A Gap Trichotomy for Boolean Constraint Problems: Extending Schaefer's Theorem

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Abstract

In this paper, we investigate "gap problems", which are promise problems where YES instances are flexibly satisfiable in a certain sense, and NO instances are not satisfiable at all. These gap problems generalise a family of constraint-related decision problems, including the constraint satisfaction problem itself, the *separation problem* (can distinct variables be validly assigned distinct values?) and the 2-*robust satisfiability problem* (does any assignment on two variables extend to a full satisfying assignment?). We establish a Gap Trichotomy Theorem, which on Boolean domains, completely classifies the complexity of the gap problems considered. As a consequence, we obtain several well-known dichotomy results, as well as dichotomies for the separation problem and the 2-robust satisfiability problem: all are either polynomial-time tractable or **NP**-complete. Schaefer's original dichotomy is a notable particular case.

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1 Introduction

Constraint Satisfaction Problems (CSPs) occur widely in practice, both as natural problems, and as an underlying framework for constraint programming; see Tsang [23]. When the template is restricted to some fixed finite domain, these problems still cover many important practical problems as well as providing an important framework for theoretical considerations in computational complexity. In the case of Boolean (2-element) domains, constraint problems coincide with the SAT variants examined by Schaefer [20]. In his paper, Schaefer proved a famous dichotomy: he showed that the complexity of CSPs over a fixed Boolean constraint language is either decidable in polynomial time or is NP-complete. Since Schaefer's seminal contribution, there have been enormous advances toward a more general dichotomy for constraint satisfaction problems on non-Boolean domains. In [10], Feder and Vardi argue that fixed template CSPs emerge as the broadest natural class for which a dichotomy might hold and propose the well-known *Dichotomy Conjecture*. Numerous extensions of Schaefer's result are now known. Amongst the broadest of these include the case of three-element domains (Bulatov [7]), List Homomorphism Problems (Bulatov [8]), and the case of directed graphs without sources and sinks (Barto, Kozik and Niven [4]).

In addition to direct extensions of Schaefer's results, many variants of constraint satisfaction problems have been shown to experience dichotomies like that of Schaefer's, such as counting CSPs [9] and balanced CSPs [21]. We explore computational complexity for notions of "flexible satisfaction": instead of asking for the existence of a single solution,

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one asks for enough solutions to satisfy a range of conditions. We focus in particular on *separability* and *robust satisfiability.* The separation problem SEP asks if it is true that, for every pair of distinct variables u and v, there is a solution giving u a different value to v. The $(k, \mathcal{F})$-robust satisfiability problem asks if every compatible partial assignment on k variables extends to a full solution. As explained in Jackson [13], the SEP condition arises naturally in universal algebraic considerations, but is also closely related to problems without a backbone: problems (typically SAT variants) where no variable is forced to take some fixed value. Implicit constraints such as these are widely associated with computational difficulty; see Monasson et al. [17] or Beacham and Culberson [5]. In the language of [5] for example, the SEP condition corresponds to the unfrozenness of equality. The $(k, \mathcal{F})$-robustness condition is an extension of a robustness condition of Abramsky, Gottlob and Kolaitis [1], who studied robust satisfiability in relation to hidden-variable models in quantum mechanics and explicitly invite the systematic study of the complexity of robust satisfiability for constraint problems. In the present article, we completely classify the complexity of the $(2, \mathcal{F})$-robust problem and the separation problem, in the case of Boolean domains.

Recall that for disjoint languages Y and N, the *promise problem* (Y, N) is the decision problem for Y, where instances are promised to lie in $Y \cup N$; see Goldreich [12] for example. The problem (Y, N) is **NP**-hard if it is **NP**-hard to decide membership in any language S containing Y and disjoint from N. In 2011, Gottlob proved **NP**-hardness of a promise problem relating to $(3k + 3)$SAT: if $\mathrm{Y}_{(k\text{-Rob})}$ denotes the set of all $(3k + 3)$SAT instances for which every possible partial assignment on k variables extends to a satisfying solution and $\mathrm{N}_{\mathrm{CSP}}$ is the set of all NO instances for $(3k + 3)$SAT, then $(\mathrm{Y}_{(k\text{-Rob})}, \mathrm{N}_{\mathrm{CSP}})$ is **NP**-hard. This promise problem can be more precisely described as a *gap problem*, because having no solutions at all is a strong shortfall relative to having k-robust satisfiability [12, p. 259].

Abramsky, Gottlob and Kolaitis [1] and then Jackson showed [13] that **NP**-hard gap problems are also to be found for some other well-known **NP**-complete problems, including 3 SAT, G3C, NAE3 SAT, and positive 1-in-3 SAT. We investigate gap problems in the Boolean case, establishing a Gap Trichotomy Theorem (Theorem 6) that provides dichotomies for these flexible satisfaction problems, as well as several known dichotomy results. A notable consequence is the recovery of Schaefer's Theorem in case of core relational structures. In addition to providing unified proofs for these dichotomies, the Gap Trichotomy Theorem reveals that whenever the constraint satisfaction problem is hard, the more general promise problem is also hard. The Gap Trichotomy Theorem also gives a continuum of examples in the style of the five examples mentioned above.

The fundamental tools used in the aforementioned extensions of Schaefer's dichotomy for Boolean CSPs to higher domains and other related computational problems has been the algebraic analysis of "polymorphisms" (see Definition 7 below). For SEP and robust satisfiability, it is necessary to move to partial polymorphisms. As a second main result, we show that the basic universal algebraic methods can nevertheless be established in this setting, see Theorem 8 below.

2 Preliminaries: Separation and Robustness

We introduce four computational problems that will be of primary focus in this article.

