Admissibility in Quantitative Graph Games*

Romain Brenguier¹, Guillermo A. Pérez², Jean-François Raskin³, and Ocan Sankur⁴

¹ University of Oxford, Oxford, UK
² Université Libre de Bruxelles (ULB), Brussels, Belgium
³ Université Libre de Bruxelles (ULB), Brussels, Belgium
⁴ CNRS, Irisa, Rennes, France

Abstract

Admissibility has been studied for games of infinite duration with Boolean objectives. We extend here this study to games of infinite duration with quantitative objectives. First, we show that, under the assumption that optimal worst-case and cooperative strategies exist, admissible strategies are guaranteed to exist. Second, we give a characterization of admissible strategies using the notion of adversarial and cooperative values of a history, and we characterize the set of outcomes that are compatible with admissible strategies. Finally, we show how these characterizations can be used to design algorithms to decide relevant verification and synthesis problems.

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1 Introduction

Two-player zero-sum graph games are the most studied mathematical model to formalize the reactive synthesis problem [15, 16]. Unfortunately, this mathematical model is often an abstraction that is too coarse. Realistic systems are usually made up of several components, all of them with their own objectives. These objectives are not necessarily antagonistic. Hence, the setting of non-zero sum graph games is now investigated in order to unleash the full potential of automatic synthesis algorithms for reactive systems, see e.g. [9, 2, 5, 6, 14, 12].

For a player with objective ϕ, a strategy σ is said to be dominated by a strategy σ’ if σ’ does as well as σ with respect to ϕ against all the strategies of the other players and strictly better for some of them. A strategy σ is admissible for a player if it is not dominated by any other of his strategies. Clearly, playing a strategy which is not admissible is sub-optimal and a rational player should only play admissible strategies. The elimination of dominated strategies can be iterated if one assumes that each player knows the other players know that only admissible strategies are played, and so on.

While admissibility is a classical notion for finite games in normal form, see e.g. [13] and pointers therein, its generalization to infinite duration games is challenging and was only considered more recently. In 2007, Berwanger was the first to show [2] that admissibility, i.e. the avoidance of dominated strategies, is well-behaved in infinite duration n-player non-zero sum turn-based games with perfect information and Boolean outcomes (two possible payoffs:

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win or lose). This framework encompasses games with omega-regular objectives. The main contributions of Berwanger were to show that

(i) in all $n$-player game structures, for all objectives, players have admissible strategies, (Berwanger even shows the existence of strategies that survive the iterated elimination of strategies)

(ii) every strategy that is dominated by a strategy is dominated by an admissible strategy, and

(iii) for finite game structures, the set of admissible strategies forms a regular set.

While the iterated admissibility formalizes a strong notion of rationality [1], it has been shown recently that the non-iterated version is strong enough to synthesize relevant strategies for non-zero sum games of infinite duration modelling reactive systems. In [11], Faella considers games played on finite graphs and focuses on the states from which one designated player cannot force a win. He compares several criteria for establishing what is the preferable behavior of this player from those states, eventually settling on the notion of admissible strategy. In [4], starting from the notion of admissible strategy, we have defined a novel rule for the compositional synthesis of reactive systems, applicable to systems made of $n$ components which have each their own objective. We have shown that this synthesis rule leads to solutions which are robust and resilient.

Here, we study the notion of admissible strategy in infinite horizon $n$-player turn-based quantitative games played on a finite game structure. We give a comprehensive picture of the properties related to the existence of such strategies and to their characterization. Contrary to the Boolean case, the number of payoffs in our setting is potentially infinite making the characterization challenging. As in [2], we assume all players have perfect information.

Main contributions. First, contrary to the Boolean case, we show that in the quantitative setting, there are dominated strategies that are not dominated by any admissible strategy (Example 9). Second, we show that the existence of worst-case optimal and cooperatively optimal strategies for all players is a sufficient condition for the existence of admissible strategies (Thm. 4). Additionally, we show that there are games without worst-case optimal or without cooperative optimal strategies that do not have admissible strategies (Lem. 3). Third, we provide a characterization of admissible strategies in terms of antagonistic and cooperative values – that are classical values defined for quantitative games – (Thm. 11) and a characterization of the outcomes compatible with admissible strategies (Thm. 13). While the first characterization allows one to precisely describe admissible strategies, the characterization of the set of outcomes is given in linear temporal logic, and is a useful tool to reason about the outcomes that can be generated by such strategies. Finally, we show how to use the aforementioned characterizations to obtain algorithms to solve relevant decision problems for games with classical quantitative measures such as $\text{Inf}$, $\text{Sup}$, $\text{LimInf}$, $\text{LimSup}$ and mean-payoff (Thms. 17, 18, and 19).

Example. Let us consider the game from Fig. 1 to illustrate several notions and decision problems introduced and solved in this paper. The game is played by two players: Player 1, who owns the square vertices, and Player 2, owner of the round vertices. The measure that we consider here is the mean-payoff. (But note that, the arguments we will develop in this example are applicable to the limit inferior and limit superior measures as well.)

First, we note that the (best) worst-case value (or, the antagonistic value) that Player 1 can force is equal to 1, while the antagonistic value for Player 2 is equal to 0. The latter values are meaningful under the hypothesis that the other player is playing fully antagonistically
and not pursuing their own objective. Now, if we account for the fact that Player 2 aims at maximizing his own payoff and so plays only admissible strategies towards this goal, then we conclude that he will never play the edge \((v_2, v_1)\). This is because, from vertex \(v_2\), Player 2 has a strategy to enforce value 2 and taking edge \((v_2, v_1)\) is unreasonable because, in the worst case, from \(v_1\) he will only obtain 0. As we show in Sec. 6, this kind of reasoning can be made formal and automated. We will show that, for games with classical quantitative measures, it can indeed be decided algorithmically if a finite memory strategy given, for instance, as a finite state Moore machine, is admissible or not.

Second, a similar but more subtle reasoning to the one presented above allows us to conclude that Player 1 will eventually play the edge \((v_1, v_2)\). Indeed, from vertex \(v_1\), Player 1 can force a payoff equal to 1 by either taking edge \((v_1, v_3)\) or \((v_1, v_2)\). Nevertheless, it is not reasonable for him to play edge \((v_1, v_3)\) because, while this choice enforces a worst-case payoff equal to 1 (the antagonistic value), playing edge \((v_1, v_2)\) is better because it ensures the same worst-case payoff and additionally leaves a possibility for Player 2 to help him by taking the cycle \(v_2-v_3\), giving him a payoff of 2. If we take into account that the adversary is playing admissible strategies, then, in the words of [4], we can solve the assume-admissible synthesis problem. In this example, we conclude that Player 1 has a strategy to enforce a payoff of 2 against all admissible strategies of Player 2. A strategy which eventually chooses edge \((v_1, v_2)\) ensures this payoff. The formalization of this reasoning and elements necessary for its automation are presented in Sec. 6.

