Report from Dagstuhl Seminar 16482

Algorithms and Effectivity in Tropical Mathematics and Beyond

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Abstract

This report documents the Dagstuhl Seminar on Algorithms and Effectivity in Tropical Mathematics and Beyond, which took place from November 27 – December 02, 2016. The report contains an executive summary as well as abstracts of the talks which reflect recent progress in the topic of the meeting.

Seminar November 27 to December 2, 2016 – http://www.dagstuhl.de/16482
1998 ACM Subject Classification F.2.2 Nonnumerical Algorithms and Problems, G.2.1 Combinatorics, I.1.2 Algorithms
Keywords and phrases Algorithms in tropical mathematics, complexity, effective bounds, optimization, zero-sum games
Digital Object Identifier 10.4230/DagRep.6.11.168
Edited in cooperation with Thorsten Jörgens

1 Executive summary

Stéphane Gaubert
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Thorsten Theobald

Tropical mathematics is a uniting name for different research directions which involve the semi-ring of real numbers endowed with the operations min, + (called the tropical semi-ring). It has emerged in several areas of computer science and of pure and applied mathematics. For the first time, this seminar brought together the computer science and the mathematics viewpoints. A focus was on effective methods, algorithms and complexity bounds in tropical mathematics, and on their relations with open questions in various areas of computer science, including optimization, game theory and circuit complexity.

One of the oldest open algorithmic challenges in tropical mathematics is the complexity of solving systems of tropical linear equalities and inequalities. It is known to be equivalent to solving mean payoff games. The solvability of these problems is among the few known problems which are contained in the intersection NP $\cap$ co-NP, but not currently known to be in P. This leads to new approaches in linear programming or convex semialgebraic programming over nonarchimedean fields.

According to the organizers' points of view the seminar was quite successful. In addition to 28 talks there were many informal discussions and exchange of ideas in small groups. We expect several new common papers of the participants conceived during the seminar. An important feature was to bring together experts with different backgrounds who often knew other participants just by their publications. According to the opinions expressed, the participants learned a lot of new things. The seminar was especially useful for the young people.

Every talk, in addition to new results, also contained open problems. This created a lot of interaction in subsequent discussions. The audience was very active, many questions were posed to the speakers during the talks and the breaks.

The talks can be conditionally partitioned into the following groups, although there were many interrelations between different groups:

- Algorithmical problems of foundations of tropical mathematics (H. Markwig, D. Maclagan, F. Rincon, V. Podolskii);
- Complexity of games and of tropical linear and convex algebra (M. Bodirsky, S. Gaubert, T. Hansen, M. Joswig, G. Loho, M. MacCaig, B. Schröter, S. Sergeev, M. Skomra);
- Algorithms and complexity bounds on tropical varieties (F. Bihan, D. Grigoriev, S. Hampe, A. Jensen, T. Jörgens, L. Tabera, T. Theobald, T. de Wolff). We mention that S. Hampe has made a demonstration of the software Polymake for computations in tropical algebra;
- Algorithms in tropical differential algebra (F. Aroca, C. Garay);
- Interactions of tropical mathematics with algorithmic issues in classical mathematics (M. Akian, X. Allamigeon, Bo Lin, M. Rojas, C. Vinzant).

During the seminar a manuscript appeared (on the Internet) with the very strong claim of a quasi-polynomial complexity algorithm for parity games. It was a lucky coincidence that so many experts with various backgrounds were present. So a special evening session on Thursday was created to analyze this result and its ramifications. This was one more highlight.

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3 Overview of Talks

3.1 Majorization inequalities for valuations of eigenvalues using tropical algebra

Marianne Akian (INRIA Saclay – Île-de-France, FR)

We consider a matrix with entries over the field of Puiseux series, equipped with its non-archimedean valuation (the leading exponent). We establish majorization inequalities relating the sequence of the valuations of the eigenvalues of a matrix with the tropical eigenvalues of its valuation matrix (the latter is obtained by taking the valuation entrywise). We also show that, generically in the leading coefficients of the Puiseux series, the precise asymptotics of eigenvalues, eigenvectors and condition numbers can be determined. For this, we apply diagonal scalings constructed from the dual variables of a parametric optimal assignment constructed from the valuation matrix.

Next, we establish an archimedean analogue of the above inequalities, which applies to matrix polynomials with coefficients in the field of complex numbers, equipped with the modulus as its valuation.

This talk covers joint works with Ravindra Bapat, Stéphane Gaubert, Andrea Marchesini, Meisam Sharify, and Francoise Tisseur.