▶ **Definition 1.** Let Γ be a set of relation symbols, each with an associated finite arity. A *template* is a pair $\mathbb{A} = \langle A; \Gamma^{\mathbb{A}} \rangle$ consisting of a *finite* set A together with an interpretation of each n-ary relation symbol $r \in \Gamma$ as a subset $r^{\mathbb{A}}$ of A^n. The set $\Gamma^{\mathbb{A}} = \{r^{\mathbb{A}} \mid r \in \Gamma\}$ is often referred to as a *constraint language over domain A*.

We define a $\Gamma^{\mathbb{A}}$*-instance* to be a triple $I = (V; A; \mathcal{C})$ consisting of a set of *variables* V, the *domain* set A, and a set of *constraints* $\mathcal{C}$. Each constraint $c \in \mathcal{C}$ is a pair $\langle s, r^{\mathbb{A}} \rangle$, where $r^{\mathbb{A}}$ is a k-ary relation in $\Gamma^{\mathbb{A}}$ and $s = (v_1, \dots, v_k)$ is a k-tuple involving variables from V. We define a *solution* of I to be any assignment $\phi\colon V \to A$ such that for each $c = \langle (v_1, \dots, v_k), r^{\mathbb{A}} \rangle$ in $\mathcal{C}$, we have $(\phi(v_1), \dots, \phi(v_k)) \in r^{\mathbb{A}}$.

Constraint satisfaction problem CSP($\mathbb{A}$) *over template* $\mathbb{A}$.
Instance: a $\Gamma^{\mathbb{A}}$-instance I.
Question: is there a solution of I?

Nontrivial satisfaction problem $\mathrm{CSP}_{\mathrm{NonTriv}}(\mathbb{A})$ *over template* $\mathbb{A}$.
Instance: a $\Gamma^{\mathbb{A}}$-instance I.
Question: is there a nontrivial solution of I?

Separation problem SEP($\mathbb{A}$) *over template* $\mathbb{A}$.
Instance: a $\Gamma^{\mathbb{A}}$-instance I.
Question: for every pair $\{v_1, v_2\}$ of distinct variables in V, is there is a solution $\phi\colon V \to A$ of I such that $\phi(v_1) \neq \phi(v_2)$?

Our fourth computational problem of interest requires some further definitions.

▶ **Definition 2.** Let R be a set of finitary relation symbols and let $X = \{x_i \mid i \in I\}$ be a set of pairwise distinct variables. A formula in the language of R is called a *primitive-positive formula* (abbreviated to pp-formula) if, for some $\ell \in \mathbb{N}_0$ and $m, n \in \mathbb{N}$, it is of the form:

$$(\exists w_1, \dots, w_\ell) \bigwedge_{i=1}^{m} \alpha_i(x_1, \dots, x_k, w_1, \dots, w_\ell),$$

where $w_1, \dots, w_\ell, x_1, \dots, x_k$ are distinct variables, and each $\alpha_i(x_1, \dots, x_k, w_1, \dots, w_\ell)$ is either of the form $y \approx z$, where $\approx$ is the symbol for the equality relation and $y, z \in \{x_1, \dots, x_k, w_1, \dots, w_\ell\}$, or of the form $(y_1, \dots, y_k) \in r$, for some k and relation $r \in R$ of arity k and $\{y_1, \dots, y_k\} \subseteq \{x_1, \dots, x_k, w_1, \dots, w_\ell\}$.

The particular case where $\ell = 0$ (that is, no quantifiers) is used later, and is called a *conjunct-atomic formula.*

▶ **Definition 3.** Let $\Gamma^{\mathbb{A}}$ be a constraint language over a finite set A and let $\mathcal{F}$ be a finite set of pp-formulæ in the language of Γ. Let $(V; A; \mathcal{C})$ be a constraint instance for $\Gamma^{\mathbb{A}}$. For a subset $S \subseteq V$, we say that an assignment $f\colon S \to A$ is $\mathcal{F}$*-compatible* if it preserves $\mathcal{F}$.

In other words, if for some $\rho(x_1, \dots, x_k) \in \mathcal{F}$ and some tuple $(s_1, \dots, s_k) \in S^k$ the formula $\rho(s_1, \dots, s_k)$ is true in $(V; A; \mathcal{C})$, then $\rho(f(s_1), \dots, f(s_k))$ must be true in $\mathbb{A}$.

In the following, we let k be a nonnegative integer and $\mathcal{F}$ be a finite set of pp-formulæ in the language of Γ.

The $(k, \mathcal{F})$*-robust satisfiability problem* $(k, \mathcal{F})$-Robust($\mathbb{A}$) *over template* $\mathbb{A}$.
Instance: a $\Gamma^{\mathbb{A}}$-instance I.
Question: does every $\mathcal{F}$-compatible assignment on k variables extend to a solution of I?

In the case where $\mathcal{F}$ consists of the set of pp-formulæ defining all projections of relations in $\Gamma^{\mathbb{A}}$, the notion of $\mathcal{F}$-compatibility has been called "local compatibility" and $(k, \mathcal{F})$-robust satisfiability called "k-robust satisfiability", see [1, §2]. In [13, Lemma 3.1], Jackson proposes that $\mathcal{F}$-compatibility is the natural localness condition in general.

3 Main results: a gap trichotomy

The first main result presents a trichotomy of computational gap theorems for Boolean constraint languages. As consequences we obtain dichotomy theorems for each of the four computational problems described above.