**Structure of the paper.** Sec. 2 contains definitions. In Sec. 3, we study conditions under which the existence of admissible strategies is guaranteed. In Sec. 4, we give a characterization of admissible strategies, and in Sec. 5, a description of the set of outcomes compatible with admissible strategies. In Sec. 6, we apply our results to solve relevant decision problems on games with classical quantitative measures.

## 2 Preliminaries

We denote by \(\mathbb{R}\) the set of real numbers, \(\mathbb{Q}\) the set of rational numbers, \(\mathbb{N}\) the set of natural numbers, and \(\mathbb{N}_{>0}\) the set of positive integers.

A **game** is a tuple \(\mathcal{G} = (P, (V_i)_{i \in P}, v_{\text{init}}, E, (\text{payoff}_i)_{i \in P})\) where:

1. \(P\) is the non-empty and finite set of players.
2. \(V \triangleq \bigcup_{i \in P} V_i\) where for every \(i \in P\), \(V_i\) is the finite set of player \(i\)’s vertices, and \(v_{\text{init}} \in V\) is the initial vertex.
3. \(E \subseteq V \times V\) is the set of edges (it is assumed, w.l.o.g., that each vertex in \(V\) has at least one outgoing edge.)
4. For every \(i \in P\), \(\text{payoff}_i\) is a payoff function from infinite paths in the digraph \((V, E)\) to \(\mathbb{R}\) that, intuitively, player \(i\) will attempt to maximize.

An outcome \(\rho\) is an infinite path in the digraph \((V, E)\), i.e., an infinite sequence of vertices \((\rho_j)_{j \in \mathbb{N}_{>0}}\) such that \((\rho_j, \rho_{j+1}) \in E\), for all \(j \in \mathbb{N}_{>0}\). A finite prefix of an outcome is called a
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A strategy profile \( \sigma \) for the set of players \( P \) is a tuple of strategies, one for each player of \( P \).

We now formally define dominance and admissibility. We recall the intuition: a player’s strategy \( \sigma \) is dominated by another strategy \( \sigma' \) of his if \( \sigma' \) yields a payoff which is as good as that of \( \sigma \) against all strategies for the other players, and is strictly better against some of them. A strategy is admissible if no other strategy dominates it. More formally, we have:

**Dominance.** A strategy \( \sigma_i \in \Sigma_i \) very weakly dominates strategy \( \sigma'_i \in \Sigma_i \), written \( \sigma_i \succ_i \sigma'_i \), if \( \forall \sigma_{-i} \in \Sigma_{-i}, \text{payoff}(\sigma_i, \sigma_{-i}) \leq \text{payoff}(\sigma'_i, \sigma_{-i}) \). Strategy \( \sigma \) weakly dominates strategy \( \sigma' \), written \( \sigma \succ \sigma' \), if \( \sigma \succ' \) and \( \neg(\sigma' \succ \sigma) \). A strategy \( \sigma \in \Sigma_i \) is weakly dominated if there exists \( \sigma' \in \Sigma_i \) such that \( \sigma' \succ \sigma \). A strategy that is not weakly dominated is admissible. We denote by \( \mathfrak{A}_i(\mathcal{G}) \) the set of all admissible strategies for player \( i \) in \( \mathcal{G} \).

Our characterizations and algorithms are based on the notions of cooperative and antagonistic values of a history. The antagonistic value, denoted \( a\text{Val}_i(\mathcal{G}, h) \), is the maximum payoff that player \( i \) can secure from \( h \) in the worst case, i.e. against all strategies of other players. The cooperative value, denoted \( c\text{Val}_i(\mathcal{G}, h) \), is the best value player \( i \) can achieve from \( h \) with the help of other players. We also define a third type of value: the antagonistic-cooperative value, denoted \( ac\text{Val}_i(\mathcal{G}, h) \), which is the maximum value player \( i \) can achieve in \( \mathcal{G} \) with the help of other players while guaranteeing the antagonistic value of the current history \( h \). Formal definitions follow.
Antagonistic & Cooperative Values. The antagonistic value of a strategy and the cooperative value of a strategy $\sigma_i$ of player $i$ in $G$, for a history $h$ are

$$aVal_i(G, h, \sigma_i) \overset{\text{def}}{=} \inf_{\tau \in \Sigma_i} \text{payoff}(\text{Out}_h(\sigma_i, \tau));$$

$$cVal_i(G, h, \sigma_i) \overset{\text{def}}{=} \sup_{\tau \in \Sigma_i} \text{payoff}(\text{Out}_h(\sigma_i, \tau)).$$

The antagonistic value of a history $h$ for player $i$, and the cooperative value of a history $h$ for player $i$ are defined as $aVal_i(G, h) \overset{\text{def}}{=} \sup_{\sigma_i \in \Sigma_i} aVal_i(G, h, \sigma_i)$, and $cVal_i(G, h) \overset{\text{def}}{=} \sup_{\sigma_i \in \Sigma_i} cVal_i(G, h, \sigma_i)$, respectively. Finally, the antagonistic-cooperative value of a history $h$ for player $i$ is

$$acVal_i(G, h) \overset{\text{def}}{=} \sup\{cVal_i(G, h, \sigma_i) \mid \sigma_i \in \Sigma_i, aVal_i(G, h, \sigma_i) \geq aVal_i(G, h)\}.$$ 

We omit $G$ when it is clear from the context.

Observe that $aVal_i(h)$ of a history is the value of a zero-sum two-player game where player $i$ is playing against players $-i$; while $cVal_i(h)$ is the value in a one-player game, when all players play together. $acVal_i(h)$ is a new notion which is the supremum of the values player $i$ can obtain when he plays worst-case optimal strategies. A strategy $\sigma_i \in \Sigma_i$ is said to be worst-case optimal for player $i$ at history $h$ if $aVal_i(h, \sigma_i) = aVal_i(h)$; it is said to be cooperatively optimal for him at history $h$ if $cVal_i(h, \sigma_i) = cVal_i(h)$. Observe that $acVal_i(h) = -\infty$ if there are no worst-case optimal strategies from $h$.