3.2 Log-barrier interior-point methods are not strongly polynomial

Xavier Allamigeon (INRIA Saclay – Île-de-France, FR)

We identify a class of path-following interior-point methods which are not strongly polynomial. This class corresponds to primal-dual interior-point methods which iterates in the so-called "wide" neighborhood of the central path arising from the logarithmic barrier. It includes short step, long step as well as predictor-corrector types of interior-point methods.

We establish a lower bound on the number of iterations of these methods when applied to some parametric families of linear programs. The lower bound is expressed in terms of the number of tropical segments in a piecewise linear curve arising from the tropicalization of the central path. In this way, we derive that the aforementioned interior point methods require $\Omega(2^d)$ iterations to solve a linear program in dimension $O(d)$ which we studied in a previous work.

3.3 The Groebner fan of a differential ideal in tag ring of differential polynomials.

Fuensanta Aroca (Universidad Nacional Autonoma – Mexico, MX)

We will start with the Newton polygon of an algebraic plane curve, the Newton polytope of an algebraic hypersurface, and the tropical variety of an ideal. Then we will see the extension of these concepts to the differential case.

3.4 Fewnomial polynomial systems in the complex, real and tropical settings

Frédéric Bihan (University Savoie Mont Blanc – Le Bourget-du-Lac, FR)

In this talk I will first give a short survey of what is known about the number of solutions of a square polynomial system. I will mainly recall classical upper bounds in the complex, real and tropical settings. Tropical polynomial systems can be used to construct real polynomial systems with many real solutions. In the transversal case, this is known as the classical patchworking theorem. The non transversal case is more delicate since we don't know which tropical solutions lift to real ones in general. I will present a new construction due to Boulos El Hilany which uses a non transversal intersection of two plane tropical curves to produce a system with a record number of real positive solutions.

3.5 Max-closed semilinear constraint satisfaction problems

Manuel Bodirsky (TU Dresden, DE)

A semilinear relation S is max-closed if it is preserved by taking the componentwise maximum. The constraint satisfaction problem for max-closed semilinear constraints is at least as hard as determining the winner in Mean Payoff Games, a notorious problem of open computational complexity. Mean Payoff Games are known to be in the intersection of NP and co-NP, which is not known for max-closed semilinear constraints. Semilinear relations that are max-closed and additionally closed under translations have been called tropically convex in the literature. One of our main results is a new duality for open tropically convex relations, which puts the CSP for tropically convex semilinear constraints in general into NP $\cap$ co-NP. This extends the corresponding complexity result for and-or precedence constraints aka the max-atoms problem. To this end, we present a characterization of max-closed semilinear relations in terms of syntactically restricted first-order logic, and another characterization in terms of a finite set of relations L that allow primitive positive definitions of all other relations in the class. We also present a subclass of max-closed constraints where the CSP is in P; this class generalizes the class of max-closed constraints over finite domains, and the feasibility problem for max-closed linear inequalities. Finally, we show that the class of max-closed semilinear constraints is maximal in the sense that as soon as a single relation that is not max-closed is added to L, the CSP becomes NP-hard.

3.6 Lopsided approximation of amoebas

Timo de Wolff (Texas A&M University – College Station, US)

The amoeba of a Laurent polynomial is the image of the corresponding hypersurface under the coordinatewise log absolute value map. We demonstrate that a theoretical amoeba approximation method due to Purbhoo can be used effectively in practice. To do this, we resolve the main bottleneck in Purbhoo's method by exploiting relations between cyclic resultants, which allow us to reduce the number of arithmetic operations needed to compute the 2^k-th cyclic resultant of a univariate degree d polynomial from $O(d2^k)$ to $O(kd^2)$. We use the same approach to give an approximation of a semi-algebraic description of the Log preimage of the amoeba of a Laurent polynomial. We also provide a `SINGULAR/SAGE` implementation of these algorithms, which shows a significant speedup when our specialized cyclic resultant computation is used, versus a general purpose resultant algorithm.

3.7 The fundamental theorem of tropical differential algebraic geometry

Cristhian Garay (Fluminense Federal University – Rio de Janeiro, BR)

Joint work of Fuensanta Aroca, Cristhian Garay, Zeinab Toghani

Let I be an ideal of the ring of Laurent polynomials $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ with coefficients in a real-valued field (K, v). The fundamental theorem of tropical algebraic geometry states the equality $\operatorname{trop}(V(I)) = V(\operatorname{trop}(I))$ between the tropicalization $\operatorname{trop}(V(I))$ of the closed subscheme $V(I) \subset (K^*)^n$ and the tropical variety $V(\operatorname{trop}(I))$ associated to the tropicalization of the ideal $\operatorname{trop}(I)$.