▶ **Notation 4.** *Let Γ be a finite set of relations on $\{0,1\}$, let $k \in \mathbb{N}$ and let $\mathcal{F}$ be a finite set of pp-formulæ in the language of Γ. We use*

- $\mathrm{N}_{\mathrm{CSP}}(\Gamma)$ *to denote the set of NO instances for* $\mathrm{CSP}(\Gamma)$*,*
- $\mathrm{N}_{\mathrm{NonTriv}}(\Gamma)$ *to denote the set of NO instances for* $\mathrm{CSP}_{\mathrm{NonTriv}}(\Gamma)$*,*
- $\mathrm{Y}_{(k,\mathcal{F})}(\Gamma)$ *to denote the set of YES instances for $(k,\mathcal{F})$-*$\mathrm{Robust}(\Gamma)$*,*
- $\mathrm{Y}_{\mathrm{SEP}}(\Gamma)$ *to denote the set of YES instances for* $\mathrm{SEP}(\Gamma)$*,*
- $\mathrm{Y}_{\mathrm{SEP}\cap(k,\mathcal{F})}(\Gamma)$ *to denote the set of instances in* $\mathrm{Y}_{\mathrm{SEP}}(\Gamma) \cap \mathrm{Y}_{(2,\mathcal{F})}(\Gamma)$*.*

When the context refers to a specific constraint language Γ, we omit Γ from this notation.

▶ **Definition 5.** If P and Q are disjoint sets of Γ-instances, we say that Γ *satisfies* $\mathrm{GAP}(P,Q)$ (or *has the gap property* $\mathrm{GAP}(P,Q)$) if the promise problem (P,Q) is **NP**-hard.

We can now state one of main results of the article.

▶ **Theorem 6** (Gap Trichotomy Theorem)**.** *Let Γ be a constraint language on $\{0,1\}$. Exactly one of the following statements is true.*

1. Γ *satisfies* $\mathrm{GAP}(\mathrm{Y}_{\mathrm{SEP}} \cap \mathrm{Y}_{(2,\mathcal{F})}, \mathrm{N}_{\mathrm{CSP}})$ *for some finite set of pp-formulæ $\mathcal{F}$.*
2. $\mathrm{CSP}(\Gamma)$ *is trivial but Γ satisfies* $\mathrm{GAP}(\mathrm{Y}_{\mathrm{SEP}} \cap \mathrm{Y}_{(2,\mathcal{F})}, \mathrm{N}_{\mathrm{NonTriv}})$ *for some finite set of pp-formulæ $\mathcal{F}$.*
3. *The satisfiability problem, $(2,\mathcal{F})$-*$\mathrm{Robust}(\Gamma)$ *and the separation problem* $\mathrm{SEP}(\Gamma)$ *are solvable in polynomial-time, for any finite set of pp-formulæ $\mathcal{F}$.*

▶ Remark. The language of polymorphisms and clone theory can be used to express precise boundaries for when each condition of the three applies to a given Γ. We give full details including which co-clones give rise to which complexity condition below; see Figure 1 and the associated discussion. An overview of the proof of the Gap Dichotomy Theorem is given in Section 7.

A number of *dichotomy* theorems are immediate consequences of the Gap Trichotomy Theorem. We list four examples.

- (Schaefer's Dichotomy Theorem [20].) Observe that $\mathrm{CSP}(\mathbb{A})$ is **NP**-complete in case 1, and polynomial time solvable in cases 2 and 3.
- (Dichotomy Theorem for $\mathrm{CSP}_{\mathrm{NonTriv}}(\mathbb{A})$.) Observe that $\mathrm{CSP}_{\mathrm{NonTriv}}(\mathbb{A})$ is **NP**-complete in cases 1 and 2, and polynomial time solvable in case 3.
- (Dichotomy Theorem for $\mathrm{SEP}(\mathbb{A})$.) Observe that $\mathrm{SEP}(\mathbb{A})$ is **NP**-complete in cases 1 and 2, and polynomial time solvable in case 3.
- (Dichotomy Theorem for $(2,\mathcal{F})$-$\mathrm{Robust}(\Gamma)$.) Observe that $(2,\mathcal{F})$-$\mathrm{Robust}(\Gamma)$ is **NP**-complete for some $\mathcal{F}$ in cases 1 and 2, and polynomial time solvable for all $\mathcal{F}$ in case 3.

4 Main results: an algebraic approach

A pivotal development in the classification of fixed template CSP complexity was the introduction of universal algebraic methods, starting with the work of Jeavons [14], Jeavons, Cohen, Gyssens [15], with the full framework presented in Bulatov, Jeavons, Krokhin [6]. The algebraic method is fundamental in Bulatov's classification of tractable CSPs on 3-element domains [7], and of list homomorphism complexity [8], in the classification of tractable CSPs over digraphs without sources and sinks [4], the classification of CSPs solvable by local consistency check algorithm [2, 3], amongst others.

The algebraic approach concerns the analysis of *polymorphisms* of the template. For some computational problems, polymorphism analysis appears too coarse; this is true for problems considered in Schnoor and Schnoor [21] as well as the problems considered in the present article. Following [21], our results are based on methods relating to partial polymorphisms.

▶ **Definition 7.** Let $k, n \in \mathbb{N}$, let $f\colon \mathrm{dom}(f) \to A$ be a n-ary partial operation, where $\mathrm{dom}(f) \subseteq A^n$, and let r be a k-ary relation on the set A. We say that f *preserves* r or r is *invariant under* f or f is a *partial polymorphism* of r, if whenever $a_1 = (a_{11}, \dots, a_{1n}), a_2 = (a_{21}, \dots, a_{2n}), \dots, a_k = (a_{k1}, \dots, a_{kn})$ are tuples in $\mathrm{dom}(f)$, then

$$(\forall i \in \{1, \dots n\}\ (a_{1i}, a_{2i}, \dots a_{ki}) \in r) \implies (f(a_1), f(a_2), \dots, f(a_k)) \in r.$$

If f is a total operation, then f is called a *polymorphism* of r. If F is set of partial operations then we say that r is *invariant under* F if r is invariant under every operation in F. We let $\mathcal{P}_A$ be the set of all non-empty, non-nullary finitary partial operations on A and $\mathcal{R}_A$ be the set of all non-empty, non-nullary finitary relations on A. Define

$$\mathrm{pPol}(R) := \{f \in \mathcal{P}_A \mid f \text{ preserves each } r \in R\}$$

for any set $R \subseteq \mathcal{R}_A$.

