# The Relational Model Is Injective for Multiplicative Exponential Linear Logic

Daniel de Carvalho

**Innopolis University, Russia**
d.carvalho@innopolis.ru

## Abstract

We prove a completeness result for Multiplicative Exponential Linear Logic (MELL): we show that the relational model is injective for MELL proof-nets, i.e. the equality between MELL proof-nets in the relational model is exactly axiomatized by cut-elimination.

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## 1 Introduction

In the seminal paper by Harvey Friedman [11], it has been shown that equality between simply-typed lambda terms in the full typed structure $\mathcal{M}_X$ over an infinite set $X$ is completely axiomatized by $\beta$ and $\eta$: we have $\mathcal{M}_X \vDash v = u \Leftrightarrow v \simeq_{\beta\eta} u$. A natural problem is to know whether a similar result could be obtained for Linear Logic.

Such a result can be seen as a "separation" theorem. To obtain such separation theorems, it is a prerequesite to have a "canonical" syntax. When Jean-Yves Girard introduced Linear Logic (LL) [12], he not only introduced a sequent calculus system but also "proof-nets". Indeed, as for LJ and LK (sequent calculus systems for intuitionnistic and classical logic, respectively), different proofs in LL sequent calculus can represent "morally" the same proof: proof-nets were introduced to find a unique representative for these proofs.

The technology of proof-nets was completely satisfactory for the multiplicative fragment without units.[1] For proof-nets having additives, contractions or weakenings, it was easy to exhibit different proof-nets that should be identified. Despite some flaws, the discovery of proof-nets was striking. In particular, Vincent Danos proved by syntactical means in [3] the confluence of these proof-nets for the Multiplicative Exponential Linear Logic fragment (MELL). For additives, the problem to have a satisfactory notion of proof-net has been addressed in [15]. For MELL, a "new syntax" was introduced in [4]. In the original syntax, the following properties of the weakening and of the contraction did not hold:

- the associativity of the contraction;
- the neutrality of the weakening for the contraction;
- the contraction and the weakening as morphisms of coalgebras.

But they hold in the new syntax; at least for MELL, we got a syntax that was a good candidate to deserve to be considered as being "canonical". Then trying to prove that any two

[1] For the multiplicative fragment with units, it has been recently shown [14] that, in some sense, no satisfactory notion of proof-net can exist. Our proof-nets have no jump, so they identify too many sequent calculus proofs, but not more than the relational semantics.


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($\eta$-expanded) MELL proof-nets that are equal in some denotational semantics are $\beta$-joinable has become sensible and had at least the two following motivations:

- to prove the canonicity of the "new syntax" (if we quotient more normal proof-nets, then we would identify proof-nets having different semantics);
- to prove by semantics means the confluence (if a proof-net reduces to two cut-free proof-nets, then they have the same semantics, so they would be $\beta$-joinable, hence equal).

The problem of *injectivity*[2] of the denotational semantics for MELL, which is the question whether equality in the denotational semantics between ($\eta$-expanded) MELL proof-nets is exactly axiomatized by cut-elimination or not, can be seen as a study of the separation property with a semantic approach. The first work on the study of this property in the framework of proof-nets is [16] where the authors deal with the translation into LL of the pure $\lambda$-calculus; it has been studied more recently for the intuitionistic multiplicative fragment of LL [17] and for differential nets [18]. For Parigot's $\lambda\mu$-calculus, see [5] and [22].

Finally the precise problem of injectivity for MELL has been adressed by Lorenzo Tortora de Falco in his PhD thesis [23] and in [24] for the (multiset based) coherence semantics and the multiset based relational semantics. He gave partial results and counter-examples for the coherence semantics: the (multiset based) coherence semantics is not injective for MELL. Also, it was conjectured that the relational model is injective for MELL. We prove the conjecture in the present paper.

In [24], a proof of the injectivity of the relational model is given for a weak fragment. But despite many efforts ([23], [24], [1], [19], [18], [20]...), all the attempts to prove the conjecture failed up to now. New progress was made in [9], where it has been proved that the relational semantics is injective for "connected" MELL proof-nets. Still, there, "connected" is understood as a very strong assumption, the set of "connected" MELL proof-nets contains the fragment of MELL defined by removing weakenings and units. Actually [9] proved a much stronger result: in the full MELL fragment two proof-nets $R$ and $R'$ with the same interpretation are the same "up to the connections between the doors of exponential boxes" (we say that they have the same LPS[3] – see Figures 8, 9 and 10 for an example of three different proof-nets having the same LPS). We wrote: "This result can be expressed in terms of differential nets: two cut-free proof-nets with different LPS have different Taylor expansions. We also believe this work is an essential step towards the proof of the full conjecture." Despite the fact we obtained a very interesting result about *all* the proof-nets (i.e. also for non-"connected" proof-nets[4]), the last sentence was a bit too optimistic, since, in the present paper, which presents a proof of the full conjecture, we could not use any previous result nor any previous technique/idea.

The result of the present paper can be seen as

- a semantic separation property in the sense of [11];
- a semantic proof of the confluence property;
- a proof of the "canonicity" of the new syntax of MELL proof-nets;
- a proof of the fact that if the Taylor expansions of two cut-free MELL proof-nets into differential nets[5] [10] coincide, then the two proof-nets coincide.

[2] The tradition of the lambda-calculus community rather suggests the word "completeness" and the terminology of category theory rather suggets the word "faithfulness", but we follow here the tradition of the Linear Logic community.

[3] The LPS of a proof-net is the graph obtained by forgetting the outline of the boxes but keeping the auxiliary doors.

[4] and even adding the MIX rule

[5] Differential proof-nets are linear approximations of proof-nets that are meant to allow the expression of the Taylor expansion of proof-nets as infinite series of their linear approximations.

Let us give one more interpretation of its signifance. First, notice that a proof of this result should consist in showing that, given two non $\beta$-equivalent proof-nets $R$ and $R'$, their respective semantics $[\![R]\!]$ and $[\![R']\!]$ are not equal, i.e. $[\![R]\!] \setminus [\![R']\!] \neq \emptyset$ or $[\![R']\!] \setminus [\![R]\!] \neq \emptyset$.[6] But, actually, we prove something much stronger: we prove that, given a proof-net $R$, there exist two points $\alpha$ and $\beta$ such that, for any proof-net $R'$, we have $\{\alpha, \beta\} \subseteq [\![R']\!] \Leftrightarrow R \simeq_\beta R'$.

Now, the points of the relational model can be seen as non-idempotent intersection types[7] (see [6] and [7] for a correspondance between points of the relational model and System R – System R has also been studied recently in [2]). And the proof given in the present paper uses the types only to derive the normalization property; actually we prove the injectivity for cut-free proof-nets in an untyped framework:[8] substituting the assumption that proof-nets are typed by the assumption that proof-nets are normalizable does not change anything to the proof.[9] In [8], we gave a semantic characterization of normalizable untyped proof-nets and we characterized "head-normalizable" proof-nets as proof-nets having a non-empty interpretation in the relational semantics. Principal typings in untyped $\lambda$-calculus are intersection types which allow to recover all the intersection types of some term. If, for instance, we consider the System $R$ of [6] and [7], it is enough to consider some *injective* 1-*point*[10] to obtain the principal typing of an untyped $\lambda$-term. But, generally, for normalizable MELL proof-nets, *injective $k$-points*, for any $k$, are not principal typings; indeed, two cut-free MELL proof-nets having the same LPS have the same injective $k$-points for any $k \in \mathbb{N}$. In the current paper we show that a 1-point and a *$k$-heterogeneous point*[11] together allow to recover the interpretation of any normalizable MELL proof-net. So, the result of the current paper can be seen as a first attempt to find a right notion of "principal typing" of intersection types in Linear Logic. As a consequence, normalization by evaluation, as in [21] for $\lambda$-calculus, finally becomes possible in Linear Logic too.

Section 2 formalizes PS's (our cut-free proof-nets). Section 3 gives a sketch of our algorithm leading from $[\![R]\!]$ to the rebuilding of $R$. Section 4 describes more precisely one step of the algorithm and states our theorem (Theorem 40): $[\![R]\!] = [\![R']\!] \Leftrightarrow R \simeq_\beta R'$, where $\simeq_\beta$ is the reflexive symmetric transitive closure of the cut-elimination relation.

▶ Notations. We denote by $\varepsilon$ the empty sequence. If $a$ is a sequence $(\alpha_1, \ldots, \alpha_n)$, then $\alpha_0 : a$ denotes the sequence $(\alpha_0, \ldots, \alpha_n)$; otherwise, it denotes the sequence $(\alpha, a)$ of length 2. The set of finite sequences of elements of some set $\mathcal{E}$ is denoted by $\mathcal{E}^{<\infty}$.

A multiset $f$ of elements of some set $\mathcal{E}$ is a function $\mathcal{E} \to \mathbb{N}$; we denote by *Supp*$(f)$ *the support of* $f$ i.e. the set $\{e \in \mathcal{E}; f(e) \neq 0\}$. A multiset $f$ is said to be *finite* if *Supp*$(f)$ is finite. The set of finite multisets of elements of some set $\mathcal{E}$ is denoted by $\mathcal{M}_{fin}(\mathcal{E})$.

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[6] The converse, i.e. two $\beta$-equivalent proof-nets have the same semantics, holds by definition of soundness.

[7] Idempotency of intersection ($\alpha \cap \alpha = \alpha$) does not hold.

[8] For cut-free proof-nets, types guarantee that they are not cyclic as graphs – instead of typing, it is enough to assume this property. Our proof even works for "non-correct" proof-structures (correctness is the property characterizing nets corresponding in a typed framework with proofs in sequent calculus): we could expect that if the injectivity of the relational semantics holds for proof-nets corresponding with MELL sequent calculus, then it still holds for proof-nets corresponding with MELL+MIX sequent calculus, since the category **Rel** of sets and relations is a compact closed category. [13] assuming correctness substituted in the proof the "bridges" of [9] by "empires".

[9] Except that we have to consider the *atomic* subset of the interpretation instead of the full interpretation (see Remark 6).

[10] An *injective $k$-point* is a point in which all the positive multisets have cardinality $k$ and in which each atom occurring in it occurs exactly twice.

[11] *$k$-heterogeneous points* are points in which every positive multiset has cardinality $k^j$ for some $j > 0$ and, for any $j > 0$, there is at most one occurrence of a positive multiset having cardinality $k^j$ – they are obtained by *$k$-heterogeneous experiments* (see our Definition 12). The terminology "$k$-heterogeneous" has been suggested to us by some reviewer to substitute the expression "$k$-injective".

If $f$ is a function $\mathcal{E} \to \mathcal{E}'$, $x_0 \in \mathcal{E}$ and $y \in \mathcal{E}'$, then we denote by $f[x_0 \mapsto y]$ the function $\mathcal{E} \to \mathcal{E}'$ defined by $f[x_0 \mapsto y](x) = \begin{cases} f(x) & \textit{if } x \neq x_0; \\ y & \textit{if } x = x_0. \end{cases}$ If $f$ is a function $\mathcal{E} \to \mathcal{E}'$ and $\mathcal{E}_0 \subseteq \mathsf{dom}(f) = \mathcal{E}$, then we denote by $f[\mathcal{E}_0]$ the set $\{f(x); x \in \mathcal{E}_0\}$.

## 2 Syntax

We introduce the syntactical objects we are interested in. As recalled in the introduction, simple types guarantee normalization, so we can limit ourselves to nets without any cut. Correctness does not play any role, that is why we do not restrict our nets to be correct and we rather consider proof-structures (*PS's*). Since our proof is easily extended to MELL with axioms, we remove them for simplicity. Moreover, since it is convenient to represent formally our proof using differential nets with boxes (*differential PS's*), we define PS's as differential PS's satisfying some conditions (Definition 4). More generally, *differential* $\circ$*-PS's* are defined by induction on the depth: Definition 1 concerns what happens at depth 0.

We define the set $\mathbb{T}$ of types as follows: $\mathbb{T} ::= 1 \mid \bot \mid (\mathbb{T} \otimes \mathbb{T}) \mid (\mathbb{T} \parr \mathbb{T}) \mid !\mathbb{T} \mid ?\mathbb{T}$. We set $\mathfrak{T} = \{\otimes, \parr, 1, \bot, !, ?, \circ\}$. Pre-contractions ($\circ$-ports) are an artefact of our inductive definition on the depth and are used to ensure the canonicity of our syntactical objects (see Example 6).