In this talk we discuss an analogous result for a differential ideal G of the ring of differential polynomials $K[[t]]\{x_1, \ldots, x_n\}$, where K is an uncountable algebraically closed field of characteristic zero. We define the tropicalization $\operatorname{trop}(\operatorname{Sol}(G))$ of the set of solutions $\operatorname{Sol}(G) \subset K[[t]]^n$ of G, and the set of solutions $\operatorname{Sol}(\operatorname{trop}(G)) \subset \mathcal{P}(\mathbb{Z}_{\geq 0})^n$ associated to the tropicalization of the ideal $\operatorname{trop}(G)$. These two sets are linked by a tropicalization morphism $\operatorname{trop} : \operatorname{Sol}(G) \longrightarrow \operatorname{Sol}(\operatorname{trop}(G))$.

We show the equality $\operatorname{trop}(\operatorname{Sol}(G)) = \operatorname{Sol}(\operatorname{trop}(G))$, answering a question recently raised by D. Grigoriev.

3.8 The equivalence between zero-sum games and tropical convexity: non-linear Perron-Frobenius and metric fixed point methods

Stéphane Gaubert (INRIA Saclay – Île-de-France, FR)

Shapley operators represent the one day evolution of the value of a zero-sum two player game. They can be characterized as being non-expansive in a weak metric (Funk's metric) which arises in the Perron-Frobenius theory of non-linear maps over cones. The sets of sub or super fixed points of Shapley operators are precisely the (closed) tropical convex sets. In this way, metric fixed point techniques can be applied to tropical convexity. We shall survey some tools, like the existence of the mean payoff for nonexpansive mappings definable in an o-minimal structure, ergodicity conditions for zero-sum games, Kohlberg's invariant half-lines, non-linear spectral radii and Collatz-Wielandt type theorems. Then, we will discuss the applications of these results to tropical convexity and zero-sum games, starting with the equivalence between zero-sum games and tropical linear feasibility problems, and ending with extensions of this equivalence to different classes of non-expansive mappings, including ones arising in risk sensitive control and entropy games.

This talk is based on the following references:

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3.9 Bounds on the number of connected components of tropical prevarieties

Dima Grigoriev (Lille I University, FR)

For a tropical prevariety in $\mathbb{R}^n$ given by a system of k tropical polynomials in n variables with degrees at most d, we prove that its number of the connected components is less than $\binom{k+7n-1}{3n} \cdot \frac{d^{3n}}{k+n+1}$. A similar bound (moreover, on the Betti numbers) for semi-algebraic sets was established in the papers by Oleinik-Petrovskii, Milnor, Thom, Basu-Pollack-Roy.

On a number of 0-dimensional connected components a better bound $\binom{k}{n} \cdot \frac{d^n}{k-n+1}$ is obtained, which extends the Bezout bound due to B. Sturmfels from the case $k = n$ to an arbitrary $k \geq n$.

Also we show that the latter bound is close to sharp, in particular, the number of connected components can depend on k.

3.10 Tropical computations in `polymake`

Simon Hampe (TU Berlin, DE)

In this talk I'll showcase some of the features of polymake that deal with tropical things. Most of these are new features of the recent polymake release 3.0. Examples include: Tropical arithmetic (Min or Max – it's your choice!), tropical convex hull computations, tropical linear spaces, tropical polynomials and tropical hypersurfaces. I will also present my own extension `a-tint`, which mostly deals with tropical intersection theory (but has a much wider range of features than that) and which has been included as a bundled extension in polymake.

3.11 An improved algorithm for feasibility of tropical linear programs

Thomas Dueholm Hansen (Aarhus University, DK)

Deciding feasibility of a tropical linear program was recently shown to be equivalent to solving a mean payoff game by Akian, Gaubert, and Guterman (2012). Allamigeon, Benchimol, Gaubert, and Joswig (2014) showed that combinatorial simplex algorithms for linear programming can solve mean payoff games. In this talk I present an improved version of the best known combinatorial simplex algorithm for linear programming, the Random-Facet pivoting rule by Kalai (1992) and Matousek, Sharir, and Welzl (1992), which due to the above reductions gives the best known algorithm for deciding whether a tropical linear program is feasible. The same improved running time is obtained for the more general class of abstract LP-type problems.