The following theorems are analogous to some of the main contributions in Bulatov et al. [6, Theorems 5.2 & 5.4], but now in the context of SEP and $(k, \mathcal{F})$-Robust and the algebra of partial polymorphisms.

For any partial algebra $\mathbf{A}$ we let $\mathsf{HS}(\mathbf{A})$ be the smallest class of partial algebras in the same signature closed under the formation of homomorphic images (H) and subalgebras (S).

▶ **Theorem 8** (HS Theorem)**.** *Let $\mathbb{A} = \langle A; R\rangle$ and $\mathbb{B} = \langle B; S\rangle$ be templates and let $\mathcal{F}$ be a finite set of pp-formulæ in the language of R. If S satisfies* $\mathrm{GAP}(\mathrm{Y}_{\mathrm{SEP}\cap(k,\mathcal{F})}, \mathrm{N}_{\mathrm{CSP}})$ *and there exist partial algebras $\mathbf{A} = \langle A; F^{\mathbf{A}}\rangle$ and $\mathbf{B} = \langle B; F^{\mathbf{B}}\rangle$ such that*

1. $F^{\mathbf{B}} \subseteq \mathrm{pPol}(S)$,
2. $\mathbf{B} \in \mathsf{HS}(\mathbf{A})$*, and*
3. $\mathrm{pPol}(R) \subseteq F^{\mathbb{A}}$,

then R satisfies $\mathrm{GAP}(\mathrm{Y}_{\mathrm{SEP}\cap(k,\mathcal{F})}, \mathrm{N}_{\mathrm{CSP}})$.

The case where $\mathbf{A} = \mathbf{B}$ corresponds to the preservation of complexity of the gap property under conjunct atomic reductions, which is critical to the proof of the Gap Trichotomy Theorem. When $\mathbf{A} \neq \mathbf{B}$, the theorem lifts the complete classification given by the Gap Trichotomy Theorem on Boolean domains to many problems on templates with non-Boolean domains. With further effort, direct products can be incorporated into Item 2 of Theorem 8, but the full version is beyond the scope of the present article, and will appear in subsequent work.

In the full version of the present article, we show that HS theorems can be obtained for other variants of the constraint satisfaction problem, namely the equivalence problem and the implication problem, but whose definitions are not given due to space constraints.

A further useful simplification in the standard CSP setting has been the restriction to so-called idempotent polymorphisms; see [6, Theorem 4.7]. A partial polymorphism $f : \mathrm{dom}(f) \to \mathbb{A}$ is *idempotent* if $f(a, \dots, a) = a$ for every $a \in A$ for which $(a, \dots, a) \in \mathrm{dom}(f)$. As a final result we show that when analysing the complexity of SEP problems we may restrict to idempotent partial polymorphisms. The full statement of this theorem can be found in the complete version of the present article.

5 Weak co-clones and strong partial clones

We now give more technical definitions that are required for the main arguments.

▶ **Definition 9.** Let $m, n \in \mathbb{N}$, let $f \in \mathcal{P}_A$ be m-ary and let $g_1, \dots, g_m \in \mathcal{P}_A$ be n-ary. The *composition* $f(g_1, \dots, g_m)$ is an n-ary partial operation defined by

$$f(g_1, \dots, g_m)(x_1, \dots, x_n) := f(g_1(x_1, \dots, x_n), \dots, g_m(x_1, \dots, x_n)),$$

where $\operatorname{dom}(f(g_1, \dots, g_m))$ is the set

$$\left\{(x_1, \dots, x_n) \in \bigcap_{i=1}^{m} \operatorname{dom}(g_i) \mid (g_1(x_1, \dots, x_n), \dots, g_m(x_1, \dots, x_n)) \in \operatorname{dom}(f)\right\}.$$

▶ **Definition 10.** Let $f, g \in \mathcal{P}_A$. We say that f is a *restriction* of g if $\operatorname{dom}(f) \subseteq \operatorname{dom}(g)$ and f agrees with g on $\operatorname{dom}(f)$.

We let $\mathcal{O}_A$ denote the subset of $\mathcal{P}_A$ consisting of total operations.

▶ **Definition 11.** Let A be a non-empty set and let $\mathcal{C} \subseteq \mathcal{O}_A$. Then $\mathcal{C}$ is a *clone* on the set A if the following two conditions hold:

1. $\mathcal{C}$ contains all projection operations: that is, for all $n \in \mathbb{N}$, the ith projection $\pi_i \colon A^n \to A$ given by $\pi_i(x_1, \dots, x_n) = x_i$ belongs to $\mathcal{C}$;
2. $\mathcal{C}$ is closed under compositions.

For a set F of total operations, $[F]$ will denote the smallest clone containing F and we refer to $[F]$ as the *clone generated by* F. The set F is sometimes called a *base* for the clone $[F]$.

▶ **Definition 12.** Let A be a non-empty set. A subset R of $\mathcal{R}_A$ is called a *co-clone* or *relational clone* if it is closed under the formation of pp-definable relations. We define $\langle R \rangle$ to be the smallest co-clone containing R and we refer to $\langle R \rangle$ as the *co-clone generated by* R. The set R is sometimes called a *base* for $\langle R \rangle$.