► **Definition 1.** A *differential ground-structure* is a 6-tuple $\mathcal{G} = (\mathcal{W}, \mathcal{P}, l, t, \mathcal{L}, \mathsf{T})$, where

- $\mathcal{P}$ is a finite set; the elements of $\mathcal{P}(\mathcal{G})$ are the *ports of* $\mathcal{G}$;
- $l$ is a function $\mathcal{P} \to \mathfrak{T}$;the element $l(p)$ of $\mathfrak{T}$ is the *label of* $p$ *in* $\mathcal{G}$;
- $\mathcal{W}$ is a subset of $\{p \in \mathcal{P}; l(p) \neq \circ\}$; the elements of $\mathcal{W}(\mathcal{G})$ are the *wires of* $\mathcal{G}$;
- $t$ is a function $\mathcal{W} \to \{p \in \mathcal{P}; l(p) \notin \{1, \bot\}\}$ such that, for any port $p$ of $\mathcal{G}$, we have $(l(p) \in \{\otimes, \parr\} \Rightarrow \mathsf{Card}\,(\{w \in \mathcal{W}; t(w) = p\}) = 2)$; if $t(w) = p$, then $w$ is a *premise of* $p$; the *arity* $a_{\mathcal{G}}(p)$ *of* $p$ is the number of its premises;
- $\mathcal{L}$ is a subset of $\{w \in \mathcal{W}; l(t(w)) \in \{\otimes, \parr\}\}$ such that $(\forall p \in \mathcal{P})$ $(l(p) \in \{\otimes, \parr\} \Rightarrow \mathsf{Card}\,(\{w \in \mathcal{L}; t(w) = p\}) = 1)$; if $w \in \mathcal{L}$ s.t. $t(w) = p$, then $w$ is *the left premise of* $p$*;*
- and $\mathsf{T}$ is a function $\mathcal{P} \to \mathbb{T}$ such that, for any $p \in \mathcal{P}$,
  - if $l(p) \in \{1, \bot\}$, then $\mathsf{T}(p) = l(p)$;
  - if $l(p) = \otimes$ (resp. $l(p) = \parr$), then, for any $w_1 \in \mathcal{W} \cap \mathcal{L}$ and any $w_2 \in \mathcal{W} \setminus \mathcal{L}$ such that $t(w_1) = p = t(w_2)$, we have $\mathsf{T}(p) = (\mathsf{T}(w_1) \otimes \mathsf{T}(w_2))$ (resp. $\mathsf{T}(p) = (\mathsf{T}(w_1) \parr \mathsf{T}(w_2))$);
  - if $l(p) = !$, then $(\exists C \in \mathbb{T})(\mathsf{T}(p) = !C \wedge (\forall w \in \mathcal{W})(t(w) = p \Rightarrow \mathsf{T}(w) = C))$;
  - and if $l(p) \in \{\circ, ?\}$, then $(\exists C \in \mathbb{T})(\mathsf{T}(p) = ?C \wedge (\forall w \in \mathcal{W})(t(w) = p \Rightarrow \mathsf{T}(w) = C))$.

We set $\mathcal{W}(\mathcal{G}) = \mathcal{W}$, $\mathcal{P}(\mathcal{G}) = \mathcal{P}$, $l_{\mathcal{G}} = l$, $t_{\mathcal{G}} = t$, $\mathcal{L}(\mathcal{G}) = \mathcal{L}$, $\mathsf{T}_{\mathcal{G}} = \mathsf{T}$. The set $\mathcal{P}^{\mathsf{f}}(\mathcal{G}) = \mathcal{P} \setminus \mathcal{W}$ is the set of *conclusions of* $\mathcal{G}$. For any $t \in \mathfrak{T}$, we set $\mathcal{P}^t(\mathcal{G}) = \{p \in \mathcal{P}; l(p) = t\}$; we set $\mathcal{P}^m(\mathcal{G}) = \mathcal{P}^{\otimes}(\mathcal{G}) \cup \mathcal{P}^{\parr}(\mathcal{G})$; the set $\mathcal{P}^e(\mathcal{G})$ of *exponential ports of* $\mathcal{G}$ is $\mathcal{P}^!(\mathcal{G}) \cup \mathcal{P}^?(\mathcal{G}) \cup \mathcal{P}^{\circ}(\mathcal{G})$.

A *ground-structure* is a differential ground-structure $\mathcal{G}$ s.t. $\mathsf{im}(t_{\mathcal{G}}) \cap (\mathcal{P}^!(\mathcal{G}) \cup \mathcal{P}^{\circ}(\mathcal{G})) = \emptyset$.

Notice that, for any differential ground-structure $\mathcal{G}$, we have $\mathcal{P}^{\circ}(\mathcal{G}) \subseteq \mathcal{P}^{\mathsf{f}}(\mathcal{G})$.

► **Example 2.** The ground-structure $\mathcal{G}$ defined by: $\mathcal{W}(\mathcal{G}) = \{p_3, p_4, p_5\}$; $\mathcal{P}(\mathcal{G}) = \{p_1, \ldots, p_5\}$; $l_{\mathcal{G}}(p_1) = \bot$, $l_{\mathcal{G}}(p_2) = \parr$, $l_{\mathcal{G}}(p_3) = ? = l_{\mathcal{G}}(p_4)$, $l_{\mathcal{G}}(p_5) = 1$;

$t_{\mathcal{G}}(p_3) = p_2 = t_{\mathcal{G}}(p_4)$, $t_{\mathcal{G}}(p_5) = p_4$; $\mathcal{L}(\mathcal{G}) = \{p_3\}$; and $\mathsf{T}_{\mathcal{G}}(p_1) = \bot$, $\mathsf{T}_{\mathcal{G}}(p_2) = (?1 \parr ?1)$, $\mathsf{T}_{\mathcal{G}}(p_3) = ?1 = \mathsf{T}_{\mathcal{G}}(p_4)$, $\mathsf{T}_{\mathcal{G}}(p_5) = 1$; is the ground-structure of the content of the box $o_1$ of $R$ (the leftmost box of Figure 1).

The content of every box of our *differential* $\circ$*-PS's* is a $\circ$*-PS*: every !-port inside is always the main door of some box.

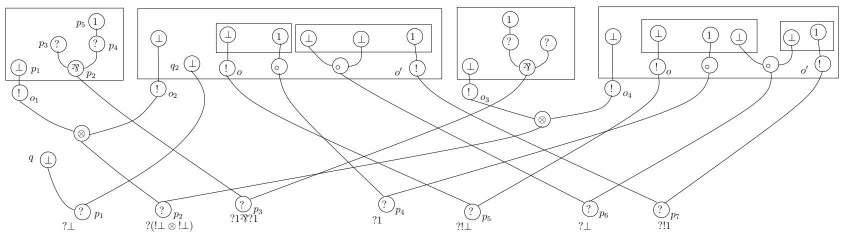

**Figure 1** PS $R$.

▶ **Definition 3.** For any $d \in \mathbb{N}$, we define, by induction on $d$, the set of *differential* $\circ$*-PS's of depth* $d$ (resp. the set of $\circ$*-PS's of depth* $d$). A *differential* $\circ$*-PS of depth* $d$ (resp. a $\circ$*-PS of depth* $d$) is a 4-tuple $S = (\mathcal{G}, \mathcal{B}_0, B, b)$, where

- $\mathcal{G}$ is a differential ground-structure (resp. a ground-structure);
- $\mathcal{B}_0 \subseteq \{p \in \mathcal{P}^!(\mathcal{G}); a_{\mathcal{G}}(p) = 0\}$ (resp. $\mathcal{B}_0 = \mathcal{P}^!(\mathcal{G})$) and is the set of *boxes of* $S$ *at depth* 0;
- $B$ is a function that associates with every $o \in \mathcal{B}_0$ a $\circ$-PS $B(o) = (\mathcal{G}(B(o)), \mathcal{B}_0(B(o)), B_{B(o)}, b_{B(o)})$ of depth $< d$ that enjoys the following property: if $d > 0$, then there exists $o \in \mathcal{B}_0$ s.t. $B(o)$ is a $\circ$-PS of depth $d-1$; the !-port $o$ is *the main door of the box* $B(o)$;
- and $b$ is a function that associates with every $o \in \mathcal{B}_0$ a function $b(o) : \mathcal{P}^{\mathsf{f}}(\mathcal{G}(B(o))) \to \{o\} \cup \mathcal{P}^?(\mathcal{G}) \cup \mathcal{P}^\circ(\mathcal{G})$ such that (resp. $\mathcal{P}^\circ(\mathcal{G}) \subseteq \bigcup_{o \in \mathcal{B}_0} \mathsf{im}(b(o))$ and), for any $o \in \mathcal{B}_0$,
  - $b(o)|_{\mathcal{P}^\circ(\mathcal{G}(B(o)))}$ is injective[12] (resp.[13] $b(o)|_{\mathcal{P}^\circ(\mathcal{G}(B(o)))} = id_{\mathcal{P}^\circ(\mathcal{G}(B(o)))}$); if $q = b(o)(p)$ with $p \in \mathcal{P}^\circ(\mathcal{G}(B(o)))$, then we set $q_{S,o} = p$;
  - $o \in \mathsf{im}(b(o))$ and $b(o)[\mathcal{P}^\circ(\mathcal{G}(B(o)))] \cap \mathcal{P}^!(\mathcal{G}) = \emptyset$;
  - for any $p \in \mathsf{dom}(b(o)) \cap \mathcal{P}^\circ(\mathcal{G}(B_R(o)))$, we have $\mathsf{T}_{\mathcal{G}}(b(o)(p)) = \mathsf{T}_{\mathcal{G}(B(o))}(p)$;
  - for any $p \in \mathsf{dom}(b(o)) \setminus \mathcal{P}^\circ(\mathcal{G}(B_R(o)))$, we have $\mathsf{T}_{\mathcal{G}}(b(o)(p)) \in \{?\mathsf{T}_{\mathcal{G}(B(o))}(p), !\mathsf{T}_{\mathcal{G}(B(o))}(p)\}$;

(resp. moreover no $p \in \mathcal{P}(\mathcal{G})$ is a sequence)[14]. For any differential $\circ$-PS $S = (\mathcal{G}, \mathcal{B}_0, B, b)$, we set $\mathcal{G}(S) = \mathcal{G}$, $\mathcal{B}_0(S) = \mathcal{B}_0$ and $\mathcal{B}(S) = \mathcal{B}_0(S) \cup \bigcup_{o \in \mathcal{B}_0(S)} \{o : o'; o' \in \mathcal{B}(B_S(o))\}$ is the set of *boxes of* $S$. We denote by $B_S$ the extension of the function $B$ that associates with each $o : o' \in \mathcal{B}(S)$, where $o \in \mathcal{B}_0(S)$, the $\circ$-PS $B_{B_S(o)}(o')$. We denote by $b_S$ the extension of the function $b$ that associates with each $o : o' \in \mathcal{B}(S)$, where $o \in \mathcal{B}_0(S)$, the function $b_{B_S(o)}(o')$. We set $\mathcal{W}_0(S) = \mathcal{W}(\mathcal{G}(S))$ and $\mathcal{P}_0(S) = \mathcal{P}(\mathcal{G}(S))$; the elements of $\mathcal{P}_0(S)$ (resp. of $\mathcal{W}_0(S)$) are the *ports of* $S$ *at depth* 0 (resp. the *wires of* $S$ *at depth* 0). For any $l \in \mathfrak{T} \cup \{m, e\}$, we set $\mathcal{P}_0^l(S) = \mathcal{P}^l(\mathcal{G}(S))$. We set $\mathcal{P}^{\mathsf{f}}(S) = \mathcal{P}^{\mathsf{f}}(\mathcal{G}(S))$, $\mathcal{P}^{\mathsf{f}}_\circ(S) = \mathcal{P}^\circ(\mathcal{G}(S))$ and $\mathcal{P}^{\mathsf{f}}_\bullet(S) = \mathcal{P}^{\mathsf{f}}(S) \setminus \mathcal{P}^{\mathsf{f}}_\circ(S)$; the elements of $\mathcal{P}^{\mathsf{f}}(S)$ are the *conclusions of* $S$ and the elements of $\mathcal{P}^{\mathsf{f}}_\circ(S)$ are the $\circ$*-conclusions of* $S$. For any relation $P \in \{\geq, =, <\}$ on $\mathbb{N}$, for any $i \in \mathbb{N}$, we set $\mathcal{B}_0^{Pi}(S) = \{o \in \mathcal{B}_0(S); depth(B_S(o))Pi\}$ and $\mathcal{B}^{Pi}(S) = \{o \in \mathcal{B}(S); depth(B_S(o))Pi\}$.

PS's are the MELL proof-nets studied in the present paper: there is no cut and no assumption of correctness property.

▶ **Definition 4.** A PS is a $\circ$-PS $R$ such that $\mathcal{P}^{\mathsf{f}}_\circ(R) = \emptyset$.

▶ **Example 5.** Consider the PS $R$ of Figure 1. We have $\mathcal{B}_0(R) = \{o_1, o_2, o_3, o_4\}$, $\mathcal{B}(R) = \{o_1, o_2, o_3, o_4, (o_2, o), (o_2, o'), (o_4, o), (o_4, o')\}$, $\mathcal{B}^{=0}(R) = \{o_1, (o_2, o), (o_2, o'), o_3, (o_4, o), (o_4, o')\}$. We have $b_R(o_2)(o) = p_5$, $b_R(o_2)(p_4) = p_4$, $b_R(o_2)(p_6) = p_6$ and $b_R(o_2)(o') = p_7$.

[12] So one cannot (pre-)contract several $\circ$-ports of the same box.
[13] This stronger condition on $\circ$-PS's is *ad hoc*, but it allows to lighten the notations.
[14] This condition on $\circ$-PS's is *ad hoc*, but it allows to simplify Definition 14.

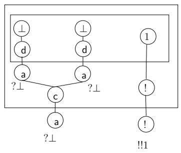

**Figure 2** $O_1$ ("old syntax").

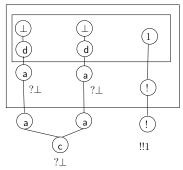

**Figure 3** $O_2$ ("old syntax").

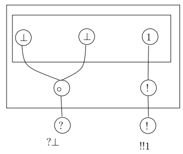

**Figure 4** PS $N$.

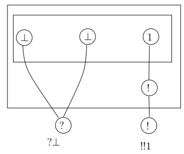

**Figure 5** Danos-Regnier-proof-net obtained from the PS $N$ by "ignoring" the $\circ$-ports.

▶ **Example 6.** In order to understand the role of the $\circ$-ports, consider how the proof-nets $O_1$ (Figure 2) and $O_2$ (Figure 3) in the "old syntax" (we denoted derelictions, contractions and auxiliary doors of the "old syntax" by $\mathsf{d}$, $\mathsf{c}$ and $\mathsf{a}$, respectively) are represented by the same PS $N$ (Figure 4). Roughly speaking, in our formalism, one pre-contracts (using $\circ$-ports) as soon as possible and one contracts (using ?-ports) as late as possible. Notice that if one "ignores" the $\circ$-ports, i.e. if, whenever a port $p$ that is not a $\circ$-port is immediately above a series of $\circ$-ports that are imediately above a contraction $q$, one draws a wire from $p$ to $q$ and one removes all the $\circ$-ports, then one obtains a cut-free proof-net of [4] without the "sequentialization condition" (see Figure 5); and, conversely, given such a proof-net, there is a unique way to add the $\circ$-ports to obtain our PS's. So our definition of PS's is exactly equivalent to the definition of cut-free proof-nets of [4] without the "sequentialization condition"; one reason to not take the same definition as the one of [4] is the desire to have an inductive definition on the depth, which, as a consequence, leads to the auxiliary notion of $\circ$-PS.

