Irrational Guards are Sometimes Needed

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Abstract

In this paper we study the art gallery problem, which is one of the fundamental problems in computational geometry. The objective is to place a minimum number of guards inside a simple polygon so that the guards together can see the whole polygon. We say that a guard at position $x$ sees a point $y$ if the line segment $xy$ is contained in the polygon.

Despite an extensive study of the art gallery problem, it remained an open question whether there are polygons given by integer coordinates that require guard positions with irrational coordinates in any optimal solution. We give a positive answer to this question by constructing a monotone polygon with integer coordinates that can be guarded by three guards only when we allow to place the guards at points with irrational coordinates. Otherwise, four guards are needed. By extending this example, we show that for every $n$, there is a polygon which can be guarded by $3n$ guards with irrational coordinates but needs $4n$ guards if the coordinates have to be rational. Subsequently, we show that there are rectilinear polygons given by integer coordinates that require guards with irrational coordinates in any optimal solution.

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Figure 1 Till, Mikkel, and Anna are meticulously guarding the polygon. They are a little irrational, but pretty optimal.

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1 Introduction

For a polygon $P$ and points $x, y \in P$, we say that $x$ sees $y$ if the line segment $xy$ is contained in $P$. A guard set $S$ is a set of points in $P$ such that every point in $P$ is seen by some point in $S$. The points in $S$ are called guards. The art gallery problem is to find a minimum cardinality guard set for a given simple polygon $P$ on $n$ vertices. Such a guard set is called optimal. The polygon $P$ is considered to be filled, i.e., it consists of a closed, simple polygonal curve in the plane and the bounded region enclosed by this curve.

This classical version of the art gallery problem has been originally formulated in 1973 by Victor Klee (see the book of O’Rourke [20, page 2]). It is often referred to as the interior-guard art gallery problem or the point-guard art gallery problem, to distinguish it from other versions that have been introduced over the years.

Chvátal proved in 1975 that $\lceil n/3 \rceil$ guards are always sufficient and sometimes necessary to guard a polygon with $n$ vertices [9]. A simpler proof was later found by Fisk [15]. Since then, the art gallery problem has been extensively studied, both from the combinatorial and the algorithmic perspective. Most of this research, however, is not focused directly on the classical art gallery problem, but on its numerous versions, including different definitions of visibility, restricted classes of polygons, restrictions on the positions of the guards, etc. For more detailed information we refer the reader to the surveys [26, 28, 20, 22].

Despite extensive research on the art gallery problem, no combinatorial algorithm for finding an optimal solution, or even for deciding whether a guard set of a given size $k$ exists, is known. The only exact algorithm is attributed to Micha Sharir (see [12]), who has shown that in $n^{O(k)}$ time one can decide whether a guard set consisting of $k$ guards exists. This result is obtained by using standard tools from real algebraic geometry [2], and it is not known how to find an optimal solution without using this powerful machinery (see [3] for an analysis of the very restricted case of $k = 2$). Some recent lower bounds [5] based on the exponential time hypothesis suggest that there might be no better exact algorithms than the one by Sharir.

To explain the difficulty in constructing exact algorithms, we want to emphasize that it is not known whether the decision version of the art gallery problem (i.e., the problem of deciding whether there is a guard set consisting of $k$ guards, where $k$ is a parameter) lies in the complexity class NP. While NP-hardness and APX-hardness of the art gallery problem have been shown for different versions of the problem [18, 25, 27, 6, 13, 21, 17], the question of whether the point-guard art gallery problem is in NP remains open. A simple way to show NP-membership would be to prove that there always exists an optimal set of guards with rational coordinates of polynomially bounded description.

Sándor Fekete posed at MIT in 2010 and at Dagstuhl in 2011 an open problem, asking whether there are polygons requiring irrational coordinates in an optimal guard set [14, 1]. The question has been raised again by Günter Rote at EuroCG 2011 [23]. It has also been mentioned by Rezende et al. [10]: “it remains an open question whether there are polygons given by rational coordinates that require optimal guard positions with irrational coordinates”.

A similar question has been raised by Friedrichs et al. [16]: “[…] it is a long-standing open problem for the more general Art Gallery Problem (AGP): For the AGP it is not known whether the coordinates of an optimal guard cover can be represented with a polynomial number of bits”.

Our results. We answer the open question of Sándor Fekete by proving the following result. Recall that a polygon $P$ is called monotone if there exists a line $l$ such that the intersection between any line orthogonal to $l$ and $P$ is either empty or a single line segment.
Theorem 1. There is a simple monotone polygon \( P \) with integer vertex coordinates such that
1. \( P \) can be guarded by 3 guards, and
2. an optimal guard set of \( P \) with guards at points with rational coordinates has size 4.