**Example 1** (Local conditions are not sufficient). The game in Fig. 2 shows that admissibility requires one to consider the values of the histories both in the past and in the future of the current history. This shows that a local condition cannot capture admissibility. In fact, consider strategy $\sigma_1$ of player 1 (who controls all square vertices) that takes the edges $(s_1, s_2), (s_4, s_6)$. If the game starts at $s_2$, $\sigma_1$ is admissible, since the choice $(s_4, s_5)$ could yield a payoff of 2 which is worse than any payoff from $s_6$. Indeed, we have that $aVal_1(s_5) < aVal_1(s_6)$. However, when the game starts at $s_1$, $\sigma_1$ is weakly dominated by the strategy that chooses $(s_1, s_3)$ since the worst payoff in the latter case is 5. In fact, when a strategy takes the edge $(s_1, s_2)$, the antagonistic value decreases from $aVal_1(s_1) = 5$ to $aVal_1(s_2) = 3$; so to be admissible, it should have a better cooperative value than 5, which is not the case if $(s_4, s_6)$ is taken. The strategy taking $(s_1, s_2), (s_4, s_5)$ is admissible. Indeed, in one outcome, the payoff is 9, which is greater than 5 as required. Thus, an admissible strategy from $s_1$ either goes to $s_3$, or goes to $s_2$ but commits to taking $(s_4, s_5)$ later.

We use temporal logic to describe sets of outcomes. We consider an extension of standard LTL with inequality conditions on payoffs for each player as in [3]. The logic, denoted LTL_{payoff}, extends LTL, and its syntax is defined as follows.

$$\varphi ::= Q \mid \neg \varphi \mid X \varphi \mid G \varphi \mid F \varphi \mid \varphi_1 \lor \varphi_2 \mid \varphi_1 \land \varphi_2 \mid \text{payoff}_i \gg v,$$
where $Q \in AP$ is a set of atomic propositions on edges, $G$ and $F$ are the standard LTL modalities, $\in\{\leq, \geq, <, >\}$, and $v \in Q$. A formula is interpreted over an outcome $\rho$ at index $k$ as follows. We have, for instance, $(\rho, k) \models Q$ if, and only if, $(\rho_k, \rho_{k+1})$ is labelled with $Q$. For convenience, we write $\rho \models \varphi$ instead of $(\rho, 1) \models \varphi$. Note that we define our predicates on edges rather than vertices; this simplifies our presentation. The semantics of the LTL modalities are standard; we refer to e.g. [3]. For payoff conditions, we have $(\rho, k) \models \text{payoff}_i \infty v \iff \text{payoff}_i(\rho \geq k) \infty v$.

Residual Games. Given game $G$, and history $h$, let us define $G_h$ as the residual game of $G$ from $h$ by modifying the initial state to last$(h)$, and the payoff functions to $\text{payoff}^f_h$ defined as follows. For all outcomes $\rho$ that start in last$(h)$, $\text{payoff}^f_h(\rho) = \text{payoff}(h \cdot \rho)$, where $h \cdot \rho = h_{\leq |h|-1} \cdot \rho$. Notice that the strategy sets of $G$ and $G_h$ are identical, and that for any $\sigma_p \in \Sigma_p$, we have $\text{Out}(G_h, \sigma_p) = \text{Out}_{\text{last}(h)}(G, \sigma_p)$.

Lemma 2. For all $h' \in \text{Hist}_{\text{last}(h)}(G)$, it holds that $a\text{Val}_1(G_h, h') = a\text{Val}_1(G, h \cdot h')$, $ac\text{Val}_1(G_h, h') = ac\text{Val}_1(G, h \cdot h')$, and $c\text{Val}_1(G_h, h') = c\text{Val}_1(G, h \cdot h')$.

3 Existence of Admissible Strategies

We start this section with two examples of quantitative games with no admissible strategies (for player 1). Then we identify a large and natural class of games for which the existence of admissible strategies is guaranteed.

Consider the games $A$ and $G$ in Fig. 4. Starting at $s_1$, the payoff of player 1, in the two games is defined as follows: an outcome that does not visit $s_3$ has a payoff equal to 0, otherwise, the payoff is equal to the number of times vertex $a$ appears in the outcome. The lemma below states that player 1 does not have admissible strategies in those two games. We sketch the proof idea.

Consider first the one-player game $A$. The antagonistic value at vertex $s_1$ is $\infty$. Any strategy which never visits $s_3$ is weakly dominated by strategies that visit $a$ at least once (i.e. with outcome $(s_1 a s_1)^+ s_2^\omega$). Furthermore, a strategy which does visit $s_3$ and $k$ times $a$ is weakly dominated by any strategy that visits $a$ at least $k + 1$ times and then goes to $s_3$.

The idea is similar for $G$ where the cooperative value at $s_1$ is $\infty$. Every strategy which does not allow outcomes visiting $s_3$ are weakly dominated by those that attempt to visit $a$ by visiting $s_2$ at least once (as from $s_2$, the other player can cooperate and visit $a$), and then go to $s_3$. Moreover, it is always possible to attempt to visit $a$ once more before going to $s_3$, thus any strategy which eventually goes to $s_3$ is also weakly dominated.

Lemma 3. Player 1 does not have admissible strategies in games $G$ and $A$.

In the two examples above, either the $a\text{Val}$ or the $c\text{Val}$ (which are both equal to $\infty$) are not achievable. This is not a coincidence. We now show that all the games that admit witnessing strategies for those values are guaranteed to have admissible strategies.
Games with strategies witnessing \(aVal\) and \(cVal\). A game is \textit{well-formed} whenever it admits witnessing strategies for \(aVal\) and \(cVal\), i.e., it satisfies:

1. For all \(i \in P\), and \(h \in \text{Hist}_{\text{out}}(G)\), \(\exists \sigma_i \in \Sigma_i, aVal_i(h, \sigma_i) = aVal_i(h)\).
2. For all \(i \in P\), and \(h \in \text{Hist}_{\text{out}}(G)\), \(\exists \sigma_i \in \Sigma_i, cVal_i(h, \sigma_i) = cVal_i(h)\).

These conditions will also be referred as Assumption 1 and 2.

We now establish the existence of admissible strategies for all well-formed games.

\textbf{Theorem 4.} In all well-formed games all players have admissible strategies.

The result follows from Lemmas. 6 and 7 below: the proof consists in showing that a particular type of admissible strategies, called \textit{strongly cooperative-optimal}, always exists. Usually, those strategies are only a strict subset of the admissible strategies available to a player. Nevertheless, they are peculiar as they are guaranteed to exist.

\textbf{Definition 5.} A strategy \(\sigma_i\) is \textit{strongly cooperative-optimal (SCO)} if for all \(h \in \text{Hist}_{\text{out}}(\sigma_i)\), if \(cVal_i(h) > aVal_i(h)\) then \(cVal_i(h, \sigma_i) = cVal_i(h)\), and if \(aVal_i(h) = cVal_i(h)\) then \(aVal_i(h, \sigma_i) = aVal_i(h)\).