We write $R \simeq R'$ (resp. $R \equiv R'$) if $R$ and $R'$ are the same differential PS's up to the names of their ports (resp. that are not conclusions):

▶ **Definition 7.** An *isomorphism $\varphi : \mathcal{G} \simeq \mathcal{G}'$ of ground-structures* is a structure-preserving bijection $\mathcal{P}_0(\mathcal{G}) \simeq \mathcal{P}_0(\mathcal{G}')$. We define, by induction on *depth*$(R)$, when $\varphi : R \simeq R'$ holds for two differential $\circ$-PS's $R$ and $R'$: it holds whenever $\varphi$ is a pair $(\varphi_\mathcal{G}, (\varphi_o)_{o\in\mathcal{B}_0(R)})$ s.t. $\varphi_\mathcal{G} : \mathcal{G}(R) \simeq \mathcal{G}(R')$, $\mathcal{B}_0(R') = \varphi_\mathcal{G}[\mathcal{P}_0^!(R)]$ and, for any $o \in \mathcal{B}_0(R)$, $\varphi_o : B_R(o) \simeq B_{R'}(\varphi_\mathcal{G}(o))$ and $(\forall q \in \mathcal{P}^\mathsf{f}(B_R(o)))b_{R'}(\varphi_\mathcal{G}(o))(\mathcal{G}(\varphi_o)(q)) = \varphi_\mathcal{G}(b_R(o)(q))$. We set $\mathcal{G}(\varphi) = \varphi_\mathcal{G}$ and, for any $o \in \mathcal{B}_0(R)$, $\varphi(o) = \varphi_o$. We write $\varphi : R \equiv R'$ if $\varphi : R \simeq R'$ s.t. $\mathcal{G}(\varphi)|_{\mathcal{P}^\mathsf{f}(R)} = id_{\mathcal{P}^\mathsf{f}(R)}$. We write $R \simeq R'$ (resp. $R \equiv R'$) if there exists $\varphi$ s.t. $\varphi : R \simeq R'$ (resp. $\varphi : R \equiv R'$).

The *arity $a_R(q)$ of a port $q$ in a differential $\circ$-PS $R$* is computed by "ignoring" the $\circ$-conclusions of the boxes of $R$:

▶ **Definition 8.** Let $R$ be a differential $\circ$-PS. We define, by induction on *depth*$(R)$, the integers $a_R(q)$ for any $q \in \mathcal{P}_0(R)$ and *cosize*$(R)$: we set $a_R(q) = a_{\mathcal{G}(R)}(q) + \sum_{o\in\mathcal{B}_0(R)} \mathsf{Card}\left(\{p \in \mathcal{P}^\mathsf{f}_\bullet(B_R(o)); b_R(o)(p) = q\}\right) + \sum_{\substack{o\in\mathcal{B}_0(R)\\ q\in b_R(o)[\mathcal{P}^\mathsf{f}_\circ(B_R(o))]}} a_{B_R(o)}(q_{R,o})$ and *cosize*$(R) = \max(\{a_R(p); p \in \mathcal{P}_0(R)\} \cup \{\mathit{cosize}(B_R(o)); o \in \mathcal{B}_0(R)\})$.

▶ **Example 9.** We have $a_R(p_6) = 4$ (and not 2) and *cosize*$(R) = 4$ (see Figure 1).

## 3 Experiments and their partial expansions

When Jean-Yves Girard introduced proof-nets in [12], he also introduced *experiments of proof-nets*. Experiments (see our Definition 10) are a technology allowing to compute pointwise the interpretation $[\![R]\!]$ of a proof-net $R$ in the model directly on the proof-net rather than through some sequent calculus proof obtained from one of its sequentializations: the set of *results* of all the experiments of a given proof-net is its interpretation $[\![R]\!]$. In an untyped framework, experiments correspond with type derivations and results correspond with intersection types.

▶ **Definition 10.** For any $C \in \mathbb{T}$, we define, by induction on $C$, the set $[\![C]\!]$: $[\![1]\!] = \{*\} = [\![\bot]\!]$; $[\![(C_1 \otimes C_2)]\!] = [\![C_1]\!] \times [\![C_2]\!] = [\![(C_1 \parr C_2)]\!]$; $[\![!C]\!] = \mathcal{M}_{fin}([\![C]\!]) = [\![?C]\!]$.

Let $R$ be a differential $\circ$-PS. We define, by induction on $depth(R)$, the set of *experiments of* $R$: it is is the set of triples $(R, e_{\mathcal{P}}, e_{\mathcal{B}})$, where $e_{\mathcal{P}}$ is a function that associates with every $p \in \mathcal{P}_0(R)$ an element of $[\![\mathsf{T}_{\mathcal{G}(R)}(p)]\!]$ and $e_{\mathcal{B}}$ is a function which associates to every $o \in \mathcal{B}_0(R)$ a finite multiset of experiments of $B_R(o)$ such that

- for any $p \in \mathcal{P}_0^m(R)$, for any $w_1, w_2 \in \mathcal{W}_0(R)$ such that $t_{\mathcal{G}(R)}(w_1) = p = t_{\mathcal{G}(R)}(w_2)$, $w_1 \in \mathcal{L}(\mathcal{G}(R))$ and $w_2 \notin \mathcal{L}(\mathcal{G}(R))$, we have $e_{\mathcal{P}}(p) = (e_{\mathcal{P}}(w_1), e_{\mathcal{P}}(w_2))$;
- for any $p \in \mathcal{P}_0^e(R)$, we have $e(p) = \sum_{\substack{w \in \mathcal{W}_0(R) \\ t_{\mathcal{G}(R)}(w) = p}} [e_{\mathcal{P}}(w)] + \sum_{o \in \mathcal{B}_0(R)} \sum_{e' \in Supp(e_{\mathcal{B}}(o))} (\sum_{\substack{q \in \mathcal{P}^{\mathsf{f}}_{\bullet}(B_R(o)) \\ b_R(o)(q) = p}} e_{\mathcal{B}}(o)(e') \cdot [e'_{\mathcal{P}}(q)] + \sum_{\substack{q \in \mathcal{P}^{\mathsf{f}}_{\circ}(B_R(o)) \\ b_R(o)(q) = p}} e_{\mathcal{B}}(o)(e') \cdot e'_{\mathcal{P}}(q))$.

For any experiment $e = (R, e_{\mathcal{P}}, e_{\mathcal{B}})$, we set $\mathcal{P}(e) = e_{\mathcal{P}}$ and $\mathcal{B}(e) = e_{\mathcal{B}}$. We set $[\![R]\!] = \{\mathcal{P}(e)|_{\mathcal{P}^{\mathsf{f}}(R)};\ e \textit{ is an experiment of } R\}$.

We encode in a more compact way the "relevant" information given by an experiment *via pseudo-experiments* and the functions $e^{\#}$:

▶ **Definition 11.** For any differential $\circ$-PS $R$, we define, by induction on $depth(R)$, the set of *pseudo-experiments of* $R$: it is the set of functions that associate with every $o \in \mathcal{B}_0(R)$ a finite set of pseudo-experiments of $B_R(o)$ and with $\varepsilon$ a pair $(R, m)$ for some $m \in \mathbb{N}$.

Given an experiment $e$ of some differential $\circ$-PS $R$, we define, by induction on $depth(R)$, a pseudo-experiment $\overline{e}$ of $R$ as follows: $\overline{e}(\varepsilon) = (R, 1)$ and $\overline{e}(o) = \bigcup_{f \in Supp(\mathcal{B}(e)(o))} \{\overline{f}[\varepsilon \mapsto (B_R(o), i)]; 1 \leq i \leq \mathcal{B}(e)(o)(f)\}$ for any $o \in \mathcal{B}_0(R)$.

Given a pseudo-experiment $e$ of a differential $\circ$-PS $R$, we define, by induction on $depth(R)$, the function $e^{\#} : \mathcal{B}(R) \to \mathcal{P}_{fin}(\mathbb{N})$ as follows: for any $o \in \mathcal{B}_0(R)$, $e^{\#}(o) = \{\mathsf{Card}\,(e(o))\}$ and, for any $o' \in \mathcal{B}(B_R(o))$, $e^{\#}(o : o') = \bigcup_{e' \in e(o)} e'^{\#}(o')$.

There are different kinds of experiments:

- In [24], it was shown that given the result of an *injective $k$-obsessional experiment* ($k$ big enough) of a cut-free proof-net in the fragment $A ::= X|?A \parr A|A \parr ?A|A \otimes A|!A$, one can rebuild the entire experiment and, so, the entire proof-net. There, "injective" means that the experiment labels two different axioms with different atoms and "obsessional" means that different copies of the same axiom are labeled by the same atom.
- In [9], it was shown that for any two cut-free MELL proof-nets $R$ and $R'$, we have $LPS(R) = LPS(R')$ iff, for $k$ big enough[15], there exist an *injective $k$-experiment* of $R$ and an *injective $k$-experiment* of $R'$ having the same result; as an immediate corollary we obtained the injectivity of the set of (recursively) connected proof-nets. There, "injective" means that not only the experiment labels two different axioms with different atoms, but it labels also different copies of the same axiom by different atoms. Given some proof-net $R$, there is exactly one injective $k$-experiment of $R$ up to the names of the atoms.

[15] Interestingly, [13], following the approach of [9], showed that, if these two proof-nets are assumed to be (recursively) connected, then we can take $k = 2$.

- In the present paper we show that, for any two PS's $R$ and $R'$, given the result $\alpha$ of a $k$-*heterogeneous experiment* of $R$ for $k$ big enough, if $\alpha \in [\![R']\!]$, then $R'$ is the same PS as $R$. The conditions on $k$ are given by the result of a 1-experiment, so we show that two (well-chosen) points are enough to determine a PS. The expression "$k$-heterogeneous" means that, for any two different occurrences of boxes, the experiment never takes the same number of copies: it takes $k^{j_1}$ copies and $k^{j_2}$ copies with $j_1 \neq j_2$ (*a contrario*, in [24] and [9], the experiments always take the same number of copies). As shown by the proof-net S of Figure 11, it is impossible to rebuild the experiment from its result, since there exist four different 4-heterogeneous experiments $e_1, e_2, e_3$ and $e_4$ such that, for any $i \in \{1, 2, 3, 4\}$, we have $e_i(p) = (*, *)$, $e_i(o_1) = [*, *, *, *]$ and $e_i(p') = [[\underbrace{*, \ldots, *}_{4^2}], \ldots, [\underbrace{*, \ldots, *}_{4^6}]]$. For instance $e_1$ takes 4 copies of the box $o_1$ and 16 copies of the box $o_2$, while $e_2$ takes 4 copies of the box $o_1$ and 64 copies of the box $o_2$.

▶ **Definition 12.** Let $k > 1$. A pseudo-experiment $e$ of a $\circ$-PS $R$ is said to be $k$-*heterogeneous* if

- for any $o \in \mathcal{B}(R)$, for any $m \in e^{\#}(o)$, there exists $j > 0$ such that $m = k^j$;
- for any $o \in \mathcal{B}_0(R)$, for any $o' \in \mathcal{B}(B_R(o))$, we have $(\forall e_1, e_2 \in e(o))\ ({e_1}^{\#}(o') \cap {e_2}^{\#}(o') \neq \emptyset \Rightarrow e_1 = e_2)$;
- and, for any $o_1, o_2 \in \mathcal{B}(R)$, we have $(e^{\#}(o_1) \cap e^{\#}(o_2) \neq \emptyset \Rightarrow o_1 = o_2)$.

An experiment $e$ is said to be $k$-*heterogeneous* if $\overline{e}$ is $k$-heterogeneous.

▶ **Example 13.** There exists a 10-heterogeneous pseudo-experiment $f$ of the proof-net $R$ of Figure 1 such that $f^{\#}(o_1) = \{10^{223}\}$, $f^{\#}(o_2) = \{10\}$, $f^{\#}(o_3) = \{10^{224}\}$, $f^{\#}(o_4) = \{100\}$, $f^{\#}((o_2, o)) = \{10^3, \ldots, 10^{12}\}$, $f^{\#}((o_2, o')) = \{10^{13}, \ldots, 10^{22}\}$, $f^{\#}((o_4, o)) = \{10^{23}, \ldots, 10^{122}\}$ and $f^{\#}((o_4, o')) = \{10^{123}, \ldots, 10^{222}\}$.

In [9], the interest for *injective* experiments came from the remark that the result of an *injective* experiment of a *cut-free* proof-net can be easily identified with a differential net of its Taylor expansion in a sum of differential nets [10] (it is essentialy the content of our Lemma 16). Thus any proof using injective experiments can be straightforwardly expressed in terms of differential nets and conversely. Since this identification is trivial, besides the idea of considering injective experiments instead of obsessional experiments, the use of the terminology of differential nets does not bring any new insight[16], it just superficially changes the presentation.That is why we decided in [9] to avoid introducing explicitely differential nets. In the present paper, we made the opposite choice for the following reason: the algorithm leading from the result of a $k$-heterogeneous experiment of $R$ to the entire rebuilding of $R$ is done in several steps: in the intermediate steps, we obtain a partial rebuilding where some boxes have been recovered but not all of them; a convenient way to represent this information is the use of "differential nets with boxes" (called "differential PS's" in the present paper) that lie between the purely linear differential proof-nets and the non-linear proof-nets. Now, the differential net representing the result and the proof-net $R$ are both instances of the more general notion of "differential nets with boxes".