The sets $\wp(\mathcal{O}_A)$ and $\wp(\mathcal{R}_A)$ are complete lattices, where $\wp()$ is the powerset operator. A well-known result of Geiger [11] states that pair of maps $\operatorname{Inv} \colon \wp(\mathcal{O}_A) \to \wp(\mathcal{R}_A)$ and $\operatorname{Pol} \colon \wp(\mathcal{R}_A) \to \wp(\mathcal{P}_A)$ form a Galois correspondence between $\wp(\mathcal{O}_A)$ and $\wp(\mathcal{R}_A)$. In particular, we have

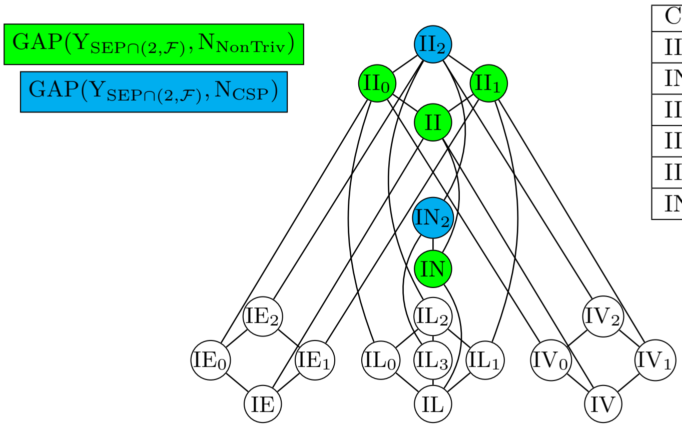
$$\begin{aligned}&\operatorname{Inv}(F) := \{r \in \mathcal{R}_A \mid r \text{ is invariant under each } f \in F\} \text{ and}\\ &\operatorname{Pol}(R) := \{f \in \mathcal{O}_A \mid f \text{ preserves each } r \in R\},\end{aligned}$$

for each $F \subseteq \mathcal{O}_A$ and each $R \subseteq \mathcal{R}_A$.

Clones on $\{0, 1\}$ were characterised by Post [18] and are usually called "Boolean clones". An upset of Post's lattice is given in Figure 1; the table included gives definitions of the shaded vertices in terms of relations invariant under basic operations. The operations c_0 and c_1 are the constant unary functions to 0 and 1, respectively, and $\neg$ is the usual negation operation on $\{0, 1\}$. Shaded vertices in Figure 1 give the precise information for the Gap Trichotomy Theorem:

- Statement 1 applies when Γ generates a co-clone containing IN_2(blue/dark grey);
- Statement 2 applies when Γ generates a co-clone containing IN, but not containing IN_2 (green/light grey);
- Statement 3 holds otherwise.

In general, it seems difficult to use pp-formulæ to transfer the complexity of problems such as SEP and $(k, \mathcal{F})$-Robust. Instead we use conjunct-atomic formulæ.


🟩 GAP(Y$_{\mathrm{SEP}\cap(2,\mathcal{F})}$, N$_{\mathrm{NonTriv}}$)
🟦 GAP(Y$_{\mathrm{SEP}\cap(2,\mathcal{F})}$, N$_{\mathrm{CSP}}$)
II$_2$
II$_0$
II$_1$
II
IN$_2$
IN
IE$_2$
IL$_2$
IV$_2$
IE$_0$
IE$_1$
IL$_0$
IL$_3$
IL$_1$
IV$_0$
IV$_1$
IE
IL
IV

Co-clone	Definition
II_2	all Boolean relations
IN_2	$\mathrm{Inv}(\{\neg\})$
II_0	$\mathrm{Inv}(\{c_0\})$
II_1	$\mathrm{Inv}(\{c_1\})$
II	$\mathrm{Inv}(\{c_0, c_1\})$
IN	$\mathrm{Inv}(\{\neg, c_0\})$

Figure 1 An upset in the Boolean co-clone lattice, with a table of polymorphism definitions for the shaded co-clones; I$\mathcal{C}$ abbreviates $\mathrm{Inv}(\mathcal{C})$, for each Boolean clone $\mathcal{C}$.

▶ **Definition 13.** A subset R of $\mathcal{R}_A$ is called a *weak co-clone* or *weak system* if it is closed under the formation of conjunct-atomic definable relations. We can define $\langle R\rangle_{\nexists}$ to be the smallest weak co-clone containing R and we refer to $\langle R\rangle_{\nexists}$ as the *weak co-clone generated by* R. The set R is sometimes called a *base* for the weak system $\langle R\rangle_{\nexists}$.

If we restrict further to conjunct atomic formulæ without equality, then we write instead $\langle R\rangle_{\nexists,\neq}$ for the smallest system containing R and say that $\langle R\rangle_{\nexists,\neq}$ is the *equality-free weak system generated by* R.

If we weaken the operators Inv and Pol to allow partial operations to be included in the definition, we obtain a refined Galois connection between the complete lattices $\wp(\mathcal{P}_A)$ and $\wp(\mathcal{R}_A)$ (see Romov [19]). In particular, sets of the form $\mathrm{Inv}(F)$ are precisely the weak co-clones, for $F \subseteq \mathcal{P}_A$. Sets of the form $\mathrm{pPol}(R)$, for $R \subseteq \mathcal{R}_A$, are called *strong partial clones*, and coincide with those subsets of $\mathcal{P}_A$ including all total projections and that are closed under composition and domain restriction. Post's lattice provides a useful approximation to the lattice of strong partial clones in the Boolean setting: for each Boolean clone $\mathcal{C}$, is it known that the set of all strong partial clones whose total operations agree with $\mathcal{C}$ forms an interval, and there are known generators for the top and bottom element in each of these intervals [21]; these are critical in the main proofs to come.