Strongly cooperative-optimal strategies are admissible because their cooperative values are always maximal, and moreover, if a payoff better than the antagonistic value cannot be achieved \((aVal_i(h) = cVal_i(h))\), then they are worst-case optimal. Any strategy which obtains a better payoff than a SCO strategy against some adversary will obtain a worse payoff against another one.

\textbf{Lemma 6.} All strongly cooperative-optimal strategies are admissible.

\textbf{Proof.} Let \(\sigma_i\) be a strongly cooperative-optimal strategy for player \(i\). Assume towards a contradiction that some \(\sigma'_i\) weakly dominates \(\sigma_i\). Let \(h\) be any minimal history compatible with \(\sigma_i\) such that \(\sigma_i(h) \neq \sigma'_i(h)\).

If \(aVal_i(h) < cVal_i(h)\), then since \(\text{last}(h)\) is controlled by player \(i\), \(aVal_i(h\sigma'_i(h)) \leq aVal_i(h) < cVal_i(h)\), and since \(\sigma_i\) is strongly cooperative optimal \(cVal_i(h\sigma_i(h), \sigma_i) = cVal_i(h)\). Therefore, as the histories \(h\sigma_i(h)\) and \(h\sigma'_i(h)\) are distinct, there is a strategy \(\tau \in \Sigma_{-i}\) such that \(\text{payoff}_i(Out_{h\sigma_i(h)}(\sigma_i, \tau)) = cVal_i(h) > aVal_i(h) \geq \text{payoff}_i(Out_{h\sigma'_i(h)}(\sigma'_i, \tau))\). This contradicts that \(\sigma'_i\) weakly dominates \(\sigma_i\).

Otherwise \(aVal_i(h) = cVal_i(h)\), then since \(\sigma_i\) is strongly cooperative optimal, for all \(\tau \in \Sigma_{-i}\), \(\text{payoff}_i(Out_h(\sigma_i, \tau)) = cVal_i(h)\) and \(\text{payoff}_i(Out_h(\sigma'_i, \tau)) \leq cVal_i(h)\). It follows that no outcome of \(\sigma'_i\) obtains a better payoff than \(\sigma_i\). We thus obtain a contradiction.

By Lem. 6, to prove the existence of admissible strategies, it suffices to prove the existence of strongly cooperative-optimal strategies. We actually give a constructive proof.

\textbf{Lemma 7.} In all well-formed games all players have SCO strategies.

Let us describe the idea of the construction. Consider any player \(i\). We define the strategy \(\sigma\) of player \(i\) as follows. For any history \(h\), if \(aVal_i(h) = cVal_i(h)\), then \(\sigma\) plays a worst-case optimal strategy from \(h\), say \(\sigma_{h,\text{wco}}\). Otherwise, we define \(\sigma\) starting from an outcome \(\rho_h\) with \(\text{payoff}_i = cVal_i(h)\), and we define \(\sigma\) is such a way that it follows \(\rho_h\). In this case, whenever another player deviates from \(\rho_h\), say, at history \(h'\), we reevaluate how to play according to whether \(aVal_i(h') < cVal_i(h')\) or \(aVal_i(h') = cVal_i(h')\). Here, the existence of \(\sigma_{h,\text{wco}}\) and that of \(\rho_h\) are guaranteed by the fact that the game is well-formed.

In subsequent sections, we consider SCO strategies in residual games \(G_h\), so let us note that these games satisfy the required assumptions if \(G\) does, which follows from Lem. 2.
Lemma 8. For any well-formed game \( G \), for all histories \( h \in \text{Hist}_{\text{out}} \), the residual game \( G_h \) is also well-formed.

We end this section with an interesting observation: an infinite weak dominance chain is not necessarily dominated by an admissible strategy, as shown in the next example. The reader should contrast the example with the fact that in the Boolean case all dominated strategies are dominated by an admissible strategy [2, Thm. 11].

Example 9 (Non-dominated weak dominance chains). There are quantitative games that have infinite dominance chains and no “maximal” admissible strategy weakly dominating them. Consider the game depicted in Fig. 3. Denote by \( \sigma_k \) the strategy of player 1 (controlling square vertices) which consists in moving from \( s_1 \) to \( s_2 \) exactly \( k \) times, and then going left (unless payoff of 2 was reached in the meantime). Then for all \( k \in \mathbb{N} \), \( \sigma^k \) is weakly dominated by \( \sigma^{k+1} \) because if the adversary decides to move right from \( s_2 \) at the \( (k+1) \)-th step, \( \sigma^{k+1} \) performs better than \( \sigma^k \), and otherwise they yield identical outcomes. It follows that all strategies \( \sigma^k \) for \( k \geq 0 \), are dominated. Here, the only admissible strategy \( \sigma^\infty \) consists in looping in the cycle forever, which does not dominate any \( \sigma^k \) since if the adversary always moves left from \( s_2 \), then \( \sigma^\infty \) yields less than \( \sigma^k \).

Remark. Above, we have defined strongly cooperative-optimal strategies that favour cooperation whenever it can have an added value. We have established that those strategies are always admissible. There are other classes of strategies that are always admissible, and we define another interesting class here. A strategy \( \sigma_i \) is a worst-case cooperative optimal strategy, if for all \( h \in \text{Hist}_{\text{out}}(\sigma_i) \):

\[
\text{aVal}_i(h, \sigma_i) = \text{cVal}_i(h, \sigma_i) = \text{aVal}_i(h) = \text{acVal}_i(h),
\]

Intuitively, we will show that a strategy is admissible if at all histories, either the strategy promises a cooperative value greater than the antagonistic value at the current vertex, or a higher cooperative value cannot be obtained without risking a lower antagonistic value (i.e. \( \text{aVal}_i(h) = \text{acVal}_i(h) \)) and the strategy is worst-case optimal.

It turns out that requiring this property at all histories ending in a player’s vertices characterize admissible strategies. We state our result in the following theorem.

Lemma 10. All worst-case cooperative optimal strategies are admissible.

However, some well-formed games do not have worst-case cooperative optimal strategies.

4 Value-based Characterization of Admissible Strategies

We present our main result, which is, a value-based characterization of admissible strategies.