The rebuilding of the proof-net $R$ is done in $d$ steps, where $d$ is the depth of $R$. We first rebuild the occurrences of the boxes of depth 0 (the deepest ones) and next we rebuild the

[16] For proof-nets with cuts, the situation is completely different: the great novelty of differential nets is that differential nets have a cut-elimination; the differential nets appearing in the Taylor expansion of a proof-net with cuts have cuts, while the semantics does not see these cuts. But the proofs of the injectivity only consider cut-free proof-nets.

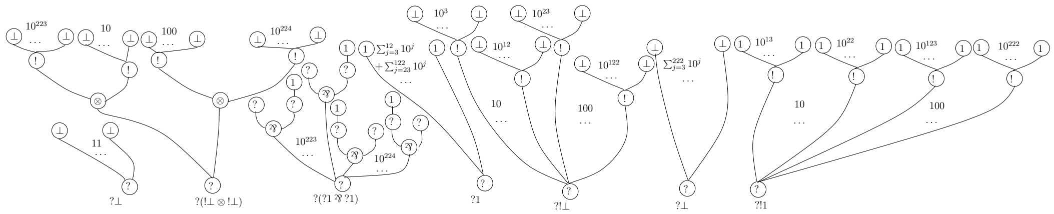

**Figure 6** $\mathcal{T}(f)[0]$.

occurrences of the boxes of depth 1 and so on... This can be formalized using differential nets (with boxes) as follows: if $e$ is an injective experiment of $R$, then $\mathcal{T}(\overline{e})[i]$ is the differential net corresponding with $e$ in which only boxes of depth $\geq i$ are expanded,[17] so $\mathcal{T}(\overline{e})[0]$ is (essentially) the same as the result of the experiment and $\mathcal{T}(\overline{e})[d] = R$; the first step of the algorithm builds $\mathcal{T}(\overline{e})[1]$ from $\mathcal{T}(\overline{e})[0]$, the second step builds $\mathcal{T}(\overline{e})[2]$ from $\mathcal{T}(\overline{e})[1]$, and so on... We thus reduced the problem of the injectivity to the problem of rebuilding $\mathcal{T}(\overline{e})[i+1]$ from $\mathcal{T}(\overline{e})[i]$ for any $k$-heterogeneous experiment $e$ ($k$ big enough).

▶ **Definition 14.** Let $R$ be a $\circ$-PS of depth $d$. Let $e$ be a pseudo-experiment of $R$. Let $i \in \mathbb{N}$. We define, by induction on $d$, a differential $\circ$-PS $\mathcal{T}(e)[i] = ((\mathcal{W}_{e,i}, \mathcal{P}_{e,i}, l_{e,i}, t_{e,i}, \mathcal{L}_{e,i}, \mathsf{T}_{e,i}), \mathcal{B}_{e,i}, B_{e,i}, b_{e,i})$ of depth $\min\{i, d\}$ s.t. $\mathcal{P}^{\mathsf{f}}(R) = \mathcal{P}^{\mathsf{f}}(\mathcal{T}(e)[i])$ and $(\forall p \in \mathcal{P}^{\mathsf{f}}(R)) l_{\mathcal{G}(R)}(p) = l_{\mathcal{G}(\mathcal{T}(e)[i])}(p)$ as follows: we set $\mathcal{P}^{\bullet}_{e,i} = \bigcup_{o_1 \in \mathcal{B}_0^{\geq i}(R)} \bigcup_{e_1 \in e(o_1)} \{(o_1, e_1) : p; p \in \mathcal{P}_{e_1,i} \setminus \mathcal{P}^{\mathsf{f}}_{\circ}(B_R(o_1))\}$;

- $\mathcal{W}_{e,i} = \mathcal{W}_0(R) \cup \mathcal{P}^{\bullet}_{e,i}$ and $\mathcal{P}_{e,i} = \mathcal{P}_0(R) \cup \mathcal{P}^{\bullet}_{e,i}$;
- $l_{e,i}(p) = \begin{cases} l_{\mathcal{G}(R)}(p) & \textit{if } p \in \mathcal{P}_0(R); \\ l_{e_1,i}(p') & \textit{if } p = (o_1, e_1) : p' \textit{ with } o_1 \in \mathcal{B}_0^{\geq i}(R); \end{cases}$
- $t_{e,i}$ is the extension of $t_{\mathcal{G}(R)}$ that associates with each $(o_1, e_1) : w' \in \mathcal{W}_{e,i}$, where $o_1 \in \mathcal{B}_0^{\geq i}(R)$, the port $\begin{cases} (o_1, e_1) : t_{e_1,i}(w') & \textit{if } w' \in \mathcal{W}_{e_1,i} \textit{ and } t_{e_1,i}(w') \notin \mathcal{P}^{\mathsf{f}}_{\circ}(B_R(o_1)); \\ b_R(o_1)(t_{e_1,i}(w')) & \textit{if } w' \in \mathcal{W}_{e_1,i} \textit{ and } t_{e_1,i}(w') \in \mathcal{P}^{\mathsf{f}}_{\circ}(B_R(o_1)); \\ b_R(o_1)(w') & \textit{if } w' \in \mathcal{P}^{\mathsf{f}}_{\bullet}(B_R(o_1)); \end{cases}$
- $\mathcal{L}_{e,i} = \mathcal{L}(\mathcal{G}(R)) \cup \bigcup_{o_1 \in \mathcal{B}_0^{\geq i}(R)} \bigcup_{e_1 \in e(o_1)} \{(o_1, e_1) : p; p \in \mathcal{L}_{e_1,i}\}$
- $\mathsf{T}_{e,i}(p) = \begin{cases} \mathsf{T}_{\mathcal{G}(R)}(p) & \textit{if } p \in \mathcal{P}_0(R); \\ \mathsf{T}_{e_1,i}(p') & \textit{if } p = (o_1, e_1) : p' \textit{ with } o_1 \in \mathcal{B}_0^{\geq i}(R); \end{cases}$
- $\mathcal{B}_{e,i} = \mathcal{B}_0^{<i}(R) \cup \bigcup_{o_1 \in \mathcal{B}_0^{\geq i}(R)} \bigcup_{e_1 \in e(o_1)} \{ (o_1, e_1) : o' ; o' \in \mathcal{B}_{e_1,i}\}$
- $B_{e,i}(o) = \begin{cases} B_R(o) & \textit{if } o \in \mathcal{B}_0^{<i}(R); \\ B_{e_1,i}(o') & \textit{if } o = (o_1, e_1) : o' \textit{ with } o_1 \in \mathcal{B}_0^{\geq i}(R); \end{cases}$
- $b_{e,i}$ is the extension of $b_R|_{\mathcal{B}_0^{<i}(R)}$ that associates with each $(o_1, e_1) : o' \in \mathcal{B}_{e,i}$, where $o_1 \in \mathcal{B}_0^{\geq i}(R)$, the function $p \mapsto \begin{cases} (o_1, e_1) : b_{e_1,i}(o')(p) & \textit{if } b_{e_1,i}(o')(p) \notin \mathcal{P}^{\mathsf{f}}_{\circ}(B_R(o_1)); \\ b_{e_1,i}(o')(p) & \textit{if } b_{e_1,i}(o')(p) \in \mathcal{P}^{\mathsf{f}}_{\circ}(B_R(o_1)). \end{cases}$

▶ **Example 15.** If $f$ is a pseudo-experiment of the proof-net $R$ of Figure 1 with $f^{\#}$ like in Example 13, then Figures 6 and 7 represent respectively $\mathcal{T}(f)[0]$ and $\mathcal{T}(f)[1]$.

The injectivity of the relational semantics for differential PS's of depth 0 is trivial (one can proceed by induction on the cardinality of the set of ports). Since $[\![\mathcal{T}(\overline{e})[0]]\!] = \{e_{|\mathcal{P}^{\mathsf{f}}(R)}\}$, one can easily identify the result $e_{|\mathcal{P}^{\mathsf{f}}(R)}$ of an experiment $e$ with the differential net $\mathcal{T}(\overline{e})[0]$:

[17] Boxes *of* depth $\geq i$ are boxes whose content is a proof-net of depth $\geq i$; the reader should not confuse *boxes of depth* $\geq i$ with *boxes at depth* $\geq i$.

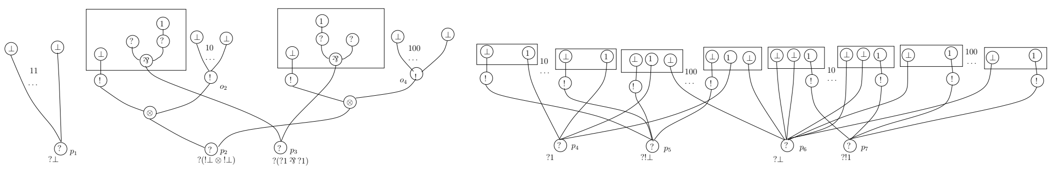

**Figure 7** $\mathcal{T}(f)[1]$.

▶ **Lemma 16.** *Let $R$ and $R'$ be two $\circ$-PS's such that $\mathcal{P}^{\mathsf{f}}(R) = \mathcal{P}^{\mathsf{f}}(R')$. Let $e$ be an experiment of $R$ and let $e'$ be an experiment of $R'$ such that $e_{|\mathcal{P}^{\mathsf{f}}(R)} = e'_{|\mathcal{P}^{\mathsf{f}}(R')}$. Then $\mathcal{T}(\overline{e})[0] \equiv \mathcal{T}(\overline{e'})[0]$.*

Now, the following fact shows that if we are able to recover $\mathcal{T}(\overline{e})[depth(R)]$ from $\mathcal{T}(\overline{e})[0]$, then we are done.

▶ **Fact 17.** *Let $R$ be a $\circ$-PS. Let $e$ be a pseudo-experiment of $R$. Then $\mathcal{T}(e)[depth(R)] = R$.*

If $e$ is a $k$-heterogeneous experiment of $R$, then, for any $i \in \mathbb{N}$, there exists a bijection $!_{e,i} : \bigcup_{o \in \mathcal{B}^{\geq i}(R)} \{\log_k(m); m \in e^{\#}(o)\} \simeq \mathcal{P}_0^!(\mathcal{T}(e)[i]) \setminus \mathcal{B}_0(\mathcal{T}(e)[i])$ such that, for any $j \in \mathsf{dom}(!_{e,i})$, we have $(a_{\mathcal{T}(e)[i]} \circ !_{e,i})(j) = k^j$. In Subsection 4.1, we will show how to recover $\bigcup_{o \in \mathcal{B}^{\geq i}(R)} \{\log_k(m); m \in e^{\#}(o)\}$ from $\mathcal{T}(e)[0]$. There are two kinds of boxes of $\mathcal{T}(e)[i+1]$ at depth 0: the "new" boxes of depth $i$ and the boxes of depth $< i$, which are the "old" boxes (i.e. that already were in $\mathcal{T}(e)[i]$) that do not go inside some "new" box:

▶ **Fact 18.** *Let $R$ be a $\circ$-PS. Let $e$ be a pseudo-experiment of $R$. Let $i \in \mathbb{N}$. Then we have*
- $\mathcal{B}_0^{<i}(\mathcal{T}(e)[i+1]) = \mathcal{B}_0(\mathcal{T}(e)[i]) \cap \mathcal{P}_0^!(\mathcal{T}(e)[i+1])$;
- $B_{\mathcal{T}(e)[i+1]}|_{\mathcal{B}_0^{<i}(\mathcal{T}(e)[i+1])} = B_{\mathcal{T}(e)[i]}|_{\mathcal{B}_0^{<i}(\mathcal{T}(e)[i+1])}$;
- *and* $b_{\mathcal{T}(e)[i+1]}|_{\mathcal{B}_0^{<i}(\mathcal{T}(e)[i+1])} = b_{\mathcal{T}(e)[i]}|_{\mathcal{B}_0^{<i}(\mathcal{T}(e)[i+1])}$.

The challenge is the rebuilding of the "new" boxes at depth 0 of depth $i$.

# 4 From $\mathcal{T}(e)[i]$ to $\mathcal{T}(e)[i+1]$

## 4.1 The outline of the boxes

In this subsection we first show how to recover the set $\bigcup_{o \in \mathcal{B}^{\geq i}(R)} \{\log_k(m); m \in e^{\#}(o)\}$ and, therefore, the set $\mathcal{P}_0^!(\mathcal{T}(e)[i]) \setminus \mathcal{B}_0(\mathcal{T}(e)[i])$ (Lemma 21). Next, we show how to determine, from $\mathcal{T}(e)[i]$, the set $\mathcal{B}_0^{=i}(\mathcal{T}(e)[i+1])$ of "new" boxes and, for any such "new" box $o \in \mathcal{B}_0^{=i}(\mathcal{T}(e)[i+1])$, the set $\mathsf{im}(b_{\mathcal{T}(e)[i+1]}(o))$ of exponential ports that are immediately below (Proposition 25). In particular, we have $\mathcal{B}_0^{=i}(\mathcal{T}(e)[i+1]) = !_{e,i}[\mathcal{N}_i(e)]$, where the set $\mathcal{N}_i(e) \subseteq \mathbb{N}$ is defined from the set $\mathcal{M}_0(e)$ of the numbers of copies of boxes taken by the pseudo–experiment $e$:

▶ **Definition 19.** Let $R$ be a differential $\circ$-PS. Let $k > 1$. Let $e$ be a $k$-heterogeneous pseudo-experiment of $R$. For any $i \in \mathbb{N}$, we define, by induction on $i$, $\mathcal{M}_i(e) \subseteq \mathbb{N} \setminus \{0\}$ and $(m_{i,j}(e))_{j \in \mathbb{N}} \in \{0, \ldots, k-1\}^{\mathbb{N}}$ as follows. We set $\mathcal{M}_0(e) = \bigcup_{o \in \mathcal{B}(R)} \{j \in \mathbb{N}; k^j \in e^{\#}(o)\}$ and we write $\mathsf{Card}\,(\mathcal{M}_i(e))$ in base $k$: $\mathsf{Card}\,(\mathcal{M}_i(e)) = \sum_{j \in \mathbb{N}} m_{i,j}(e) \cdot k^j$; we set $\mathcal{M}_{i+1}(e) = \{j > 0; m_{i,j}(e) \neq 0\}$.