An interesting consequence of Theorem 1 is that there is no optimal guard set of \( P \) among a candidate set of guard positions consisting of intersections between extensions of chords and edges of \( P \). It does not help to expand the candidate set by adding a line through each pair of candidates, thus creating new intersections to be added to the set of candidates, or to repeat this procedure any finite number of iterations, since all candidate points created by such a process must inevitably have rational coordinates. This shows that algorithms based on this procedure, as well as other algorithms for the art gallery problem which consider only rational points as possible guard positions, will in general not find an optimal guard set.

We then extend Theorem 1 by providing a family of polygons for which the ratio between the size of an optimal rational guard set and the size of an optimal set with irrational guards allowed is \( \frac{4}{3} \).

Theorem 2. There is a family of simple polygons \( (P_n)_{n \in \mathbb{Z}^+} \) with integer vertex coordinates such that
1. \( P_n \) can be guarded by \( 3n \) guards, and
2. an optimal guard set of \( P_n \) with guards at points with rational coordinates has size \( 4n \).
Moreover, the coordinates of the points defining the polygons \( P_n \) are polynomial in \( n \).

We show that the phenomenon with guards at irrational coordinates occurs already in the much simpler class of rectilinear polygons, i.e., polygons where each edge is parallel to the \( x \)-axis or to the \( y \)-axis.

Theorem 3. There is a rectilinear polygon \( P_R \) with vertices at integer coordinates satisfying the following properties.
1. \( P_R \) can be guarded by 9 guards.
2. An optimal guard set of \( P_R \) with guards at points with rational coordinates has size 10.

The Structure of the Paper. Section 2 contains the description of a monotone polygon \( P \) with vertices at points with rational coordinates that can be guarded by three guards only if the guards are placed at points with irrational coordinates. In Section 3, we describe the intuition behind our construction, and explain how we have found the polygon \( P \). The formal proof of Theorems 1 and 2 is then provided in Section 4. In Section 5, we present the rectilinear polygon \( P_R \) from Theorem 3 requiring guards with irrational coordinates in an optimal guard set. Finally, in Section 6 we suggest some open problems for future research.

2 The Polygon

In Figure 2 we present the polygon \( P \). In Section 4 we will prove that \( P \) can be guarded by three guards only when we allow the guards to be placed at points with irrational coordinates.

The polygon \( P \) is constructed as follows. We start with a basic rectangle \([0, 20] \times [0, 4] \subset \mathbb{R}^2\). Then, we append to it six triangular pockets (colored with green in the figure), which are triangles defined by the following coordinates:

- \( T_1^L \) : \{(2, 4), (2, 4.5), (2.1, 4)\},
- \( T_1^R \) : \{(2, 0), (2, −0.5), (1.9, 0)\},
- \( T_2^L \) : \{(16 \frac{1}{2}, 4), (17 \frac{1}{2}, 4.15), (17 \frac{1}{2}, 4)\},
- \( T_2^R \) : \{(3.5, 0), (3, −0.15), (3, 0)\},
- \( T_3^L \) : \{(19, 4), (19, 4.5), (19.1, 4)\},
- \( T_3^R \) : \{(19, 0), (19, −0.5), (18.9, 0)\}.

SoCG 2017
Figure 2 The polygon $P$. We will show that $P$ can be guarded by three guards only when we allow the guards to be placed at points with irrational coordinates. For practical reasons, the blue rectangular pockets are drawn shorter than they actually are.
Next, we append three rectangular pockets (colored with blue in the figure, for practical reasons these pockets are drawn in the figure shorter than they actually are), which are rectangles defined in the following way.

- $R_ℓ$: $[-10,0] \times [1.7,1.8]$, $R_r$: $[20,30] \times [0.5,0.6]$, and $R_m$: $[10.5,10.6] \times [4,8]$.

Last, we append four quadrilateral pockets (colored with red in the figure), which are defined by points with the following coordinates:

- Top-left pocket $P_ℓ^t$: $(4, 4), (4, \frac{280}{27}), (8, \frac{294}{27}), (8,4)$
- Top-right pocket $P_r^t$: $(12,4), (12, \frac{2486}{275}), (16,\frac{1776}{275}), (16,4)$
- Bottom-left pocket $P_b^t$: $(4,0), (4, -\frac{19}{12}), (8,-\frac{18}{12}), (8,0)$
- Bottom-right pocket $P_r^b$: $(12,0), (12, -\frac{34}{12}), (16, -\frac{36}{12}), (16,0)$

The polygon $P$ is clearly monotone. We will denote by $e_ℓ^t$, $e_r^t$, $e_b^t$, and $e_b^r$ the non-axis-parallel edge within each of the four quadrilateral pockets, respectively.