For any game \( G \), and player \( i \), let us define the following property, denoted \( \star(h, \sigma) \), for a given strategy \( \sigma \in \Sigma_i(G) \) and history \( h \):

\[
\text{cVal}_i(h, \sigma) > \text{aVal}_i(h)
\]

or

\[
\text{aVal}_i(h, \sigma) = \text{cVal}_i(h, \sigma) = \text{aVal}_i(h) = \text{acVal}_i(h),
\]

Intuitively, we will show that a strategy is admissible if at all histories, either the strategy promises a cooperative value greater than the antagonistic value at the current vertex, or a higher cooperative value cannot be obtained without risking a lower antagonistic value (i.e. \( \text{aVal}_i(h) = \text{acVal}_i(h) \)) and the strategy is worst-case optimal.

It turns out that requiring this property at all histories ending in a player’s vertices characterize admissible strategies. We state our result in the following theorem.

Theorem 11. Under Assumption 1, for any game \( G \), player \( i \), and \( \sigma_i \in \Sigma_i(G) \), \( \sigma_i \) is admissible if, and only if, for all \( h \in \text{Hist}_{\text{out}}(G, \sigma_i) \) with last(h) \( \in V_i \), \( \star(h, \sigma_i) \) holds.

It will be useful to consider the negation of \( \star(h, \sigma) \), which we simplify as follows:
Lemma 12. For all histories $h$ and strategy $\sigma$, the negation of $\prec(h, \sigma)$ is equivalent to
\[
\text{cVal}(h, \sigma) \leq \text{aVal}_i(h) \land \text{aVal}_i(h, \sigma) < \text{aVal}_i(h) \tag{3}
\]
\[
\lor \quad \text{cVal}(h, \sigma) = \text{aVal}_i(h, \sigma) = \text{aVal}_i(h) \land \text{acVal}_i(h) > \text{aVal}_i(h). \tag{4}
\]

Proof of Thm. 11. We prove the contrapositive. Assume that $\exists h \in \text{Hist}_\text{vna}(G, \sigma_i)$, last($h$) $\in V_i$ and $\neg \prec(h, \sigma_i)$. Then by Lem. 12, either (3) or (4) holds for $(h, \sigma_i)$.

Assume (3) holds for $(h, \sigma_i)$. By Assumption 1, there exists a worst-case optimal strategy $\sigma^\text{wco}_h$ from $h$, with $\text{aVal}_i(h, \sigma^\text{wco}_h) = \text{aVal}_i(h)$. Define $\sigma'_i = \sigma_i[h \leftarrow \sigma^\text{wco}_h]$. We claim that $\sigma'_i$ weakly dominates $\sigma_i$. In fact, for any $\sigma_{-i} \in \Sigma_{-i}(G)$ with $h \notin \text{Hist}_\text{vna}(G, \sigma_{-i})$, we have $\text{Out}_\text{vna}(G, \sigma_i, \sigma_{-i}) = \text{Out}_\text{vna}(G, \sigma'_i, \sigma_{-i})$. For any $\sigma_{-i} \in \Sigma_{-i}(G)$ compatible with $h$, both outcomes go through $h$. By definition of $\sigma'_i$, $\text{Out}_\text{vna}(G, \sigma'_i, \sigma_{-i}) = h_{\leq |h|-1} \text{Out}_\text{h}(G, \sigma^\text{wco}_h, \sigma_{-i})$. Therefore, we have that $\text{payoff}_i(\text{Out}_\text{vna}(G, \sigma'_i, \sigma_{-i})) \geq \text{aVal}_i(h)$ by definition of $\sigma^\text{wco}_h$. The latter is greater than $\text{cVal}_i(h, \sigma_i)$ from (3), so greater than $\text{payoff}_i(\text{Out}_\text{vna}(G, \sigma_i, \sigma_{-i}))$ by definition of $\text{cVal}_i(h)$. Thus, $\sigma'_i$ very weakly dominates $\sigma_i$. Since by assumption, $\text{aVal}_i(h, \sigma_i) < \text{aVal}_i(h)$, and $h$ is compatible with $\sigma_i$, there is a strategy $\sigma_{-i} \in \Sigma_{-i}$ such that $h \subseteq_{\text{pref}} \text{Out}_\text{vna}(G, \sigma_i, \sigma_{-i})$ and $\text{payoff}_i(\text{Out}_\text{vna}(G, \sigma_i, \sigma_{-i})) < \text{aVal}_i(h)$. As shown before, $\text{aVal}_i(h) \leq \text{payoff}_i(\text{Out}_\text{vna}(G, \sigma'_i, \sigma_{-i}))$. Hence, $\sigma'_i$ weakly dominates $\sigma_i$.

Assume now that (4) holds. Consider $\varepsilon > 0$ small enough so that $\text{acVal}_i(h) > \text{aVal}_i(h)+\varepsilon$. By definition of $\text{acVal}_i(h)$, there exists a strategy $\tau_i \in \Sigma_i$ such that $\text{cVal}_i(h, \tau_i) \geq \text{aVal}_i(h)+\varepsilon$, and moreover $\text{aVal}_i(h, \tau_i) \geq \text{aVal}_i(h)$. Consider $\tau_{-i} \in \Sigma_{-i}$ compatible with $h$ such that $\text{payoff}_i(\text{Out}_\text{h}(G, h, \tau_i \tau_{-i})) \geq \text{cVal}_i(h, \tau_i) - \frac{\varepsilon}{2} \geq \text{aVal}_i(h) + \frac{\varepsilon}{2} > \text{aVal}_i(h)$. Note that such a $\tau_{-i}$ exists by definition of $\text{cVal}_i(h, \tau_i)$. It follows that $\sigma_i[h \leftarrow \tau_i] \subseteq_{\text{pref}} \text{Out}_\text{h}(G, \sigma_i, \tau_{-i})$ weakly dominates $\sigma_i$. In fact, the outcomes are identical for any outcome not compatible with $h$. For any $\sigma_{-i}$ compatible with $h$, we have $\text{payoff}_i(\text{Out}_\text{vna}(G, \sigma_i, \sigma_{-i})) = \text{aVal}_i(h)$ by (4). Moreover, $\text{payoff}_i(\text{Out}_\text{vna}(G, \sigma'_i, \sigma_{-i})) \geq \text{aVal}_i(h)$ since at $h$ we have that $\text{aVal}_i(h, \tau_i) \geq \text{aVal}_i(h)$; thus $\text{aVal}_i(h, \sigma'_i) \geq \text{aVal}_i(h)$. Furthermore, we have $\text{payoff}_i(\text{Out}_\text{vna}(G, \sigma'_i, \tau_{-i})) \geq \text{aVal}_i(h)$ and $\text{payoff}_i(\text{Out}_\text{vna}(G, \sigma_i, \tau_{-i})) < \text{aVal}_i(h)$.