For any $i \in \mathbb{N}$, we set $\mathcal{N}_i(e) = \mathcal{M}_i(e) \setminus \mathcal{M}_{i+1}(e)$.

Notice that all the sets $\mathcal{M}_i(e)$ and $\mathcal{N}_i(e)$ can be computed from $\mathcal{T}(e)[0]$, since we have $\mathcal{M}_0(e) = \{a_{\mathcal{T}(e)[0]}(p); p \in \mathcal{P}_0^!(\mathcal{T}(e)[0])\}$.

▶ **Example 20.** If $f$ is a 10-heterogeneous pseudo-experiment as in Example 13, then $\mathcal{M}_0(f) = \{1, \ldots, 224\}$. We have $\mathsf{Card}\,(\mathcal{M}_0(f)) = 4 + 2 \cdot 10^1 + 2 \cdot 10^2$, hence $\mathcal{M}_1(f) = \{1, 2\}$ and $\mathcal{N}_0(f) = \{3, \ldots, 224\}$. We have $\mathsf{Card}\,(\mathcal{M}_1(f)) = 2$, hence $\mathcal{M}_2(f) = \emptyset$ and $\mathcal{N}_1(f) = \{1, 2\}$.

The following lemma shows that, for any $k$-heterogeneous pseudo-experiment $e$ of $R$, for any $i \in \mathbb{N}$, the function $!_{e,i}$ is actually a bijection $\mathcal{M}_i(e) \to \mathcal{P}_0^!(\mathcal{T}(e)[i]) \setminus \mathcal{B}_0(\mathcal{T}(e)[i])$ such that, for any $j \in \mathcal{M}_i(e)$, we have $(a_{\mathcal{T}(e)[i]} \circ !_{\mathcal{T}(e)[i]})(j) = k^j$.

▶ **Lemma 21.** *Let $R$ be a $\circ$-PS. Let $k >$ Card$(\mathcal{B}(R))$. For any $k$-heterogeneous pseudo-experiment $e$ of $R$, for any $i \in \mathbb{N}$, we have $\mathcal{M}_i(e) = \bigcup_{o \in \mathcal{B}^{\geq i}(R)} \{j \in \mathbb{N}; k^j \in e^{\#}(o)\}$, hence $\mathcal{N}_i(e) = \bigcup_{o \in \mathcal{B}^{=i}(R)} \{j \in \mathbb{N}; k^j \in e^{\#}(o)\}$.*

▶ **Example 22** (Continuation of Example 20)**.** We thus have $\mathcal{M}_1(f) = \{1, 2\}$ and $\mathcal{P}_0^!(\mathcal{T}(f)[i]) \setminus \mathcal{B}_0(\mathcal{T}(f)[1]) = \{o_2, o_4\}$ with $a_{\mathcal{T}(f)[1]}(o_2) = 10^1$ and $a_{\mathcal{T}(f)[1]}(o_4) = 10^2$ (see Figure 7).

The set $\mathcal{K}_{k,\mathcal{N}_i(e)}(S)$ of "critical ports" is a set of exponential ports that will play a crucial role in our algorithm.

▶ **Definition 23.** Let $S$ be a differential $\circ$-PS. Let $k > 1$. For any $p \in \mathcal{P}_0(S)$, we define the sequence $(m_{k,j}(S)(p))_{j \in \mathbb{N}} \in \{0, \ldots, k-1\}^{\mathbb{N}}$ as follows: $a_S(p) = \sum_{j \in \mathbb{N}} m_{k,j}(S)(p) \cdot k^j$. For any $j \in \mathbb{N}$, we set $\mathcal{K}_{k,j}(S) = \{p \in \mathcal{P}_0(S); m_{k,j}(S)(p) \neq 0\} \cap \mathcal{P}^e(\mathcal{G}(S))$ and, for any $J \subseteq \mathbb{N}$, we set $\mathcal{K}_{k,J}(S) = \bigcup_{j \in J} \mathcal{K}_{k,j}(S)$.

▶ **Example 24.** We have $\mathcal{K}_{10,1}(S) = \{p_1, p_4, p_5, p_6, p_7, o_2\}$ and $\mathcal{K}_{10,2}(S) = \{p_4, p_5, p_6, p_7, o_4\}$, where $S$ is the PS of Figure 7. So we have $\mathcal{K}_{10,\{1,2\}}(S) = \{p_1, p_4, p_5, p_6, p_7, o_2, o_4\}$.

Critical ports are defined by their arities. We show that they are exponential ports that are immediately below the "new" boxes:

▶ **Proposition 25.** *Let $R$ be a $\circ$-PS. Let $k >$ Card$(\mathcal{B}(R))$, cosize$(R)$. Let $e$ be a $k$-heterogeneous pseudo-experiment of $R$ and let $i \in \mathbb{N}$. Then we have $\mathcal{B}_0^{=i}(\mathcal{T}(e)[i+1]) = !_{e,i}[\mathcal{N}_{i,e}]$. Furthermore, for any $j \in \mathcal{N}_i(e)$, we have $\mathsf{im}(b_{\mathcal{T}(e)[i+1]}(!_{e,i}(j))) = \mathcal{K}_{k,j}(\mathcal{T}(e)[i])$ and, if $!_{e,i}(j) \notin \mathcal{B}_0^{=i}(R)$, then there exist $o_1 \in \mathcal{B}_0^{\geq i+1}(R)$ and $e_1 \in e(o_1)$ such that $j \in \mathcal{N}_i(e_1)$. In particular, we have $\mathcal{K}_{k,\mathcal{N}_i(e)}(\mathcal{T}(e)[i]) \subseteq \mathcal{P}_0^e(\mathcal{T}(e)[i+1])$.*

In particular, this proposition highlights one more essential difference between the $k$-experiments of [23, 24, 9, 13] and our $k$-heterogeneous experiments. There, such a $k$-experiment labelling some contraction $p$ with a multiset of cardinality $\sum_j m_j \cdot k^j$ (where $0 \leq m_j < k$ for any $j$) gives the information that immediately above the contraction $p$ there are exactly $m_{j_0}$ series of exactly $j_0$ auxiliary doors. Here, whenever a $k$-heterogeneous experiment labels some contraction $p$ with a multiset of cardinality $\sum_j m_j \cdot k^j$ (where $0 \leq m_j < k$ for any $j$), the integer $j_0$ is not related to the number of auxiliary doors in series anymore; it corresponds, in the case $m_{j_0} > 0$, with the existence of a box, whose some occurrence takes $k^{j_0}$ copies of its content, having, among all its auxiliary doors, exactly $m_{j_0}$ auxiliary doors that are, each of them, *the first one* (i.e. the deepest one) of a series of auxiliary doors immediately above the contraction $p$.

▶ **Example 26** (Continuation of Example 22)**.** We thus have $!_{f,1}[\mathcal{N}_1(f)] = \{o_2, o_4\}$; and indeed $o_2$ and $o_4$ are the boxes of depth 1 at depth 0 of $\mathcal{T}(f)[2] \equiv R$ (see Figure 1). Moreover we have $\mathcal{K}_{10,1}(\mathcal{T}(f)[1]) = \{p_1, p_4, p_5, p_6, p_7, o_2\}$ and $\mathcal{K}_{10,2}(\mathcal{T}(f)[1]) = \{p_4, p_5, p_6, p_7, o_4\}$; and indeed, in Figure 1, we have $\mathsf{im}(b_{\mathcal{T}(f)[2]}(o_2)) = \{p_1, p_4, p_5, p_6, p_7, o_2\}$ and $\mathsf{im}(b_{\mathcal{T}(f)[2]}(o_4)) = \{p_4, p_5, p_6, p_7, o_4\}$.

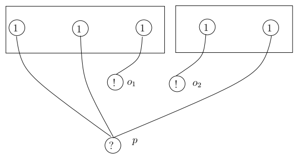

**Figure 8** $R_1$.

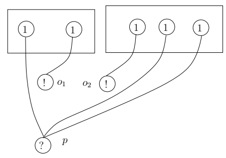

**Figure 9** $R_2$.

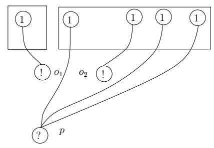

**Figure 10** $R_3$.

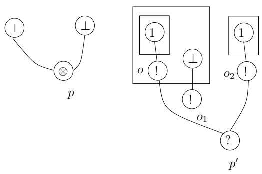

**Figure 11** PS $S$.

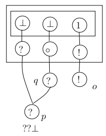

**Figure 12** PS $R'$.

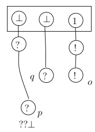

**Figure 13** PS $U$.

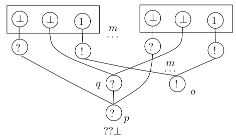

**Figure 14** PS $\mathcal{T}(e')[1]$.

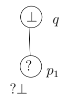

**Figure 15** PS $T$.

As the following example shows, the information we obtain is already strong, but not strong enough.

▶ **Example 27.** The PS's $R_1$, $R_2$ and $R_3$ of Figures 8, 9 and 10 respectively have the same LPS. But if we know that $p \in \mathsf{im}(b_R(o_1))$, then we know that $R \neq R_3$. Still we are not able to distinguish between $R_1$ and $R_2$.

## 4.2 Connected components

In order to rebuild the content of the boxes, we introduce our notion of *connected component* (Definition 32), which uses the auxiliary notions of *substructure* (Definition 28) and *connected substructure* (Definition 31). A differential $\circ$-PS $R$ is a substructure of a differential $\circ$-PS $S$ (we write $R \sqsubseteq S$) if $R$ is obtained from $S$ by erasing some ports and wires. More precisely:

▶ **Definition 28.** Let $R$ and $S$ be two differential $\circ$-PS's. Let $\mathcal{Q} \subseteq \mathcal{P}_0^e(S)$. We write $R \sqsubseteq_{\mathcal{Q}} S$ to denote that $\mathcal{P}_0(R) \subseteq \mathcal{P}_0(S)$, $\mathcal{W}_0(R) = \{w \in \mathcal{W}_0(S) \cap (\mathcal{P}_0(R) \setminus \mathcal{Q}); t_{\mathcal{G}(S)}(w) \in \mathcal{P}_0(R)\}$, $l_{\mathcal{G}(R)} = l_{\mathcal{G}(S)}|_{\mathcal{P}_0(R)}$, $t_{\mathcal{G}(R)} = t_{\mathcal{G}(S)}|_{\mathcal{W}_0(R)}$, $\mathcal{L}(\mathcal{G}(R)) = \mathcal{L}(\mathcal{G}(S)) \cap \{w \in \mathcal{W}_0(S); t_{\mathcal{G}(S)}(w) \in \mathcal{P}_0^m(R)\}$, $\mathsf{T}_{\mathcal{G}(R)} = \mathsf{T}_{\mathcal{G}(S)}|_{\mathcal{P}_0(R)}$, $\mathcal{B}_0(R) = \mathcal{B}_0(S) \cap \mathcal{P}_0(R)$, $B_R = B_S|_{\mathcal{B}_0(R)}$ and $b_R = b_S|_{\mathcal{B}_0(R)}$. We write $R \sqsubseteq S$ if there exists $\mathcal{Q}$ such that $R \sqsubseteq_{\mathcal{Q}} S$.

▶ Remark 1. Let $R \sqsubseteq S$ and $\mathcal{Q} \subseteq \mathcal{P}_0^e(S)$. We have $R \sqsubseteq_{\mathcal{Q}} S$ iff $\mathcal{Q} \cap \mathcal{P}_0(R) \subseteq \mathcal{P}^{\mathsf{f}}(R)$.

▶ Remark 2. If $R, R' \sqsubseteq_{\mathcal{Q}} S$ and $\mathcal{P}_0(R) = \mathcal{P}_0(R')$, then $R = R'$.

Let us explain with the following example why we sometimes need to erase some wires (so the notion of $\sqsubseteq_\emptyset$ is not enough).

▶ **Example 29.** We need to erase some wires whenever there exist a box $o$ and $p, q \in \mathsf{im}(b_{R'}(o))$ such that $t_{\mathcal{G}(R')}(q) = p$. Consider, for instance, Figure 12. If $e'$ is a $k$-heterogeneous pseudo-experiment of $R'$, then we want to be able to consider the PS $U$ of Figure 13 as a substructure of $\mathcal{T}(e')[1]$, so we need to erase the wire $q$; we thus have $U \sqsubseteq_{\{p,q,o\}} \mathcal{T}(e')[1]$.

The relation $\circ\!\!-_S$ formalizes the notion of “connectedness" between two ports of $S$ at depth 0. But be aware that, here, “connected” has nothing to do with “connected” in the sense of [9]: here, any two doors of the same box are always “connected”.

▶ **Definition 30.** Let $S$ be a differential $\circ$-PS. We define the binary relation $\circ\!\!-_S$ on $\mathcal{P}_0(S)$ as follows: for any $p, p' \in \mathcal{P}_0(S)$, we have $p \circ\!\!-_S p'$ iff ($p \in \mathcal{W}_0(S)$ and $p' = t_{\mathcal{G}(S)}(p)$) or ($p' \in \mathcal{W}_0(S)$ and $p = t_{\mathcal{G}(S)}(p')$) or $((\exists o \in \mathcal{B}_0(S))\{p, p'\} \subseteq \mathsf{im}(b_S(o)))$.