### 3 Intuition

In this section, we explain the key ideas behind the construction of the polygon $P$. Our presentation is informal, but it resembles the work process that lead to the construction of $P$ more than the formal proof of Theorem 1 in Section 4 does. Here we omit all “scary” computations and focus on conveying the big picture. In the end of this section, we also explain how we actually constructed the polygon $P$.

Define a rational point to be a point with two rational coordinates. An irrational point is a point that is not rational. A rational line is a line that contains two rational points. An irrational line is a line that is not rational.

**Forcing a Guard on a Line Segment.** Consider the drawing of the polygon $P$ in Figure 2. We will now explain an idea of how three pairs of triangular pockets, $(T_ℓ^t, T_b^t)$, $(T_r^t, T_b^b)$, and $(T_r^m, T_b^m)$, can enforce three guards on three line segments within $P$.

Consider the two triangular pockets in Figure 3a. The blue line segment contains one edge of each of these pockets, and the interiors of the pockets are at different sides of the line segment. A guard which sees the point $t$ must be placed within the orange triangular region, and a guard which sees $b$ must be placed within the yellow triangular region. Thus, a single guard can see both $t$ and $b$ only if it is on the blue line segment $t_b$, which is the intersection of the two regions.

Consider now the case that we have $k$ pairs of triangular pockets and no two regions corresponding to different pairs of pockets intersect. In order to guard the polygon with $k$ guards, there must be one guard on the line segment corresponding to each pair. Our
Irrational Guards are Sometimes Needed

**Figure 4** Left: The guard \( g_2 \) must be inside the triangular region (or to the left of it) in order to guard the entire part of the polygon that is not seen by \( g_1 \). Right: All possible positions of the point \( i \) define a simple curve \( C \).

The polygon \( \mathcal{P} \) has three such pairs of pockets (see Figure 3b), and it can be checked that the corresponding regions do not intersect. Note that in this way we can only enforce a guard to be on a rational line as the line contains vertices of the polygon, which are rational points.

**Restricting a Guard to a Region Bounded by a Curve.** For the following discussion, see Figure 4 and notation therein. We want to guard the polygon from Figure 4 using two guards, \( g_1 \) and \( g_2 \). We assume that \( g_1 \) is forced to lie on the blue vertical line segment \( l \).

Consider some position of \( g_1 \) on \( l \) such that \( g_1 \) can see at least one point of the top edge \( e_t \) of the top quadrilateral pocket and at least one point of the bottom edge \( e_b \) of the bottom quadrilateral pocket. Let \( p_t \) and \( p_b \) denote the leftmost points seen by \( g_1 \) on \( e_t \) and \( e_b \), respectively. Observe that \( p_t \) moves to the right if \( g_1 \) moves up and to the left if \( g_1 \) moves down. The point \( p_t \) behaves in the opposite way when \( g_1 \) is moved. Consider some fixed position of \( g_1 \) on the blue line segment, and the corresponding positions of \( p_t \) and \( p_b \). Let \( b \) be the bottom right corner of the top pocket and \( d \) the top right corner of the bottom pocket. Let \( i \) be the intersection point of the line containing \( p_t \) and \( b \) with the line containing \( p_b \) and \( d \). The points \( b, d, i \) define a triangular region \( \Delta \). It is clear that if we place the guard \( g_2 \) anywhere inside \( \Delta \), then \( g_1 \) and \( g_2 \) will together see the entire polygon. On the other hand, if we place \( g_2 \) to the right of \( \Delta \), then \( g_1 \) and \( g_2 \) will not see the entire polygon, as some part of the top or the bottom pocket will not be seen.

Now, let us move the guard \( g_1 \) along \( l \). Each position of \( g_1 \) yields an intersection point \( i \). We denote the union of all these intersection points by \( \mathcal{C} \) (see the right picture in Figure 4). It is easy to see that \( \mathcal{C} \) is a simple curve.