Assume that for all $h \in \text{Hist}_\text{vna}(G, \sigma_i)$ with last($h$) $\in V_i$, we have $\prec(h, \sigma_i)$, and that $\sigma_i$ is weakly dominated by some strategy $\sigma'_i$. We will show a contradiction.

Let $\sigma_{-i}$ be a strategy in $\Sigma_{-i}(G)$ and $\rho = \text{Out}_\text{vna}(G, \sigma_i, \sigma_{-i})$ and $\rho'_i = \text{Out}_\text{vna}(G, \sigma'_i, \sigma_{-i})$. If $\rho = \rho'$ then $\text{payoff}_i(\rho') \leq \text{payoff}_i(\rho)$ and otherwise let $j$ be the first index where they differ, and $h = \rho_{<j-1} = \rho'_{<j-1}$. We have that $h$ is compatible with both strategies, last($h$) $\in V_i$ and $\sigma_i(h) \neq \sigma'_i(h)$.

If (1) holds, that is, $\text{cVal}_i(h, \sigma_i) > \text{aVal}_i(h)$, consider $\varepsilon > 0$ such that $\text{cVal}_i(h, \sigma_i) > \text{aVal}_i(h) + \varepsilon$, and a strategy $\sigma'_{-i} \in \Sigma_{-i}$ which ensures that $\text{payoff}_i(\text{Out}_\text{h}(G, \sigma'_i, \sigma'_{-i})) \geq \text{cVal}_i(h, \sigma_i) - \frac{\varepsilon}{2},$ and $\text{payoff}_i(\text{Out}_\text{h}(G, \sigma_i, \sigma'_{-i})) \leq \text{aVal}_i(h, \sigma'_i) + \frac{\varepsilon}{2}$. Such a strategy profile $\sigma'_{-i}$ exists since $h\sigma'_i(h)$ and $h\sigma_i(h)$ are distinct, and since last($h$) $\in V_i$. The latter also implies that $\text{aVal}_i(h) \geq \text{aVal}_i(h, \sigma'_i)$. It thus follows that
\[
\text{payoff}_i(\text{Out}_\text{vna}(G, \sigma_i, \sigma_{-i}[h \leftarrow \sigma'_{-i}])) > \text{payoff}_i(\text{Out}_\text{vna}(G, \sigma'_i, \sigma_{-i}[h \leftarrow \sigma'_{-i}]))
\]
contradicting the fact that $\sigma'_i$ weakly dominates $\sigma_i$.

Therefore (2) must hold, and $\text{acVal}_i(h) = \text{aVal}_i(h)$. If there exists $j \geq |h|$ such that $\text{aVal}_i(\rho'_{<j}) < \text{aVal}_i(h)$, then there exists $\varepsilon > 0$ and a strategy profile $\sigma'_{-i} \in \Sigma_{-i}$ compatible with $h$ which ensures that $\text{payoff}_i(\text{Out}_\text{vna}(G, \sigma'_i, \sigma'_{-i})) \leq \text{aVal}_i(\rho'_{<j}) + \varepsilon < \text{aVal}_i(h) \leq \text{payoff}_i(\text{Out}_\text{vna}(G, \sigma_i, \sigma'_{-i}))$. This contradicts $\sigma'$ weakly dominating $\sigma$. Hence for all $j \geq |h|$, $\text{aVal}_i(\rho'_{<j}) \geq \text{aVal}_i(h)$. Now, observe that $\text{payoff}_i(\rho') \leq \text{acVal}_i(h)$. In fact, one can construct a strategy $\tau$, which, from $h$ follows $\rho'$, and in case another player does not respect $\rho'$, switches to a worst-case optimal strategy ensuring $\text{aVal}_i(\rho'_{<j}) \geq \text{aVal}_i(h)$. It follows
that \( \text{payoff}(\rho') \leq \text{cVal}_i(h, \tau) \leq \text{acVal}_i(h) \). Furthermore, by (2), \( \text{payoff}_i(\rho) \geq \text{acVal}_i(h) = \text{aVal}_i(h, \sigma_i) \), so \( \text{payoff}_i(\rho') \leq \text{payoff}_i(\rho) \). This being true for all strategies of \( \Sigma_{-i} \), proves that \( \sigma_i \) very weakly dominates \( \sigma'_i \) and contradicts that \( \sigma'_i \) weakly dominates \( \sigma_i \).

\[ \blacktriangleleft \]

## 5 Characterization of the Outcomes of Admissible Strategies

Observe that the characterization of Thm. 11 does not immediately yield an effective representation of the set of admissible strategies. In order to reason about the possible behaviors observable in a game under admissible strategies we are interested in describing the set of outcomes that can be observed when all players play admissible strategies. In this section, for each player, we give a linear temporal logic description of the outcomes that are each compatible with at least one admissible strategy.

Note that our main goal is to obtain such a characterization in full generality, for all well-formed games so we defer computability considerations to the next section. We will then see how the three types of values can be computed at all histories.

Let us fix a game \( G \), and player \( i \). We present the intuition of the characterization. If an outcome \( \rho \) is compatible with an admissible strategy, say \( \sigma_i \), then all prefixes \( h \) with \( \text{last}(h) \in V_i \) must satisfy (1) or (2). Given \( h \), if (1) holds, then two things can happen. Either \( \text{payoff}_i(\rho) > \text{Val}_i(h) \), and thus \( \rho \) witnesses \( \text{cVal}_i(h, \sigma_i) > \text{aVal}_i(h) \), or this is not the case but there is another outcome \( \rho' \) compatible with \( \sigma_i \) ending \( h \) with \( \text{payoff}_i(\rho') > \text{Val}_i(h) \). Notice how the longest common prefix of \( \rho \) and \( \rho' \) ends always with a vertex in \( V_{-i} \) since both outcomes are compatible with \( \sigma_i \). If (2) holds at \( h \), then, in particular, \( \text{payoff}_i(\rho) = \text{Val}_i(h) \), and, moreover, \( \text{aVal}_i \) remains constant at all prefixes of \( \rho \) extending \( h \). The last observation simply follows from \( \text{aVal}_i(h, \sigma_i) = \text{aVal}_i(h) \) which is implied by (2).

**Extended LTL\(_{\text{payoff}}\).** Let \( \text{aValues}_i = \{ \text{Val}_i(h) \mid h \text{ is a history} \} \) be the set of antagonistic values of player \( i \). We will now define atomic propositions attached to edges of a game. Formally, we have a **labelling function** \( \lambda : E \rightarrow \mathcal{P}(\text{AP}) \) which assigns to every edge a set of propositions from \( \text{AP} \). The set \( \text{AP} \) includes the proposition \( V_i \) whose truth value, for every edge \( e = (u, v) \), is determined as follows: \( V_i \in \lambda(u, v) \iff u \in V_i \).