The talk is based on joint work with Uri Zwick, and in particular the paper *An Improved Version of the Random-Facet Pivoting Rule for the Simplex Algorithm* (2015).

3.12 Homotopy continuation for tropical polynomial systems

Anders Jensen (Aarhus University, DK)

In this talk we solve systems of n tropical polynomial equations in n unknowns by homotopy continuation, meaning that solutions are tracked as the coefficients of the system vary generically. Symbolically perturbing the coefficients gives an algorithm which is memoryless and exact, while it at the same time has complexity bounds similar to those of a recent algorithm by Malajovich. As in the numerical algebraic geometry case, all isolated tropical solutions are obtained. We report on running times for our implementation in the setting of mixed volume computation.

3.13 Linear programs over fields of Puiseux fractions

Michael Joswig (TU Berlin, DE)

Joint work of Xavier Allamigeon, Pascal Benchimol, Stéphane Gaubert, Michael Joswig, Georg Loho, Benjamin Lorenz, Benjamin Schröter

The formal Puiseux series with complex coefficients form the field which is probably the most popular for tropicalization. We start out by giving a short survey on known variations. For applications to linear programming we are particularly interested in real variants, as these admit an ordering. One example is the field of univariate rational functions with real coefficients, which was already studied by Hilbert. This construction can be extended to rational or even real exponents, and we call the resulting expressions *Puiseux fractions*. They have a natural finite encoding, which makes them useful for computations. In fact, they have been implemented in the software system `polymake`. Further, they have natural primal and dual valuations, such that they are particularly well suited to situations where min and max occur together. We show that generalizations of the classical linear programs of Klee and Minty can be interpreted over Puiseux fractions. Moreover, via Puiseux fractions we investigate a family of linear programs whose central paths exhibit an unusually large total curvature.

3.14 Imaginary projections of polynomials

Thorsten Jörgens (Goethe-Universität – Frankfurt am Main, DE)

Joint work of Thorsten Jörgens, Thorsten Theobald, Timo de Wolff

For a polynomial $f \in \mathbb{C}[z_1, \ldots, z_n]$, its amoeba $\mathcal{A}(f)$ is defined as the image of its variety in the complex torus $(\mathbb{C}^*)^n$ under the log-absolute map $\log|\cdot|$, respectively as projection to the real part of the complex logarithm, $\mathrm{Re} \circ \log_{\mathbb{C}}(\cdot)$. Its behavior at infinity described by the logarithmic limit set

$$\lim_{r\to\infty} \left(\frac{1}{r}\mathcal{A}(f) \cap \mathbb{S}^{n-1} \right)$$

is one way to define a tropical variety.

We replace the underlying map by the projection of the variety onto its imaginary part. It turns out that the occurring structure, its imaginary projection $\mathcal{I}(f)$, has many similarities to amoebas, such as the convexity of the complement components. But there are also some differences, for example the limit set

$$\lim_{r\to\infty} \left(\frac{1}{r}\mathcal{I}(f) \cap \mathbb{S}^{n-1} \right)$$

is not always polyhedral. In this talk, we consider this structure from several viewpoints and discuss the situation at infinity.

3.15 Computing linear systems on metric graphs

Bo Lin (University of California – Berkeley, US)

The linear system $|D|$ of a divisor D on a metric graph has the structure of a cell complex. And the set $R(D)$ of corresponding tropical rational functions has the structure of a tropical semimodule. We introduce the anchor divisors and anchor cells in it – they serve as the landmarks for us to compute the f-vector of the complex and find all cells in the complex.

Then we compute the minimal set of generators of $R(D)$ using the landmarks. We apply these methods to some examples – namely the canonical linear systems of some small trivalent graphs.

3.16 Abstract tropical linear programming

Georg Loho (TU Berlin, DE)

Analogously to classical oriented matroid programming, we give the first feasibility algorithm for signed tropical oriented matroids, which encode the structure of tropical linear inequality systems. The feasibility problem for these systems is of special interest as it is in NP $\cap$ co-NP but no polynomial time algorithm is known.

To encode the combinatorial structure of tropical linear inequality systems, we equip tropical oriented matroids, which were introduced to study the combinatorics of tropical point configurations, with an additional sign information. We generalize the feasibility problem for tropical linear inequality systems to signed tropical oriented matroids. This allows us to formulate pivoting between basic covectors which only depends on the combinatorial structure and not on the coefficients of the inequalities.