▶ **Definition 14.** Let A be a non-empty set, let $\mathcal{C}$ be a clone on A and let Γ be a set of finitary relations on A. We call Γ a *weak base* for the co-clone $\mathrm{Inv}(\mathcal{C})$ if $\mathcal{I}_{\cup}(\mathcal{C}) = \mathrm{pPol}(\Gamma)$.

We will often present relations in a matrix form. The representation is not unique, but it is succinct. For a k-ary relation $r = \{a_1, \ldots, a_m\}$ on a non-empty set A with $|r| = m$, the *matrix representation of* r is the $m \times k$ matrix $M = (a_{ij})$ over A whose ith row is the tuple a_i. (Non-uniqueness follows because the ordering $a_1, \ldots, a_m$ is arbitrary.)

▶ **Definition 15.** Define Cols_3 to be the following 8-ary relation over $\{0, 1\}$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

▶ **Definition 16.** Let $\mathcal{C}$ be a Boolean clone and let r be a relation on A. Define $\mathcal{C}(r)$ to be the smallest relation containing r that is invariant under every operation in $\mathcal{C}$. We refer to $\mathcal{C}(r)$ as the *$\mathcal{C}$-closure* of the relation r.

Using the work of Schnoor and Schnoor [21] and Schnoor [22, Table 3.1], the following construction gives weak-bases for each of the Boolean co-clones shaded in Figure 1.

▶ **Proposition 17** ([21, Theorem 4.11], [22, Table 3.1]). *Let* $\mathrm{I}\mathcal{C}$ *be any of the Boolean co-clones listed in the table within Figure 1. Then* $\mathcal{C}(\mathrm{Cols}_3)$ *is a weak-base for* $\mathrm{I}\mathcal{C}$*.*

For example, to construct a weak base for the Boolean co-clone $\mathrm{IN}_2 = \mathrm{Inv}(\{\neg\})$, we simply close the relation Cols_3 under $\neg$. Thus,

$$\mathrm{N}_2(\mathrm{Cols}_3) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Weak bases that generate not only the smallest weak system but also the smallest equality-free weak-system of relations generating the same co-clone will be crucial in the classification of the problems considered. Schnoor and Schnoor [21, Definition 5.1] give an irredundancy condition ensuring conjunct-atomic definability without equality. We omit the definition, but observe that all six of the relations in Proposition 17 are irredundant.

▶ **Theorem 18** ([21, Corollary 5.6]). *Let* A *be a non-empty set, let* $\mathcal{C}$ *be a clone on* A *and let* Γ *be an irredundant weak base for the co-clone* $\mathrm{Inv}(\mathcal{C})$*. If* Γ' *is set of relations on* A *such that* $\langle\Gamma'\rangle = \mathrm{Inv}(\mathcal{C})$*, then* $\langle\Gamma\rangle_{\nexists,\neq} \subseteq \langle\Gamma'\rangle_{\nexists,\neq}$*.*

The next two sections are dedicated to establishing the Gap Trichotomy Theorem.

6 Towards a dichotomy: gap properties

We begin with three results that are crucial for establishing gap properties. The first result is an abridged version of [13, Theorem 6.1]. We let $\mathbb{2}$ denote the positive 1-in-3 SAT template $\langle\{0,1\};\{(1,0,0),(0,1,0),(0,0,1)\}\rangle$.

▶ **Theorem 19** ([13]). *Let* $\mathcal{K}$ *be the set consisting of all positive* 1*-in-*3 SAT *instances* I *with the following properties:*

- *no variable appears more than once in each constraint tuple of* I*,*
- I *is* 2*-robustly positive* 1*-in-*3 *satisfiable.*

Then the positive 1*-in-*3 SAT *relation has* $\mathrm{GAP}(\mathcal{K}, \mathrm{N}_{\mathrm{CSP}})$*.*

The next lemma summarises the basic method employed in Abramsky, Gottlob and Kolaitis [1] and Jackson [13]. It is essentially the definition of reduction for promise problems; see [12, Definition 3], for example.

▶ **Lemma 20.** *Let* Γ *and* Γ' *be finite sets of relations on* $\{0,1\}$*. Let* A *and* B *be disjoint sets of* Γ*-instances and let* X *and* Y *be disjoint sets of* Γ'*-instances. Further, let* Γ *have the gap property* $\mathrm{GAP}(A,B)$*. If there is a polynomial-time computable function* $f\colon \mathcal{I}_\Gamma \to \mathcal{I}_{\Gamma'}$ *satisfying:*

1. $I \in A \Rightarrow f(I) \in X$,
2. $I \in B \Rightarrow f(I) \in Y$,

then Γ' *satisfies* $\mathrm{GAP}(X,Y)$*. In particular,* Γ' *has the gap property* $\mathrm{GAP}(f(A), f(B))$*.*

It is well known that the complexity of $\mathrm{CSP}(\Gamma)$ depends only on the co-clone generated by Γ, see [14, Theorem 3.4] or alternatively [6, Theorem 2.16] for a proof explicitly using pp-formulæ. We now give an analogous result that says when analysing the the complexity of SEP and $(2, \mathcal{F})$-robust satisfiability we need only consider relations up to equality-free conjunct-atomic definability. We first give a preliminary lemma.