We consider LTL\(_{\text{payoff}}\) with atomic propositions as defined above and additional propositions \( \text{aVal}_q^i, \text{acVal}_q^i, \) and \( \text{gAlt}_q^i \) defined for all \( q \in \text{aValues}_i \). The semantics of these are straightforward: for an outcome \( \rho \) and \( k \in \mathbb{N}_{\geq 0} \) we have

\[
\begin{align*}
(\rho, k) \models & \text{aVal}_q^i \iff \text{aVal}_i(\rho \leq k) = q, \\
(\rho, k) \models & \text{acVal}_q^i \iff \text{acVal}_i(\rho \leq k) = q, \text{ and} \\
(\rho, k) \models & \text{gAlt}_q^i \iff \rho_k \in V_{-i} \land \exists v' \neq \rho_k : (\rho_k, v') \in E \land \text{cVal}_i(\rho \leq k \cdot v') > q,
\end{align*}
\]

with the convention that, when \( k \) is omitted, we assume it is 1.

As mentioned earlier, we consider two cases depending on whether (1) or (2) hold. Thus, let us define the corresponding two sub-formulas:

\[
\begin{align*}
\varphi_1 & \equiv \bigvee_{q \in \text{aValues}_i} \left( \text{aVal}_q^i \land \left( \text{payoff}_i > q \lor \text{F}(\text{gAlt}_q^i) \right) \right), \text{ and} \\
\varphi_2 & \equiv \bigvee_{q \in \text{aValues}_i} \left( \text{aVal}_q^i \land \text{acVal}_q^i \land \text{payoff}_i = q \land \text{G} \left( \text{aVal}_q^i \right) \right).
\end{align*}
\]

We define the following formula which will be shown to capture the outcomes of admissible strategies: \( \Phi_{\text{adm}}^i \equiv \text{G} \left( \neg V_i \lor \varphi_1 \lor \varphi_2 \right) \).
Theorem 13. For any well-formed game $G$, outcome $\rho$ satisfies $\Phi_{\text{adm}}^i$ if, and only if, it is compatible with an admissible strategy for player $i$.

We give the idea of the proof. For any outcome $\rho$ compatible with an admissible strategy $\sigma$, for player $i$. We show that for any prefix $h$ of $\rho$ with last($h$) $\in V_i$, $(\rho, |h|)$ satisfies either $\varphi_1$ or $\varphi_2$. In fact, by Thm. 11, either (1) or (2) hold, and we show that these correspond to $\varphi_1$ and $\varphi_2$.

Conversely, for any $\rho$ satisfying $\Phi_{\text{adm}}^i$, we construct an admissible strategy $\sigma$, for player $i$ compatible with $\rho$. The strategy follows $\rho$, and in case of deviation, it switches immediately either to an SCO — which is guaranteed to exist — or to a worst-case optimal strategy, depending on whether $\varphi_1$ or $\varphi_2$ holds at the current history. The resulting strategy is proven to be admissible.

Assuming prefix-independence. Before concluding this section, let us consider the consequences of further assuming that our payoff function is prefix-independent.

3. For all $i \in P$, for all outcomes $\rho$, it holds that $\forall j \in \mathbb{N}, \text{payoff}_i((\rho_k)_{k \geq j}) = \text{payoff}_i(\rho)$.

Observe that, under Assumption 3, the set $\text{aValues}_i$ can be equivalently defined as $\{\text{aVal}_i(v) | v \in V\}$ and is thus finite. One can also extend the labelling $\lambda$ and set of atomic propositions $AP$ such that, for every edge $e = (u, v)$ and $q \in \text{aValues}_i$:

\[
\begin{align*}
\text{aVal}_i^q \in \lambda(u, v) & \iff \text{aVal}_i(u) = q, \\
\text{acVal}_i^q \in \lambda(u, v) & \iff \text{acVal}_i(u) = q, \text{ and} \\
g\text{Alt}_i^q \in \lambda(u, v) & \iff u \in V_{-i} \land \exists v' \neq v : (u, v') \in E \land \text{cVal}_i(v') > q.
\end{align*}
\]

It immediately follows that:

Lemma 14. Under Assumption 3, for all $i \in P$, $\Phi_{\text{adm}}^i$ is expressible in $\text{LTL}^{\text{payoff}}$.

6 Applications and Future Works

In this section, we show how to apply Theorem 11 (value-based characterization of admissible strategies) and Theorem 13 (characterization of the set of outcomes of admissible strategies) to solve relevant verification and synthesis problems.

Classical payoff functions. So far, we have assumed that games were equipped for each player $i \in P$ with a payoff function. To define payoff functions, we proceed as usual by first assigning weights to edges of the game graph using weight functions $w_i : E \rightarrow \mathbb{R}$, one for each player $i \in P$. With the weight function $w_i$, we associate to each outcome $\rho = p_1p_2\ldots p_n$, an infinite sequence of rational values $w_1(\rho) = w_i(p_1p_2)w_i(p_2p_3)\ldots w_i(p_np_{n+1})\ldots$, and we aggregate this sequence of values with measures such as $\text{Inf}$, $\text{Sup}$, $\text{LimInf}$, $\text{LimSup}$, and mean payoff ($\text{MP}$ and $\overline{\text{MP}}$). It is well known, see e.g. [7] and [18], that all the payoff functions defined above satisfy Assumptions 1-2. By Theorem 4, we get the following.

Lemma 15. In games with payoff functions from $\text{Inf}$, $\text{Sup}$, $\text{LimInf}$, $\text{LimSup}$, $\text{MP}$, and $\overline{\text{MP}}$, all players have admissible strategies.

It is also known that, in games defined with the payoff functions considered here, the antagonistic and cooperative values ($\text{cVal}$ and $\text{aVal}$) are computable. One can also show that $\text{acVal}$ is computable for prefix-independent payoff functions. Indeed, this value of a
vertex coincides with the $cVal$ inside the sub-graph induced by the vertices with the optimal antagonistic value. Furthermore, using a classical transformation on the game structure, we can guarantee that all payoff functions. We thus obtain the following result, by Lemma 14.

- **Lemma 16.** In games with payoff functions from Inf, Sup, LimInf, LimSup, MP, and MP, the formulas $\Phi_{\text{adm}}^i$ for all $i \in P$ are effectively computable, finite, and expressible in $\text{LTL}_{\text{payoff}}$.

We will now consider several problems of interest which can be solved using the characterizations that we have developed in the previous sections. All the results are applicable to the measures concerned by Lemmas 15 and 16.