3.17 On integer points in tropical polytopes

Marie MacCaig (Ecole Polytechnique – Palaiseau, FR)

Joint work of Peter Butkovic, Stephane Gaubert, Marie MacCaig

We provide a brief overview of algorithms and complexity relating to finding integer points in tropical polytopes, discussing results for both tropical polytopes describes by vertices, and tropical polytopes described by inequalities.

For the question of existence of an integer point, pseudopolynomial algorithms exist but the complexity is unresolved. We outline a special case which is polynomially solvable and additionally show that adding a simple restriction to the solution set makes the problem NP-hard.

Further we discuss the complexity of counting the number of integer points in a tropical polytope, and the related problem of calculating the volume. We prove that, for tropical polytopes described by inequalities, these problems are #P-hard and, for tropical polytopes described by vertices, even approximation is hard.

3.18 Tropical commutative algebra

Diane Maclagan (University of Warwick – Coventry, GB)

In joint work with Felipe Rincón we have introduced a special class of ideals, called tropical ideals, in the semiring of tropical polynomials, motivated by describing subschemes of tropical toric varieties. We show that these ideals have a well-behaved Groebner theory, with a finite Groebner complex. Consequences include a simple version of the weak Nullstellensatz. The definition involves the theory of valuated matroids.

3.19 Computing faithful tropicalizations of curves: a moduli space perspective

Hannah Markwig (Universität Tübingen, DE)

Joint work in progress with Maria Angelica Cueto, Ralph Morrison and Ilya Tyomkin.

The tropicalization of a subvariety of a torus (or, more generally, of a toric variety) in general depends on the chosen embedding. A faithful tropicalization can be viewed as an embedding for which the tropicalization captures important geometric properties. We demonstrate how one can use modifications to compute faithful tropicalizations for curves for the case of elliptic curves embedded as plane cubics. We also discuss the situation of curves of genus two, three and four, and the relation to moduli spaces of (plane) tropical curves.

3.20 Polynomials in tropical algebra and related algorithmic problems

Vladimir Podolskii (Steklov Institute – Moscow, RU)

In this talk we will review some recent results on tropical polynomials including studies of systems of tropical linear polynomials and a tropical analog of Hilbert's Nullstellensatz.

We will discuss solvability problem and equivalence problem for tropical linear systems. We will discuss their computational complexity. We will also consider the problem of computing the dimension of the solution set of a given tropical linear systems.

For general polynomial systems we will discuss several versions of Hilbert's Nullstellensatz. We will present effective versions of these theorems. The upper bounds on the degrees of polynomials in effective tropical Nullstellensätze have matching lower bounds.

3.21 Tropical ideals

Felipe Rincon (University of Oslo, NO)

Joint work of Diane Maclagan, Felipe Rincon

We study a special class of ideals, called tropical ideals, in the semiring of tropical polynomials, with the goal of developing a useful and solid algebraic foundation for tropical geometry. The class of tropical ideals strictly includes the tropicalizations of classical ideals, and it satisfies many desirable properties that mimic the classical setup. In particular, every tropical ideal has an associated variety, which we prove is always a finite polyhedral complex. In addition we show that tropical ideals satisfy the ascending chain condition, even though they are typically not finitely generated, and also the weak Nullstellensatz.

3.22 Tropical intersection multiplicity and complexity

J. Maurice Rojas (Texas A&M University – College Station, US)

Joint work of Pascal Koiran, J. Maurice Rojas, Natacha Portier

We recall the Shub-Smale τ-Conjecture, which is a statement on integer roots of polynomials that implies a variant of P$\neq$NP. This conjecture remains open but, via some p-adic tricks, we give two new statements that are potentially easier to prove and still imply new separations of complexity classes. Also, these new variants of the τ-Conjecture lead to interesting open questions on the notion of intersection multiplicity for tropical varieties over p-adic fields. A key subtlety is that current tropical intersection products have not so far focussed on the number of distinct valuations (without multiplicity) in the tropically degenerate case.

3.23 Tropical linear spaces and k-splits

Benjamin Schröter (TU Berlin, DE)

Tropical (uniform) linear spaces are polyhedral complexes, which are dual to a subdivision of the hypersimplex into matroid polytopes. We give explicit constructions for such subdivisions, in particular, for coarsest nontrivial subdivisions. Our focus will be on those tropical linear spaces, whose bounded cells form a k-simplex. These are the so-called k-splits of the hypersimplex. From those tropical linear spaces we derive a new class of matroids, called split matroids. As an application we obtain a dimension bound for the moduli space of those tropical linear spaces, the Dressian.