▶ **Lemma 21.** *Let $\Gamma^{\mathbb{A}}$ be a constraint language over a finite set A and let $R^{\mathbb{A}}$ be a finite set of relations in $\langle \Gamma^{\mathbb{A}} \rangle_{\not\exists}$. There is a polynomial-time construction that transforms any instance $I = (V; A; \mathcal{C})$ of $\mathrm{CSP}(R^{\mathbb{A}})$ into an instance I' of $\mathrm{CSP}(\Gamma^{\mathbb{A}})$ on the same variables, and moreover, the solutions of I are exactly the solutions of I'.*

▶ **Theorem 22.** *Let $\Gamma^{\mathbb{A}}$ be a constraint language over a set A, let $R^{\mathbb{A}}$ be any finite set of relations in $\langle \Gamma^{\mathbb{A}} \rangle_{\not\exists, \neq}$, let $\mathcal{F}$ be a finite set of pp-formulæ in the language of R and let $k \in \mathbb{N}$. There is a polynomial-time computable function that reduces*

1. *$\mathrm{CSP}(R^{\mathbb{A}})$ to $\mathrm{CSP}(\Gamma^{\mathbb{A}})$,*
2. *$\mathrm{SEP}(R^{\mathbb{A}})$ to $\mathrm{SEP}(\Gamma^{\mathbb{A}})$, and*
3. *$(k, \mathcal{F})$-$\mathrm{Robust}(R^{\mathbb{A}})$ to $(k, \mathcal{G})$-$\mathrm{Robust}(\Gamma^{\mathbb{A}})$, for some finite set $\mathcal{G}$ of pp-formulæ in the language of Γ.*

Proof. The reduction from $\mathrm{CSP}(R^{\mathbb{A}})$ to $\mathrm{CSP}(\Gamma^{\mathbb{A}})$ is obtained immediately from Lemma 21. This proves (1).

Since the solutions of I in $\mathrm{CSP}(R^{\mathbb{A}})$ are precisely the solutions of I' in $\mathrm{CSP}(\Gamma^{\mathbb{A}})$, it follows that separating solutions of I are exactly the separating solutions of I'. Hence I is a YES instance of $\mathrm{SEP}(R^{\mathbb{A}})$ if and only if I' is a YES instance of $\mathrm{SEP}(\Gamma^{\mathbb{A}})$. This establishes (2).

For (3), consider $r \in R$ of arity ℓ and abstractly expressible by an equality-free conjunct-atomic formula $r(x_1, \ldots, x_\ell)$ in the language of Γ. For each pp-formula $\rho(w_1, \ldots, w_m) \in \mathcal{F}$, we construct a pp-formula $\rho_\Gamma(w_1, \ldots, w_m)$ in the language of Γ in the following way: replace every occurrence of an ℓ-ary relation symbol r in ρ with its conjunct-atomic defining formula $r(x_1, \ldots, x_\ell)$. Let $\mathcal{G} = \{\rho_\Gamma \mid \rho \in \mathcal{F}\}$. Then since $\rho(a_1, \ldots, a_m)$ is true in $\langle A; R^{\mathbb{A}} \rangle$ if and only if $\rho_\Gamma(a_1, \ldots, a_m)$ is true in $\langle A; \Gamma^{\mathbb{A}} \rangle$ for $(a_1, \ldots, a_m) \in A^m$ and $\rho(w_1, \ldots, w_m) \in \mathcal{F}$, it follows that the $\mathcal{F}$-compatible assignments on k variables of I are exactly the $\mathcal{G}$-compatible assignments on k variables of I'. Thus, since the solutions of I are precisely the solutions of I', by Lemma 21, it then follows that I is a YES instance of $(k, \mathcal{F})$-$\mathrm{Robust}(R^{\mathbb{A}})$ if and only if I' is a YES instance of $(k, \mathcal{G})$-$\mathrm{Robust}(\Gamma^{\mathbb{A}})$. ◀

Theorem 22 holds more generally: with some caveats and proper amendment to the proof, the assumption that $R \subseteq \langle \Gamma \rangle_{\not\exists, \neq}$ can be weakened to $R \subseteq \langle \Gamma \rangle_{\not\exists}$. However, this result is not required for establishing our main theorems.

7 Proof of the Gap Trichotomy Theorem

In this section, we establish gap properties for relations generating the Boolean co-clones II_2, IN_2, II_0, II_1, II or IN. These co-clones are shaded in Figure 1. Each co-clone must be considered separately, however the proofs follow the same structure: we first establish a gap property for the irredundant weak-base and then use the fact that gap properties are preserved by the $\langle - \rangle_{\not\exists, \neq}$-operator. We sketch details only in the case of II_2.

7.1 The Boolean co-clone II_2 and IN_2

By Proposition 17, the relation $\mathrm{I}_2(\mathrm{Cols}_3)$ of Definition 15 is an irredundant weak base for II_2.

▶ **Proposition 23.** *The relation* $\mathrm{I}_2(\mathrm{Cols}_3)$ *satisfies* $\mathrm{GAP}(\mathrm{Y}_{\text{SEP}\cap\text{2-Rob}}, \mathrm{N}_{\text{CSP}})$.

Proof outline. We apply Lemma 20, reducing from $\mathrm{GAP}(\mathrm{Y}_{\text{SEP}\cap\text{2-Rob}}, \mathrm{N}_{\text{CSP}})$ for positive 1-in-3SAT. The result will then follow from Theorem 19.

Given an instance $I = (V; \{0,1\}; \mathcal{C})$ of positive 1-in-3 SAT, construct an instance $I^\star = (V^\star; \{0,1\}; \mathcal{C}^\star)$ of II_2 - SAT in the following way.

1. First let $\overline{V} = \{\bar{v} \mid v \in V\}$ be a disjoint copy of V, and construct $V^\star = V \cup \overline{V} \cup \{\top, \bot\}$, where $\top, \bot \notin V \cup \overline{V}$,
2. for each constraint $\langle (x,y,z), +1\text{in}3\,\text{SAT}\rangle$ in $\mathcal{C}$, we include the constraint $\langle (x,y,z,\bar{x},\bar{y},\bar{z},\bot,\top), \mathrm{I}_2(\mathrm{Cols}_3)\rangle$ in $\mathcal{C}^\star$.