**Deciding the admissibility of a finite memory strategy.** As a first example, we consider the problem of deciding, given a game structure $\mathcal{G}$, and a (finite memory) strategy $\sigma_i$ for player $i \in P$ described as a finite state transducer $M_i$, if $\sigma_i$ is admissible in $\mathcal{G}$.

To solve this problem, we rely on Theorem 11 and proceed as follows. First, we compute for each vertex $v$ of the game $\mathcal{G}$, the values $aVal_i(\mathcal{G}, v)$, $cVal_i(\mathcal{G}, v)$, and $acVal_i(\mathcal{G}, v)$. Second, we construct the synchronized product between the transducer $M_i$ that defines the strategy $\sigma_i$ and the game $\mathcal{G}$. States in this product are of the form $(v, m)$ where $v$ is a vertex of $\mathcal{G}$ and $m$ is a (memory) state of the transducer $M_i$. Third, we compute for each state $(v, m)$ the values $aVal_i(\mathcal{G}, (v, m), \sigma_i)$, $cVal_i(\mathcal{G}, (v, m), \sigma_i)$, and $acVal_i(\mathcal{G}, (v, m), \sigma_i)$. Finally, we verify that there is no reachable vertex $(v, m)$ in the product where condition (1) or condition (2) are falsified. We then obtain the following theorem:

- **Theorem 17.** Given a game $\mathcal{G}$ and a finite memory strategy $\sigma_i$ for player $i \in P$ specified as a finite state transducer $M_i$, we can decide if $\sigma_i$ is an admissible strategy for player $i$ in $P\text{Time}$ for measures Inf, Sup, LimInf, LimSup; in $\text{NP} \cap \text{coNP}$ for MP, and MP.

**Model-checking under admissibility.** We now turn to the following problem. Given a game structure $\mathcal{G}$ and a $\text{LTL}_{\text{payoff}}$ formula $\varphi$, decide if all outcomes of the game that are compatible with the admissible strategies of all players satisfy $\varphi$, i.e. if $\bigwedge_{i \in P} \text{Out}(\mathcal{G}, \mathcal{A}_i(\mathcal{G})) \models \varphi$. This problem was introduced in the Boolean setting in [5] and allows one to check that a property is induced by the rationality of the players in a game.

- **Theorem 18.** For all measures Inf, Sup, LimInf, LimSup, MP, MP, one can decide, given game $\mathcal{G}$ and $\text{LTL}_{\text{payoff}}$ formula $\varphi$, whether $\bigwedge_{i \in P} \text{Out}(\mathcal{G}, \mathcal{A}_i(\mathcal{G})) \models \varphi$.

**Proof Sketch.** For each player $i \in P$, consider the formula $\Phi_{\text{adm}}^i$ from Theorem 13, which describes the set $\text{Out}(\mathcal{G}, \mathcal{A}_i(\mathcal{G}))$. The formula is finite and constructible by Lemma 16. The problem now amounts to verifying if $\mathcal{G}$ satisfies the specification $(\bigwedge_{i \in P} \Phi_{\text{adm}}^i) \implies \varphi$. For all payoff functions, except mean-payoff, this can be reduced to model checking an $\text{LTL}$ formula (since the measures are regular). For MP and MP, the result follows from [3] which shows that the model checking problem against $\text{LTL}_{\text{payoff}}$ is decidable.

**Quantitative assume-admissible synthesis.** In [4], a new rule for reactive synthesis in non-zero sum $n$-player games was proposed. The setting there is similar to the setting considered here but it is Boolean: each player $i \in P$ has his own omega-regular objective $O_i \subseteq V^\omega$. The synthesis rule asks if player $i \in P$ has a strategy to enforce its own objective $O_i$ against admissible strategies of the other players. In other words, the rule asks for the existence of worst-case optimal strategies against rational adversaries.
The quantitative extension of this problem asks given a game $G$, a player $i \in P$, and an LTL payoff formula $\phi$. Using Theorem 13, we can reduce this query to a plain two-player zero-sum game on the game structure $G$ with objective:

$$\exists \sigma \in \Sigma_i, \forall \tau \in \Sigma_{-i}, \Phi^i_{\text{adm}} \land (\bigwedge_{j \in P \setminus \{i\}} \Phi^j_{\text{adm}}) \implies \varphi$$

Since for Inf, Sup, LimInf, LimSup, $\Phi^i_{\text{adm}}$ and $\varphi$ are omega-regular, the problem reduces to deciding the winner in a two-player zero-sum game with omega-regular objectives. As a consequence, we obtain the following theorem:

**Theorem 19.** The quantitative assume-admissible synthesis problem for player $i \in P$ is decidable for measures Inf, Sup, LimInf, LimSup.

For the measures $MP$, $\overline{MP}$, we obtain objectives in which mean-payoff constraints and omega-regular constraints are mixed. On the one hand, those objectives are outside known decidable classes of objectives treated in [8] and in [10]. On the other hand, the undecidability results obtained in [17] do not apply to them. This motivates further research on zero-sum two player games with a mix of mean-payoff and omega-regular objectives.

**Towards iterative elimination.** Once we have computed the admissible strategies for each player, we restrict each player to these strategies, and repeat the computation of the admissible strategies in the restricted game. This can be iterated several times and gives a process that is called iterative elimination of dominated strategies, and well known in game theory. This process is difficult to analyze for mean-payoff, because objectives of different players interfere in non-trivial ways and games with Boolean combinations of mean-payoff objectives are undecidable [17]. However it seems feasible for regular payoffs, such as Inf, Sup, LimInf and LimSup, for which we can construct parity automata recognizing outcomes with payoff $q$. Given $i \geq 0$, we can actually compute a parity automaton accepting the set of outcomes of $S^i$ which is the set of strategies that remain after $i$ steps of elimination. We summarize here the ingredients but leave the details for future work. Assume we have a parity automaton representing the outcomes of $S^i$. Note that for $i = 0$ this is simply all outcomes. If the payoffs are regular, then we can compute values $c\text{Val}_i(h, S^i)$, $a\text{Val}_i(h, S^i)$ and $ac\text{Val}_i(h, S^i)$, which correspond to cooperative, antagonistic, and antagonist-cooperative values when players only play strategies from $S^i$. We can then use these values as atomic propositions for a LTL payoff formulas similar to $\Phi^i_{\text{adm}}$ of Section 5, which characterizes outcomes of strategies of $S^{i+1}$. In the case of regular payoffs this yields a parity automaton which represents the outcomes of $S^{i+1}$. This procedure can then be repeated to compute outcomes that are possible under iterative elimination.

**References**

Admissibility in Quantitative Graph Games