▶ **Definition 31.** Let $S$ and $T$ be two differential $\circ$-PS's. Let $\mathcal{Q} \subseteq \mathcal{P}_0^e(S)$ such that $T \sqsubseteq_{\mathcal{Q}} S$. We write $T \trianglelefteq_{\mathcal{Q}} S$ if, for any $p, p' \in \mathcal{P}_0(T)$, there exists a finite sequence $(p_0, \ldots, p_n)$ of elements of $\mathcal{P}_0(T)$ such that $p_0 = p$, $p_n = p'$ and, for any $j \in \{0, \ldots, n-1\}$, we have $p_j \frown_S p_{j+1}$ and $(p_j \in \mathcal{Q} \Rightarrow j = 0)$.

▶ Remark 3. If $T \trianglelefteq_{\mathcal{Q}} S$, $\mathcal{Q}' \subseteq \mathcal{P}_0^e(S)$ and $\mathcal{P}_0(T) \cap \mathcal{Q}' \subseteq \mathcal{Q}$, then $T \trianglelefteq_{\mathcal{Q}'} S$.

The sets $\mathcal{S}_S^k((\mathcal{Q}, \mathcal{Q}_0))$ of "components $T$ of $S$ above $\mathcal{Q}$ and $\mathcal{Q}_0$ that are connected *via* other ports than $\mathcal{Q}$ and such that $cosize(T) < k$" will play a crucial role in the algorithm of the rebuilding of $\mathcal{T}(\overline{e})[i+1]$ from $\mathcal{T}(\overline{e})[i]$. The reader already knows that, here, "connected" has nothing to do with the "connected proof-nets" of [9]: there, the crucial tool used was rather the "bridges" that put together two doors of the same copy of some box only if they are connected in the LPS of the proof-net.

▶ **Definition 32.** Let $k \in \mathbb{N}$. Let $S$ be a differential $\circ$-PS. Let $\mathcal{Q} \subseteq \mathcal{P}_0^e(S)$ and $\mathcal{Q}_0 \subseteq \mathcal{P}_0(S)$. We set $\mathcal{S}_S^k((\mathcal{Q}, \mathcal{Q}_0)) = \left\{ T \trianglelefteq_{\mathcal{Q}} S;\ \begin{array}{c} cosize(T) < k \text{ and } \mathcal{P}^{\mathsf{f}}(T) \subseteq \mathcal{Q} \cup \mathcal{Q}_0 \text{ and} \\ \mathcal{P}_0(T) \setminus \mathcal{Q} \neq \emptyset \text{ and } (\forall p \in \mathcal{P}_0(T))(\forall q \in \mathcal{P}_0(S)) \\ ((p \frown_S q \text{ and } q \notin \mathcal{P}_0(T)) \Rightarrow p \in \mathcal{Q}) \end{array} \right\}$. We write also $\mathcal{S}_S^k(\mathcal{Q})$ instead of $\mathcal{S}_S^k((\mathcal{Q}, \emptyset))$ and $\mathcal{C}^k(S) = \mathcal{S}_S^k((\mathcal{P}_\circ^{\mathsf{f}}(S), \mathcal{P}_\bullet^{\mathsf{f}}(S)))$.

A port at depth 0 of $S$ that is not in $\mathcal{Q}$ cannot belong to two different components:

▶ **Fact 33.** *Let $k \in \mathbb{N}$. Let $S$ be a differential $\circ$-PS. Let $\mathcal{Q}, \mathcal{Q}_0 \subseteq \mathcal{P}_0(S)$. Let $T, T' \in \mathcal{S}_S^k((\mathcal{Q}, \mathcal{Q}_0))$ such that $(\mathcal{P}_0(T) \cap \mathcal{P}_0(T')) \setminus \mathcal{Q} \neq \emptyset$. Then $T = T'$.*

▶ **Example 34.** We have $\mathsf{Card}\left(\mathcal{S}_S^{10}(\{p_1, p_4, p_5, p_6, p_7, o_2\})\right) = 241$ and $\mathsf{Card}\left(\mathcal{S}_S^{10}(\{p_4, p_5, p_6, p_7, o_4\})\right) = 320$, where $S$ is the PS of Figure 7.

The operator $\sum$ glues together several $\circ$-PS's that share only $\circ$-conclusions:

▶ **Definition 35.** Let $\mathcal{U}$ be a set of $\circ$-PS's. We say that $\mathcal{U}$ is *gluable* if, for any $R, S \in \mathcal{U}$ s.t. $R \neq S$, we have $\mathcal{P}_0(R) \cap \mathcal{P}_0(S) \subseteq \mathcal{P}_\circ^{\mathsf{f}}(R) \cap \mathcal{P}_\circ^{\mathsf{f}}(S)$. If $\mathcal{U}$ is gluable, then $\sum \mathcal{U}$ is the $\circ$-PS such that $\mathcal{P}_\circ^{\mathsf{f}}(\sum \mathcal{U}) = \bigcup \{\mathcal{P}_\circ^{\mathsf{f}}(R); R \in \mathcal{U}\}$ obtained by glueing all the elements of $\mathcal{U}$.

The set $\mathcal{C}^k(R)$ (for $k$ big enough) is an alternative way to describe a $\circ$-PS $R$:

▶ **Fact 36.** *Let $R$ be a $\circ$-PS. Let $k > cosize(R)$. We have $R = \sum \mathcal{C}^k(R)$.*

Definition 37 allows to formalize the operation of "putting a connected component inside a box", which will be useful for building the boxes of depth $i$ of $\mathcal{T}(\overline{e})[i+1]$: from some *boxable* differential $\circ$-PS $R \sqsubseteq \mathcal{T}(\overline{e})[i]$, we build a $\circ$-PS $\overline{R}$ such that, for some $o \in \mathcal{B}_0^{=i}(\mathcal{T}(\overline{e})[i+1])$, there exists $T \in \mathcal{C}^k(B_{\mathcal{T}(\overline{e})[i+1]}(o))$ such that $T \simeq \overline{R}$.

▶ **Definition 37.** Let $R$ be a differential $\circ$-PS. If $\mathcal{P}^{\mathsf{f}}(R) \subseteq \mathcal{P}_0^e(R)$, $\mathcal{P}^{\mathsf{f}}(R) \cap \mathcal{B}_0(R) = \emptyset$ and $\mathcal{P}_0^!(R) \setminus \mathcal{B}_0(R) \subseteq \mathcal{P}^{\mathsf{f}}(R)$, then one says that $R$ is *boxable* and we define a $\circ$-PS $\overline{R}$ s.t. $\mathcal{P}_0(\overline{R}) \subseteq \mathcal{P}_0(R)$, $\mathcal{P}_\bullet^{\mathsf{f}}(\overline{R}) \subseteq \mathcal{W}_0(R)$ and $\mathcal{P}_\circ^{\mathsf{f}}(\overline{R}) = \mathcal{P}^{\mathsf{f}}(R) \cap \mathcal{P}_0(\overline{R})$ as follows:

- $\mathcal{P}_0(\overline{R}) = \mathcal{W}_0(R) \cup \bigcup_{o \in \mathcal{B}_0(R)} (\mathsf{im}(b_R(o)) \cap \mathcal{P}^{\mathsf{f}}(R))$;
- $\mathcal{W}_0(\overline{R}) = \{w \in \mathcal{W}_0(R); t_{\mathcal{G}(R)}(w) \in \mathcal{W}_0(R)\}$
- $l_{\mathcal{G}(\overline{R})}(p) = \begin{cases} l_{\mathcal{G}(R)}(p) & \text{if } p \in \mathcal{W}_0(R); \\ \circ & \text{otherwise;} \end{cases}$
- $t_{\mathcal{G}(\overline{R})} = t_{\mathcal{G}(R)}|_{\mathcal{W}_0(\overline{R})}$; $\mathcal{L}(\mathcal{G}(\overline{R})) = \mathcal{L}(\mathcal{G}(R)) \cap \mathcal{W}_0(\overline{R})$; $\mathcal{B}_0(\overline{R}) = \mathcal{B}_0(R)$; $b_{\overline{R}} = b_R$.

If $\mathcal{U}$ is a set of boxable differential $\circ$-PS's, then we set $\overline{\mathcal{U}} = \{\overline{R}; R \in \mathcal{U}\}$.

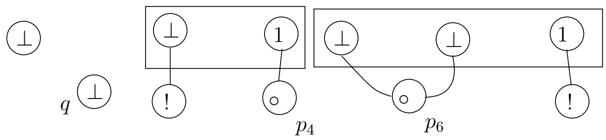

**Figure 16** The $\circ$-PS $\sum \overline{\mathcal{U}_{j_0}}$ of Example 39.

In the proof of the following proposition, we finally describe the complete algorithm leading from $\mathcal{T}(e)[i]$ to $\mathcal{T}(e)[i+1]$. Informally: for every $j_0 \in \mathcal{N}_i(e)$, for every equivalence class $\mathfrak{T} \in \mathcal{S}^k_{\mathcal{T}(e)[i]}(\mathcal{K}_{k,j_0}(\mathcal{T}(e)[i]))_{/\equiv}$, if $\mathsf{Card}\,(\mathfrak{T}) = \sum_{j \in \mathbb{N}} m_j \cdot k^j$ (with $0 \leq m_j < k$), then we remove $m_{j_0} \cdot k^{j_0}$ elements of $\mathfrak{T}$ from $\mathcal{G}(\mathcal{T}(e)[i])$ and we put $m_{j_0}$ such elements inside the (new) box $!_{e,i}(j_0)$ of depth $i$. For every $j_0 \in \mathcal{N}_i(e)$, the set $\mathcal{U}_{j_0}$ of the proof is the union of the sets of such $m_{j_0}$ elements for all the equivalence classes $\mathfrak{T} \in \mathcal{S}^k_{\mathcal{T}(e)[i]}(\mathcal{K}_{k,j_0}(\mathcal{T}(e)[i]))_{/\equiv}$.

▶ **Proposition 38.** *Let $R$ and $R'$ be two PS's. Let $k >$ cosize$(R)$, cosize$(R')$, $\mathsf{Card}\,(\mathcal{B}(R))$, $\mathsf{Card}\,(\mathcal{B}(R'))$. Let $e$ be a $k$-heterogeneous pseudo-experiment of $R$ and let $e'$ be a $k$-heterogeneous pseudo-experiment of $R'$ s.t. $\mathcal{T}(e)[0] \equiv \mathcal{T}(e')[0]$. Then, for any $i \in \mathbb{N}$, we have $\mathcal{T}(e)[i] \equiv \mathcal{T}(e')[i]$.*

**Proof (Sketch).** By induction on $i$. We assume that $\mathcal{T}(e)[i] \equiv \mathcal{T}(e')[i]$. We set $\mathcal{M} = \mathcal{M}_i(e) = \mathcal{M}_i(e')$ and $\mathcal{N} = \mathcal{N}_i(e) = \mathcal{N}_i(e')$.

Let $S \equiv \mathcal{T}(e)[i]$. There is a bijection $! : \mathcal{M} \to \mathcal{P}^!_0(S) \setminus \mathcal{B}_0(S)$ such that, for any $j \in \mathcal{M}$, we have $(a_S \circ !)(j) = k^j$. For any $j \in \mathcal{N}$, we set $\mathcal{K}_j = \mathcal{K}_{k,j}(S)$ and $\mathcal{T}_j = \mathcal{S}^k_S(\mathcal{K}_j)$. We set $\mathcal{T} = \bigcup_{j \in \mathbb{N}} \mathcal{T}_j$. For any $T \in \mathcal{T}$, we define $(m^T_j)_{j \in \mathbb{N}} \in \{0, \ldots, k-1\}^{\mathbb{N}}$ as follows: $\mathsf{Card}\,(\{T' \in \mathcal{T}; T \equiv T'\}) = \sum_{j \in \mathbb{N}} m^T_j \cdot k^j$. We set $\mathcal{P} = \{p \in \mathcal{P}_0(S);\ p \notin \bigcup_{j \in \mathcal{N}} \bigcup_{T \in \mathcal{S}^k_S(\mathcal{K}_j)} \mathcal{P}_0(T)\}$. For any $j \in \mathcal{N}$, we are given $\mathcal{U}_j \subseteq \mathcal{T}_j$ such that, for any $T \in \mathcal{T}_j$, we have $\mathsf{Card}\,(\{T' \in \mathcal{U}_j; T' \equiv T\}) = m^T_j$. Let $S'$ be some differential PS such that $\mathcal{G}(S_{\restriction \mathcal{P}}) \sqsubseteq_\emptyset \mathcal{G}(S') \sqsubseteq_\emptyset \mathcal{G}(S)$, where $S_{\restriction \mathcal{P}}$ is the unique $S_0 \sqsubseteq_\emptyset S'$ s.t. $\mathcal{P}_0(S_0) = \mathcal{P}$, and:

- $\bigcup_{j_0 \in \mathcal{N}} \mathcal{T}'_{j_0} \subseteq \bigcup_{j_0 \in \mathcal{N}} \mathcal{T}_{j_0}$
- for any $T \in \mathcal{T}$, we have $\mathsf{Card}\,\left(\{T' \in \bigcup_{j \in \mathbb{N}} \mathcal{S}^k_{S'}(\mathcal{K}_j); T' \equiv T\}\right) = \sum_{j \notin \mathcal{N}} m^T_j \cdot k^j$;
- $\mathcal{B}_0(S') = (\mathcal{B}_0(S) \cap \mathcal{P}_0(S')) \cup ![\mathcal{N}]$
- for any $o_1 \in \mathcal{B}_0(S) \cap \mathcal{P}_0(S')$, we have $B_{S'}(o_1) = B_S(o_1)$ and $b_{S'}(o_1) = b_S(o_1)$
- for any $j \in \mathcal{N}$, there exists $\rho^j : B_{S'}(!(j)) \simeq \sum \overline{\mathcal{U}_j}$ such that $b_{S'}(!(j))(q) = \begin{cases} \mathcal{G}(\rho^j)(q) & \text{if } q \in \mathcal{P}^f_\circ(B_{S'}(!(j))); \\ t_{\mathcal{G}(S)}(\mathcal{G}(\rho^j)(q)) & \text{if } q \in \mathcal{P}^f_\bullet(B_{S'}(!(j))). \end{cases}$

Then one can show that $\mathcal{T}(e)[i+1] \equiv S' \equiv \mathcal{T}(e')[i+1]$. ◀

▶ **Example 39.** Consider Figure 7. We set $j_0 = 1$. We have $o_2 = !_{f,1}(j_0)$. We set $\mathcal{T}_{j_0} = \mathcal{S}^{10}_{\mathcal{T}(f)[1]}(\mathcal{K}_{10,1}(\mathcal{T}(f)[1]))$. Let $T$ be the $\circ$-PS of Figure 15. We have $T \in \mathcal{T}_{j_0}$ and $\mathsf{Card}\,(\{T' \in \mathcal{T}; T' \equiv T\}) = 1 \cdot 10^0 + 1 \cdot 10^1$. So we can take $\mathcal{U}_{j_0}$ in such a way that $\{T' \in \mathcal{U}_{j_0}; T' \equiv T\} = \{T\}$. There exists $\rho^{j_0} : B_{\mathcal{T}(f)[2]}(o_2) \simeq \sum \overline{\mathcal{U}_{j_0}}$ (see Figure 16) such that $b_{\mathcal{T}(f)[2]}(o_2)(q_2) = p_1 = t_{\mathcal{G}(\mathcal{T}(f)[1])}(\mathcal{G}(\rho^{j_0})(q_2))$ (see Figure 1): we set $\mathcal{G}(\rho^{j_0})(q_2) = q$.