Any solution φ of I in CSP(2) can be extended to a solution $\varphi^\star$ of $I^\star$ in the following way. For each $v \in V$, define $\varphi^\star(v) := \varphi(v)$, $\varphi^\star(\bar{v}) = \neg \circ \varphi(v)$, $\varphi^\star(\bot) = 0$ and $\varphi^\star(\top) = 1$, where $\neg$ is the usual Boolean complement. For the converse direction, observe that the projection $\pi_{\{1,2,3\}}(\mathrm{I}_2(\mathrm{Cols}_3)) = +1\text{in}3\,\text{SAT}$, thus if ψ is a solution of $I^\star$ in II_2 - SAT, then the restriction $\psi\restriction_V$ is a solution of I in CSP(2). Hence we have shown that any solution ϕ of I extends uniquely to a solution $\phi^\star$ of $I^\star$.

Now assume that I lies in the class $\mathcal{K}$ of Theorem 19. Since I is 2-robustly satisfiable and no variable appears more than once in each constraint tuple, it follows that I has the following properties.

(♡) For every pair of distinct variables x and y in V, there are solutions φ_1, φ_2 and φ_3 of I satisfying

$$(\varphi_1(x), \varphi_1(y)) = (0,0),$$
$$(\varphi_2(x), \varphi_2(y)) = (0,1),$$
$$(\varphi_3(x), \varphi_3(y)) = (1,0).$$

(♢) If x and y do not appear in a common constraint tuple, then there is a solution φ_4 of I satisfying $(\varphi_4(x), \varphi_4(y)) = (1,1)$.

These conditions can be used to show that $I^\star \in \mathrm{Y}_{\text{SEP}\cap\text{2-Rob}}(\mathrm{II}_2\text{ - SAT})$: there are a number of cases according to the different combinations of containments in $V, \overline{V}, \{\top, \bot\}$ for a given pair of variables $\{u, v\} \subseteq V^\star$. ◀

The following theorem now follows from Theorems 18 and 22.

▶ **Theorem 24.** *Let* Γ *be a finite constraint language on* $\{0,1\}$ *such that* $\langle\Gamma\rangle = \mathrm{II}_2$. *Then* Γ *has the gap property* $\mathrm{GAP}(\mathrm{Y}_{\text{SEP}\cap(2,\mathcal{G})}, \mathrm{N}_{\text{CSP}})$, *for some finite set* $\mathcal{G}$ *of pp-formulæ in the language of* Γ, *and consequently both* $(2,\mathcal{G})$-Robust(Γ) *and* SEP(Γ) *are* **NP**-*complete.*

A similar approach gives an analogous theorem for constraint languages generating IN_2: the reduction can be taken from either of positive 1-in-3SAT directly, or by following from the reduction in Proposition 23. Together with Theorem 24, these two cases cover Statement 1 of the Gap Trichotomy Theorem 6.

7.2 The Boolean co-clones with trivial CSPs but hard $\text{CSP}_{\text{NonTriv}}$

This section relates to the clones $\mathrm{II}_0, \mathrm{II}_1, \mathrm{II}, \mathrm{IN}$ corresponding to Statement 2 in the Gap Dichotomy Theorem 6. In these cases, we can reuse the same fundamental construction used for II_2 and IN_2. The proof proceeds as follows: to lie in Y_{SEP} or $\mathrm{Y}_{(2,\mathcal{F})}$, it is necessary to have an assignment in which $\bot$ and $\top$ take different values. We then argue that this forces solutions into II_2 or IN_2.

7.3 Proving tractablity

We establish a theorem that covers all cases that are solvable in polynomial-time. The proof relies on a result of Jackson [13, Proposition 3.2], which says that a constraint language Γ on a finite set A is polynomial-time equivalent to Γ_{Con}, with respect to Turing reductions, for each of $\mathrm{SEP}(\mathbb{A})$ and $(k, \mathcal{F})$-Robust.

▶ **Theorem 25.** *Let Γ be a constraint language on $\{0, 1\}$. If $\mathrm{IN} \not\subseteq \langle\Gamma\rangle$, then the computational problems $\mathrm{SEP}(\Gamma)$ and $(2, \mathcal{F})$-$\mathrm{Robust}(\Gamma)$ are solvable in polynomial-time.*

Proof. When $\mathrm{IN} \not\subseteq \langle\Gamma\rangle$, it follows from Post's co-clone lattice (see Figure 1), that $\mathrm{IN} \not\subseteq \langle\Gamma \cup \{(0), (1)\}\rangle$ and then it is known that the constraint problem $\mathrm{CSP}(\Gamma \cup \{(0), (1)\})$ is tractable; this can be found in Schaefer's original argument for example; see [20, Lemma 4.1]. Then from [13, Proposition 3.2], the problems $\mathrm{SEP}(\Gamma)$ and $(2, \mathcal{F})$-$\mathrm{Robust}(\Gamma)$ are solvable in polynomial-time. ◀

8 Proof of the HS Theorem

We give a brief overview of the proof for the HS Theorem. The result is established by carrying the gap property through items 1, 2 and 3 of Theorem 8. The main difficulties arise from items 2 (HS) and 3 (restricted pp-definability), requiring a series of polynomial-time reductions. The constructions used for substructures and homomorphisms are based on those in [6, 16], given in the standard CSP setting. In the case of taking substructures, SEP and $(2, \mathcal{F})$-Robust carry through using the standard construction (the local compatibility condition $\mathcal{F}$ is changed during the reduction). The homomorphism case however requires proper amendment, including the addition of extra relations and non-trivial usage of the gap property; the main complication arising from SEP. For item 3, we require a more general version of Theorem 22. The addition of equality presents complications for $(2, \mathcal{F})$-robust satisfiability, and we again require the use of a gap property to carry through the reduction. The proof for SEP is similar however, a further slight variation of the proof is necessary.

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