3.24 Tropical linear algebra with the Lukasiewicz T-norm

Sergei Sergeev (University of Birmingham, GB)

The max-T semiring is defined as the unit interval $[0, 1]$ equipped with the arithmetics $a + b = \max(a, b)$ and ab being a T-norm. Linear algebra and convex geometry over such semiring can be developed in the usual way [1]. We then focus on the case of Lukasiewicz T-norm $\max(0, a + b - 1)$. We observe that any problem of the max-Lukasiewicz linear algebra can be equivalently formulated as a problem of the tropical (max-plus) linear algebra.

Based on this equivalence, we develop a theory of the matrix powers and the eigenproblem over the max-Lukasiewicz semiring [2].

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3.25 Tropical spectrahedra and stochastic mean payoff games

Mateusz Skomra (Ecole Polytechnique – Palaiseau, FR)

Joint work of Xavier Allamigeon, Stéphane Gaubert, Mateusz Skomra

Semidefinite programming (SDP) is a fundamental tool in convex and polynomial optimization. It consists in minimizing the linear functions over the spectrahedra (sets defined by linear matrix inequalities). In particular, SDP is a generalization of linear programming. In practice, approximate solutions to SDPs are obtained using interior point methods, but solving those programs exactly is a challenging task. One of the basic algorithmic questions associated with SDP is to decide whether a spectrahedron is nonempty. It is unknown whether this problem belongs to NP in the Turing machine model, and the state-of-the-art algorithms that certify the (in)feasibility of spectrahedra are based on cylindrical decomposition or the critical points method. We study the nonarchimedean analogue of this problem, replacing the field of real numbers by the field of Puiseux series. We introduce the notion of *tropical spectrahedra*, and show that, under genericity conditions, these objects can be described explicitly by systems of polynomial inequalities in the tropical semiring. Furthermore, we demonstrate a subclass of tropical spectrahedra which encode Shapley operators associated with stochastic mean payoff games. As a result, we show that a large class of semidefinite feasibility problems defined over Puiseux series can be solved efficiently using combinatorial algorithms designed for stochastic games.

3.26 Tropical discriminants in positive characteristic

Luis Tabera (University of Cantabria, ES)

Joint work of Alicia Dickenstein, María Isabel Herrero, Luis Tabera

In this talk I will present some results on tropical A-discriminants in positive characteristic and the p-adic case. Unlike the resultant, the A-discriminant depends on the characteristic of the underlying algebraic field. I will also show a discussion of all the cells in a tropical Severi variety in characteristic zero and show that characteristic p discriminants helps us understand some cells of these tropical Severi varieties.

3.27 From combinatorial tropical intersections to the mixed Ehrhart polynomial

Thorsten Theobald (Goethe-Universität – Frankfurt am Main, DE)

Joint work of Christian Haase, Martina Juhnke-Kubitzke, Raman Sanyal, Thorsten Theobald

For lattice polytopes $P_1, \ldots, P_k \subseteq \mathbb{R}^d$, Bihan [1] introduced the discrete mixed volume $\mathrm{DMV}(P_1, \ldots, P_k)$ in analogy to the classical mixed volume. Departing from combinatorial questions on the intersection of the tropical hypersurfaces defined by tropical polynomials $f_1, ..., f_k$ in d variables with Newton polytopes $P_1, \ldots, P_k$ (as studied in [3]), the associated mixed Ehrhart polynomial $\mathrm{ME}_{P_1,\ldots,P_k}(n) = \mathrm{DMV}(nP_1, \ldots, nP_k)$ arises.

In the talk we discuss the mixed Ehrhart polynomial as well as its occurrence in tropical geometry. We provide a characterization of all mixed Ehrhart coefficients in terms of the classical multivariate Ehrhart polynomial. We also show that for large enough dilates $rP_1, \ldots, rP_k$ the corresponding mixed h^*-polynomial has only real roots and as a consequence the mixed h^*-vector becomes non-negative.

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3.28 Tropical hyperbolic polynomials

Cynthia Vinzant (North Carolina State University – Raleigh, US)

Hyperbolic and stable polynomials are real multivariate polynomials with certain topological properties. Their hypersurfaces bound convex cones, and optimization over these cones generalizes linear and semidefinite programming. There have been several interesting developments relating stable polynomials to matroids and other tropical objects. I will give an introduction to these polynomials and their connections to optimization and tropical geometry and present some related open questions.

Participants

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