## 4.3 Injectivity

▶ **Theorem 40.** *Let $R$ and $R'$ be two PS's s.t. $\mathcal{P}^f(R) = \mathcal{P}^f(R')$. If $[\![R]\!] = [\![R']\!]$, then $R \equiv R'$.*

**Proof.** We set $d = \max\{depth(R), depth(R')\}$. For any $k > 1$, there exist a $k$-heterogeneous experiment $e$ of $R$ and a $k$-heterogeneous experiment $e'$ of $R'$[18] such that $e_{|\mathcal{P}^{\mathsf{f}}(R)} = e'|_{\mathcal{P}^{\mathsf{f}}(R')} \in [\![R]\!] \cap [\![R']\!]$. By Lemma 16, we have $\mathcal{T}(\overline{e})[0] \equiv \mathcal{T}(\overline{e'})[0]$. Therefore if $k > cosize(R), \mathsf{Card}\,(\mathcal{B}(R))$, then, by Proposition 38, we have $\mathcal{T}(\overline{e})[d] \equiv \mathcal{T}(\overline{e'})[d]$. Now, by Fact 17, we have $R = \mathcal{T}(\overline{e})[d] \equiv \mathcal{T}(\overline{e'})[d] = R'$. ◀

▶ Remark 4. From any 1-point of $[\![R]\!]$ (i.e. the result of some 1-experiment of $R$) one can recover $cosize(R)$ and $\mathsf{Card}\,(\mathcal{B}(R))$. This remark shows that for characterizing $R$, two points are enough: a 1-point of its interpretation, from which one can bound $cosize(R)$ and $\mathsf{Card}\,(\mathcal{B}(R))$, and a $k$-heterogeneous point of its interpretation with $k > cosize(R), \mathsf{Card}\,(\mathcal{B}(R))$.

▶ Remark 5. If we want to extend our theorem to PS's with axioms, then we assume that the interpretation of any ground type is an infinite set and we consider a $k$-heterogeneous experiment $e$ that is "injective" in the sense that every atom (the atoms are the elements of the interpretations of the ground types) occurring in $e[\mathcal{P}^{\mathsf{f}}(R)]$ occurs exactly twice.

▶ Remark 6. In an untyped framework with axioms, we need to add the constraint on the injective $k$-heterogeneous point one considers to be $[\![R]\!]$-atomic, i.e. a point of $[\![R]\!]$ that cannot be obtained from another point of $[\![R]\!]$ by some substitution that is not a renaming. Atomic points are results of atomic experiments (experiments that label axioms with atoms) – the converse does not necessarily hold.

▶ Conclusion. We showed the injectivity of the relational semantics for MELL proof-nets by showing the injectivity of the Taylor expansion for cut-free MELL proof-nets, i.e. two different cut-free MELL proof-nets have different Taylor expansions; and, more precisely, we showed that, for any cut-free MELL proof-net, two simple differential nets in its Taylor expansion are enough to recover the entire proof-net. As a reviewer pointed out, it is worth noticing that the same proof should work in presence of cuts, i.e. our proof is a proof of a stronger result: the Taylor expansion of MELL proof-nets is injective (and two simple differential nets in the Taylor expansion of any MELL proof-net are enough to recover the entire proof-net).

## References


1 Pierre Boudes. Unifying static and dynamic denotational semantics. Technical report, Institut de Mathématiques de Luminy, 2004.

2 Antonio Bucciarelli, Delia Kesner, and Simona Ronchi Della Rocca. The inhabitation problem for non-idempotent intersection types. In Josep Diaz, Ivan Lanese, and Davide Sangiorgi, editors, *Theoretical Computer Science – 8th IFIP TC 1/WG 2.2 International Conference, TCS 2014, Rome, Italy, September 1-3, 2014. Proceedings*, volume 8705 of *Lecture Notes in Computer Science*, pages 341–354. Springer, 2014.

3 Vincent Danos. *La Logique Linéaire appliquée à l'étude de divers processus de normalisation.* Ph.D. thesis, Université Paris 7, 1990.

4 Vincent Danos and Laurent Regnier. Proof-nets and the Hilbert space. In *Advances in Linear Logic*, volume 222 of *London Math. Soc. Lecture Note Ser.* Cambridge University Press, 1995.

5 René David and Walter Py. $\lambda\mu$-calculus and Böhm's theorem. *Journal of Symbolic Logic*, 66(1):407–413, March 2001.


[18] This is not necessary true for the multiset based coherent semantics.

6 Daniel de Carvalho. *Sémantiques de la logique linéaire et temps de calcul*. Ph.D. thesis, Université Aix-Marseille II, 2007.

7 Daniel de Carvalho. Execution time of lambda-terms via denotational semantics and intersection types. RR 6638, INRIA, 2008. To appear in Mathematical Structures in Computer Science.

8 Daniel de Carvalho, Michele Pagani, and Lorenzo Tortora de Falco. A semantic measure of the execution time in linear logic. *Theoretical Computer Science*, 412/20:1884–1902, 2011.

9 Daniel de Carvalho and Lorenzo Tortora de Falco. The relational model is injective for multiplicative exponential linear logic (without weakenings). *Annals of Pure and Applied Logic*, 163(9):1210–1236, 2012.

10 Thomas Ehrhard and Laurent Regnier. Differential interaction nets. *Theoretical Computer Science*, 364:166–195, 2006.

11 Harvey Friedman. Equality between functionals. *Lecture Notes in Mathematics*, 453, 1975.

12 Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.

13 Giulio Guerrieri, Lorenzo Tortora De Falco, and Luc Pellissier. Injectivity of relational semantics for (connected) mell proof-nets via taylor expansion, 2014. URL: `https://hal.archives-ouvertes.fr/hal-00998847`.

14 Willem Heijltjes and Robin Houston. No proof nets for MLL with units: proof equivalence in MLL is pspace-complete. In Thomas A. Henzinger and Dale Miller, editors, *Joint Meeting of the Twenty-Third EACSL Annual Conference on Computer Science Logic (CSL) and the Twenty-Ninth Annual ACM/IEEE Symposium on Logic in Computer Science (LICS), CSL-LICS'14, Vienna, Austria, July 14-18, 2014*, pages 50:1–50:10. Association for Computing Machinery, 2014.

15 Dominic J. D. Hughes and Rob J. van Glabbeek. Proof nets for unit-free multiplicative-additive linear logic (extended abstract). In *Proceedings of the Eighteenth Annual IEEE Symposium on Logic in Computer Science (LICS 2003)*, pages 1–10. IEEE Computer Society Press, 2003.

16 Gianfranco Mascari and Marco Pedicini. Head linear reduction and pure proof net extraction. *Theor. Comput. Sci.*, 135(1), 1994.

17 Satoshi Matsuoka. Weak typed böhm theorem on imll. *Annals of Pure and Applied Logic*, 145(1):37–90, 2007.

18 Damiano Mazza and Michele Pagani. The separation theorem for differential interaction nets. In *Proceedings of the 14th LPAR International Conference*, volume 4790 of *LNAI*. Springer, 2007.

19 Michele Pagani. Proofs, denotational semantics and observational equivalences in multiplicative linear logic. *Mathematical Structures in Computer Science*, 17(2):341–359, 2007. `doi:10.1017/S0960129506005652`.

20 Michele Pagani and Christine Tasson. The taylor expansion inverse problem in linear logic. In *Proceedings of the 24th ANNUAL IEEE Symposium on Logic in Computer Science*, 2009.

21 Simona Ronchi Della Rocca. Principal type scheme and unification for intersection type discipline. *Theoretical Computer Science*, 59:181–209, 1988.

22 Alexis Saurin. Separation with streams in the lambda-mu-calculus. In *Proceedings of the 20th Annual IEEE Symposium on Logic in Computer Science*, 2005.

23 Lorenzo Tortora de Falco. *Réseaux, cohérence et expériences obsessionnelles*. Ph.D. thesis, Université Paris 7, January 2000.

24 Lorenzo Tortora de Falco. Obsessional experiments for linear logic proof-nets. *Mathematical Structures in Computer Science*, 13(6):799–855, 2003.

## A With axioms (Remark 5)

With axioms, we need to slightly modify Definition 11, since different experiments can induce the same pseudo-experiment:

▶ **Definition 41.** Given an experiment $e$ of some differential $\circ$-PS $R$, we define, by induction on $depth(R)$, a pseudo-experiment $\overline{e}$ of $R$ as follows: $\overline{e}(\varepsilon) = (R, 1)$ and

$$\overline{e}(o) = \left\{ \overline{f}[\varepsilon \mapsto (B_R(o), i)]; f \in Supp(\mathcal{B}(e)(o)) \textit{ and } 1 \leq i \leq \sum_{\substack{g \in Supp(\mathcal{B}(e)(o)) \\ \overline{g} = \overline{f}}} \mathcal{B}(e)(o)(g) \right\}$$

for any $o \in \mathcal{B}_0(R)$.

Notice that, if there is no axiom, Definitions 11 and **??** induce the same pseudo-experiment $\overline{e}$ for an experiment $e$.

## B Untyped framework (Remark 6)

Since there is no type, we define (differential) ground-structures *via* the auxiliary definition of *(differential) pre-ground-structures*. We set $\mathfrak{T}' = \{\otimes, \invamp, 1, \bot, !, ?, \circ, ax\}$.

▶ **Definition 42.** A *differential pre-ground-structure* is a 6-tuple $\mathcal{G} = (\mathcal{W}, \mathcal{P}, l, t, \mathcal{L}, \mathcal{A})$, where

- $\mathcal{P}$ is a finite set; the elements of $\mathcal{P}(\mathcal{G})$ are the *ports of* $\mathcal{G}$;
- $l$ is a function $\mathcal{P} \to \mathfrak{T}'$;the element $l(p)$ of $\mathfrak{T}'$ is the *label of* $p$ *in* $\mathcal{G}$;
- $\mathcal{W}$ is a subset of $\{p \in \mathcal{P}; l(p) \neq \circ\}$; the elements of $\mathcal{W}(\mathcal{G})$ are the *wires of* $\mathcal{G}$;
- $t$ is a function $\mathcal{W} \to \{p \in \mathcal{P}; l(p) \notin \{1, \bot, ax\}\}$ such that, for any port $p$ of $\mathcal{G}$, we have $(l(p) \in \{\otimes, \invamp\} \Rightarrow \mathsf{Card}(\{w \in \mathcal{W}; t(w) = p\}) = 2)$; if $t(w) = p$, then $w$ is a *premise of* $p$; the *arity* $a_{\mathcal{G}}(p)$ *of* $p$ is the number of its premises;
- $\mathcal{L}$ is a subset of $\{w \in \mathcal{W}; l(t(w)) \in \{\otimes, \invamp\}\}$ such that $(\forall p \in \mathcal{P})$ $(l(p) \in \{\otimes, \invamp\} \Rightarrow \mathsf{Card}(\{w \in \mathcal{L}; t(w) = p\}) = 1)$; if $w \in \mathcal{L}$ s.t. $t(w) = p$, then $w$ is *the left premise of* $p$;
- and $\mathcal{A}$ is a partition of $\{p \in \mathcal{P}; l(p) = ax\}$ such that, for any $a \in \mathcal{A}$, $\mathsf{Card}(a) = 2$; the elements of $\mathcal{A}$ are the *axioms of* $\mathcal{G}$.

We set $\mathcal{W}(\mathcal{G}) = \mathcal{W}$, $\mathcal{P}(\mathcal{G}) = \mathcal{P}$, $l_{\mathcal{G}} = l$, $t_{\mathcal{G}} = t$, $\mathcal{L}(\mathcal{G}) = \mathcal{L}$, $\mathcal{A}(\mathcal{G}) = \mathcal{A}$. The set $\mathcal{P}^{\mathrm{f}}(\mathcal{G}) = \mathcal{P} \setminus \mathcal{W}$ is the set of *conclusions of* $\mathcal{G}$. For any $t \in \mathfrak{T}'$, we set $\mathcal{P}^t(\mathcal{G}) = \{p \in \mathcal{P}; l(p) = t\}$; we set $\mathcal{P}^m(\mathcal{G}) = \mathcal{P}^{\otimes}(\mathcal{G}) \cup \mathcal{P}^{\invamp}(\mathcal{G})$; the set $\mathcal{P}^e(\mathcal{G})$ of *exponential ports of* $\mathcal{G}$ is $\mathcal{P}^!(\mathcal{G}) \cup \mathcal{P}^?(\mathcal{G}) \cup \mathcal{P}^\circ(\mathcal{G})$.

