSM). The system H$^\Box$ applies only to $\mathcal{ML}$ formulas and is hence sound by Corollary 2.2. The system L is easily confirmed sound, and the soundness of S is proven as in Proposition 4.3. So we prove only the axioms M.

(Lin$\Box$): Assume $(\mathcal{K}, T) \vDash \Box\sim\varphi$. Then the unique successor team $R[T]$ does not satisfy φ. So it is not the case that $(\mathcal{K}, R[T]) \vDash \varphi$, hence $(\mathcal{K}, T) \nvDash \Box\varphi$ by definition of $\Box$. The other direction is similar. The flatness axiom (F$\Diamond$) follows from the definition of a successor team. (E$\Box$), (I$\Box$), (Dis$\Box$), (DisΔ) are clear, and as well are (Nec$\Box$) and (NecΔ): If a formula φ is a theorem and hence holds in all teams, then it certainly holds for the image team of any team and all successor teams in general.

It remains to prove (D$\Diamond\otimes$).

"$\multimap$": Assume $\mathcal{K} = (W, R, V)$ and $(\mathcal{K}, T) \vDash \Diamond(\varphi \otimes \psi)$. Then T has a successor team T' which satisfies $\varphi \otimes \psi$, i.e., there are S' and U' such that $T' = S' \cup U'$, $(\mathcal{K}, S') \vDash \varphi$ and $(\mathcal{K}, U') \vDash \psi$. We have to find a split (S, U) of T such that $(\mathcal{K}, S) \vDash \Diamond\varphi$ and $(\mathcal{K}, U) \vDash \Diamond\psi$. Define $S := \{ v \in T \mid vRu, u \in S' \}$ and $U := \{ v \in T \mid vRu, u \in U' \}$. Now every world $v \in T$ has at least one successor $u \in T'$ as T' is a successor team of T. Since $S' \cup U' = T'$, it is either $u \in S'$ or $u \in U'$. Hence by definition of S and U, v is in S or U. So $T \subseteq S \cup U$, and by definition of S and U then $T = S \cup U$. For proving $(\mathcal{K}, S) \vDash \Diamond\varphi$ and $(\mathcal{K}, U) \vDash \Diamond\psi$ it remains to show that S' is really a successor team of S (the proof for U is similar). Certainly every $v \in S$ must have at least one successor in S' by definition of S. Also every $u \in S'$ has at least one predecessor in S: As $S' \subseteq T'$ and T' is a successor team of T, it holds that u has a predecessor in T, say, v, but then $v \in T$, vRu and $u \in S'$, so $v \in S$ by definition of S. So S' is a successor team of S.

"$\Leftarrow$": Assume $(\mathcal{K}, T) \vDash \Diamond\varphi \otimes \Diamond\psi$ witnessed by $T = S \cup U$, $(\mathcal{K}, S') \vDash \varphi$ and $(\mathcal{K}, U') \vDash \psi$, S', U' being successor teams of S and U. Then $T' = S' \cup U'$ is a successor team of T which witnesses $(\mathcal{K}, T) \vDash \Diamond(\varphi \otimes \psi)$. $(\mathcal{K}, S' \cup U') \vDash \varphi \otimes \psi$ is clear, so we prove that T' is an actual successor team of T. If $v \in T$ then $v \in S$ or $v \in U$, so v has a successor u in S' or U', but either way in T'. If $u \in T'$ then $u \in S'$ or $u \in U'$, so u has a predecessor v in S or U and hence in T. ◀