A *pre-ground-structure* is a differential pre-ground-structure $\mathcal{G}$ such that $\mathsf{im}(t_{\mathcal{G}}) \cap (\mathcal{P}^!(\mathcal{G}) \cup \mathcal{P}^\circ(\mathcal{G})) = \emptyset$.

A *differential ground-structure* (resp. a *ground-structure*) is a differential pre-ground structure (resp. a pre-ground structure) $\mathcal{G}$ such that the reflexive transitive closure $<_{\mathcal{G}}$ of the binary relation $<$ on $\mathcal{P}(\mathcal{G})$ defined by $p < p'$ *iff* $p = t_{\mathcal{G}}(p')$ is antisymmetric.

For the semantics of PS's, we are given a set $A$ that does not contain any couple nor any 3-tuple and such that $* \notin A$. We define, by induction on $n$, the set $D_{A,n}$ for any $n \in \mathbb{N}$:

- $D_{A,0} = \{+, -\} \times (A \cup \{*\})$
- $D_{A,n+1} = D_{A,0} \cup (\{+, -\} \times D_{A,n} \times D_{A,n}) \cup (\{+, -\} \times \mathcal{M}_{fin}(D_{A,n}))$

We set $D_A = \bigcup_{n \in \mathbb{N}} D_{A,n}$.

Definition 44 is an adaptation of Definition 10 in an untyped framework.

▶ **Definition 43.** For any $\alpha \in D_A$, we define $\alpha^\perp \in D_A$ with $+^\perp = -$ and $-^\perp = +$:
- if $\alpha \in A \cup \{*\}$ and $\delta \in \{+,-\}$, then $(\delta, \alpha)^\perp = (\delta^\perp, \alpha)$;
- if $\alpha = (\delta, \alpha_1, \alpha_2)$ with $\delta \in \{+,-\}$ and $\alpha_1, \alpha_2 \in D_A$, then $\alpha^\perp = (\delta^\perp, {\alpha_1}^\perp, {\alpha_2}^\perp)$;
- if $\alpha = (\delta, [\alpha_1, \ldots, \alpha_m])$ with $\delta \in \{+,-\}$ and $\alpha_1, \ldots, \alpha_m \in D_A$, then $\alpha^\perp = (\delta^\perp, [{\alpha_1}^\perp, \ldots, {\alpha_m}^\perp])$.

▶ **Definition 44.** For any differential $\circ$-PS $R$, we define, by induction on *depth*$(R)$ the set of *experiments of* $R$: it is is the set of triples $(R, e_\mathcal{P}, e_\mathcal{B})$, where $e_\mathcal{P}$ is a function $\mathcal{P}_0(R) \to D_A \cup \mathcal{M}_{fin}(D_A)$ and $e_\mathcal{B}$ is a function which associates to every $o \in \mathcal{B}_0(R)$ a finite multiset of experiments of $B_R(o)$ such that
- for any $\{p, q\} \in \mathcal{A}(\mathcal{G}(R))$, we have $e_\mathcal{P}(p) = \alpha$, $e_\mathcal{P}(q) = \alpha^\perp$ for some $\alpha \in D_A$;
- for any $p \in \mathcal{P}_0^\otimes(R)$ (resp. $p \in \mathcal{P}_0^{⅋}(R)$), for any $w_1, w_2 \in \mathcal{W}_0(R)$ such that $t_{\mathcal{G}(R)}(w_1) = p = t_{\mathcal{G}(R)}(w_2)$, $w_1 \in \mathcal{L}(\mathcal{G}(R))$ and $w_2 \notin \mathcal{L}(\mathcal{G}(R))$, we have $e_\mathcal{P}(p) = (+, e_\mathcal{P}(w_1), e_\mathcal{P}(w_2))$ (resp. $e_\mathcal{P}(p) = (-, e_\mathcal{P}(w_1), e_\mathcal{P}(w_2))$);
- for any $p \in \mathcal{P}_0^1(R)$ (resp. $p \in \mathcal{P}_0^\perp(R)$), we have $e_\mathcal{P}(p) = (+, *)$ (resp. $e_\mathcal{P}(p) = (-, *)$);
- for any $p \in \mathcal{P}_0^e(R)$, we have $e(p) = \begin{cases} a & \textit{if } p \in \mathcal{P}_\circ^\mathsf{f}(R); \\ (-, a) & \textit{if } p \in \mathcal{P}_0^?(R); \\ (+, a) & \textit{if } p \in \mathcal{P}_0^!(R); \end{cases}$ where

$$a = \sum_{\substack{w \in \mathcal{W}_0(R) \\ t_{\mathcal{G}(R)}(w) = p}} [e_\mathcal{P}(w)] + \sum_{o \in \mathcal{B}_0(R)} \sum_{e' \in Supp(e_\mathcal{B}(o))} \left( \sum_{\substack{q \in \mathcal{P}_\bullet^\mathsf{f}(B_R(o)) \\ b_R(o)(q) = p}} e_\mathcal{B}(o)(e') \cdot [e'_\mathcal{P}(q)] + \sum_{\substack{q \in \mathcal{P}_\circ^\mathsf{f}(B_R(o)) \\ b_R(o)(q) = p}} e_\mathcal{B}(o)(e') \cdot e'_\mathcal{P}(q) \right)$$

For any experiment $e = (R, e_\mathcal{P}, e_\mathcal{B})$, we set $\mathcal{P}(e) = e_\mathcal{P}$ and $\mathcal{B}(e) = e_\mathcal{B}$.

For any differential $\circ$-PS $R$, we set $[\![R]\!]_A = \{\mathcal{P}(e)|_{\mathcal{P}^\mathsf{f}(R)}; e \textit{ is an experiment of } R\}$.

▶ **Definition 45.** Let $r \in \mathcal{M}_{fin}(D_A)$. We say that $r$ is *injective* if, for every $\gamma \in A$, there are at most two occurrences of $\gamma$ in $r$.

For any set $\mathcal{P}$, for any function $x : \mathcal{P} \to D_A$, we say that $x$ is *injective* if $\sum_{p \in \mathcal{P}} [x(p)]$ is injective. An experiment $e$ of a differential $\circ$-PS $S$ is said to be *injective* if $\mathcal{P}(e)|_{\mathcal{P}^\mathsf{f}(R)}$ is injective.

▶ **Definition 46.** Let $\sigma : A \to D_A$. For any $\alpha \in D_A$, we define $\sigma \cdot \alpha \in D_A$ as follows:
- if $\alpha \in A \cup \{*\}$, then $\sigma \cdot (+, \alpha) = \sigma(\alpha)$ and $\sigma \cdot (-, \alpha) = \sigma(\alpha)^\perp$;
- if $\delta \in \{+,-\}$ and $\alpha_1, \alpha_2 \in D_A$, then $\sigma \cdot (\delta, \alpha_1, \alpha_2) = (\delta, \sigma \cdot \alpha_1, \sigma \cdot \alpha_2)$;
- if $\delta \in \{+,-\}$ and $\alpha_1, \ldots, \alpha_m \in D_A$, then $\sigma \cdot (\delta, [\alpha_1, \ldots, \alpha_m]) = (\delta, [\sigma \cdot \alpha_1, \ldots, \sigma \cdot \alpha_m])$.

For any set $\mathcal{P}$, for any function $x : \mathcal{P} \to D_A$, we define a function $\sigma \cdot x : \mathcal{P} \to D_A$ by setting: $(\sigma \cdot x)(p) = \sigma \cdot x(p)$ for any $p \in \mathcal{P}$.

▶ Remark 7. For any functions $\sigma, \sigma' : A \to D_A$, for any function $x : \mathcal{P} \to D_A$, we have $\sigma \cdot (\sigma' \cdot x) = (\sigma \cdot \sigma') \cdot x$.

▶ **Definition 47.** Let $S$ be a differential $\circ$-PS. Let $e$ be an experiment of $S$. Let $\sigma : A \to D_A$. We define, by induction of *depth*$(S)$, an experiment $\sigma \cdot e$ of $S$ by setting $\mathcal{P}(\sigma \cdot e) = \sigma \cdot \mathcal{P}(e)$ and $\mathcal{B}(\sigma \cdot e)(o) = \sum_{e_1 \in Supp(\mathcal{B}(e)(o_1))} \mathcal{B}(e)(o_1)(e_1) \cdot [\sigma \cdot e_1]$ for any $o_1 \in \mathcal{B}_0(S)$.

Since we deal with untyped proof-nets, we cannot assume that the proof-nets are $\eta$-expanded and that experiments label the axioms only by atoms. That is why we introduce the notion of *atomic experiment*:

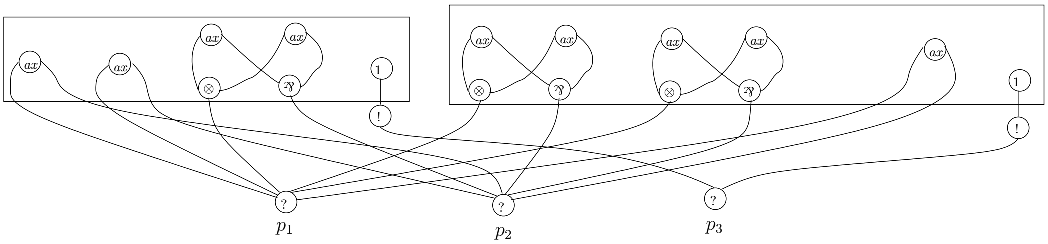


**Figure 17** PS $R''$.

▶ **Definition 48.** For any differential $\circ$-PS $R$, we define, by induction on *depth*$(R)$, the set of *atomic experiments of* $R$: it is the set of experiments $e$ of $R$ such that
- for any $\{p, q\} \in \mathcal{A}(\mathcal{G}(R))$, we have $\mathcal{P}(e)(p), \mathcal{P}(e)(q) \in \{+, -\} \times A$;
- and, for any $o_1 \in \mathcal{B}_0(R)$, the multiset $\mathcal{B}(e)(o_1)$ is a multiset of atomic experiments of $B_R(o_1)$.

▶ **Definition 49.** Let $\mathcal{P}$ be a set. Let $\mathcal{D} \subseteq (D_A)^{\mathcal{P}}$. Let $x \in \mathcal{D}$, we say that $x$ is $\mathcal{D}$-*atomic* if we have $(\forall \sigma \in (D_A)^A)(\forall y \in \mathcal{D})(\sigma \cdot y = x \Rightarrow (\forall \gamma \in At(y))\sigma(\gamma) \in A)$, where $At(y)$ is the set of atoms occurring in $\mathsf{im}(y)$.

We denote by $\mathcal{D}_{At}$ the subset of $\mathcal{D}$ consisting of the $\mathcal{D}$-atomic elements of $\mathcal{D}$.

For any PS $R$, any $[\![R]\!]_A$-atomic injective point is the result of some atomic experiment of $R$:

▶ **Fact 50.** *Let $R$ be a $\circ$-PS. Let $x \in ([\![R]\!]_A)_{At}$ injective. Then there exists an atomic experiment $e$ of $R$ such that $e_{|\mathcal{P}^f(R)} = x$.*

The converse does not necessarily hold: for some PS's $R$, there are results of atomic injective experiments of $R$ that are not $[\![R]\!]_A$-atomic. Indeed, consider Figure 17.

There exists an atomic injective experiment $e$ of $R''$ such that
- $\mathcal{P}(e)(p_1) = (-, [(+, \gamma_1), \dots, (+, \gamma_7), (+, (+, \gamma_8), (+, \gamma_9)), \dots, (+, (+, \gamma_{22}), (+, \gamma_{23}))])$,
- $\mathcal{P}(e)(p_2) = (-, [(-, \gamma_1), \dots, (-, \gamma_7), (-, (-, \gamma_8), (-, \gamma_9)), \dots, (-, (-, \gamma_{22}), (-, \gamma_{23}))])$
- and $\mathcal{P}(e)(p_3) = (-, [(+, [(+, *), (+, *)]), (+, [(+, *), (+, *), (+, *)])])$,

where $\{\gamma_1, \dots, \gamma_{23}\} \subseteq A$. But $e_{|\{p_1,p_2,p_3\}}$ is not in $([\![R'']\!]_A)_{At}$: there exists an atomic injective experiment $e'$ of $R'$ such that
- $\mathcal{P}(e')(p_1) = (-, [(+, \gamma_1), \dots, (+, \gamma_8), (+, (+, \gamma_{10}), (+, \gamma_{11})), \dots, (+, (+, \gamma_{22}), (+, \gamma_{23}))])$,
- $\mathcal{P}(e')(p_2) = (-, [(-, \gamma_1), \dots, (-, \gamma_8), (-, (-, \gamma_{10}), (-, \gamma_{11})), \dots, (-, (-, \gamma_{22}), (-, \gamma_{23}))])$
- and $\mathcal{P}(e')(p_3) = (-, [(+, [(+, *), (+, *)]), (+, [(+, *), (+, *), (+, *)])])$;

we set $\sigma(\gamma) = \begin{cases} \gamma & \textit{if } \gamma \in A \setminus \{\gamma_8\}; \\ (+, (+, \gamma_8), (+, \gamma_9)) & \textit{if } \gamma = \gamma_8; \end{cases}$ we have $\sigma \cdot e'|_{\{p_1,p_2,p_3\}} = e_{|\{p_1,p_2,p_3\}}$.

But it does not matter, because there are many enough atomic points:

▶ **Fact 51.** *Let $R$ be a $\circ$-PS. For any $y \in [\![R]\!]_A$, there exist $x \in ([\![R]\!]_A)_{At}$ and $\sigma : A \to D_A$ such that $\sigma \cdot x = y$.*