Note that \( g_2 \) sees a larger part of both pockets if it is moved horizontally to the left and a smaller part of both pockets if it is moved horizontally to the right. Consider a fixed position of \( g_2 \) or to the right of the segment \( bd \). Let \( g_2' \) be the horizontal projection of \( g_2 \) on \( \mathcal{C} \). Let \( g_1 \) be the unique position on \( l \) such that \( g_1 \) and \( g_2' \) see all of the polygon. If \( g_2 \) is to the left of \( \mathcal{C} \), \( g_2' \) sees less of the pockets than \( g_2 \), so \( g_1 \) and \( g_2 \) can together see everything. If \( g_2 \) is to the right of \( \mathcal{C} \), \( g_2 \) sees less of the pockets than \( g_2' \) and neither the top nor the bottom pocket are completely guarded by \( g_1 \) and \( g_2 \). For any higher placement of \( g_1 \) even less of the top pocket is guarded and for any lower placement of \( g_1 \) even less of the bottom pocket is guarded. Thus, there exists no placement of \( g_1 \) such that both pockets are completely guarded by \( g_1 \) and \( g_2 \). We summarize our reasoning in the following observation.

**Observation 4.** Consider a fixed position of \( g_2 \) on or to the right of the segment \( bd \). There exists a position of \( g_1 \) on \( l \) such that the entire polygon is seen by \( g_1 \) and \( g_2 \) if and only if \( g_2 \) lies on or to the left of the curve \( \mathcal{C} \).
Restricting a Guard to a Single (Irrational) Point. For this paragraph, let us consider the polygon $\mathcal{P}$ introduced in Section 2, and consider a guard set for $\mathcal{P}$ consisting of three guards. The polygon $\mathcal{P}$ is drawn in Figure 5 with additional labels and information. The three guards $g_\ell, g_m, g_r$ are forced by the triangular pockets to lie on the three green line segments $l_\ell, l_m, l_r$, respectively. Additionally, the three rectangular pockets $R_\ell, R_m, R_r$ force the guards to lie within one of two or three short intervals within each line segment. (These properties of our construction will be discussed in more detail in Section 4.) With these restrictions, we will show that for the three guards to see the whole polygon, it must hold that the guards $g_\ell$ and $g_m$ can together see the left pockets $P_\ell^l$ and $P_\ell^b$ and the guards $g_m$ and $g_r$ can together see the right pockets $P_r^l$ and $P_r^b$.

The curve $c_\ell$ bounds from the right the feasible region for the guard $g_m$ such that $g_\ell$ and $g_m$ can together see the left pockets $P_\ell^l$ and $P_\ell^b$. Similarly, the curve $c_r$ bounds from the left the feasible region for the guard $g_m$ such that $g_r$ and $g_m$ can together see the right pockets $P_r^l$ and $P_r^b$. Thus, the only way that $g_\ell, g_m, g_r$ can see the whole polygon is when $g_m$ is within the grey region between $c_\ell$ and $c_r$. Our idea is to define the line segment $l_m$ so that it contains an intersection point of $c_\ell$ and $c_r$ while not entering the interior of the grey region. A simple computation with sage [11] outputs equations defining the two curves:

$$c_\ell : 138x^2 - 568xy - 1071y^2 - 3018x + 8828y + 15312 = 0,$$

$$c_r : 138x^2 - 156xy - 356y^2 - 1791x + 3296y + 1620 = 0.$$  

One can easily verify that the point $p = (3.5 + 5\sqrt{2}, 1.5\sqrt{2}) \approx (10.57, 2.12)$ lies on both curves and also on the line $l_m = \{(x, y) : y = 0.3x - 1.05\}$. Therefore, $p$ is a feasible (and at the same time irrational) position for the guard $g_m$. Moreover, by plotting $c_\ell, c_r$, and $l_m$ in $\mathcal{P}$ as in Figure 5, we get an indication that as we traverse $l_m$ from left to right, at the point $p$ we exit the area where $g_m$ and $g_\ell$ can guard together the two left pockets and at the same time we enter the area where $g_m$ and $g_r$ can guard together the two right pockets. Thus, the only feasible position for the guard $g_m$ is the irrational point $p$. A formal proof will be given in Section 4.

Searching for the Polygon. The simplicity of the ideas behind our construction does not reflect the difficulty of finding the exact coordinates for the polygon $\mathcal{P}$. The reader might for instance presume that most other choices of horizontal pockets would work if the line segment $l_m$ is changed accordingly. However, this is not the case.
It is easy to construct the pockets so that the corresponding curves \( c_l \) and \( c_r \) intersect at some point \( p \). We expect \( p \) to be an irrational point in general since the curves \( c_l \) and \( c_r \) are defined by two second degree polynomials, as indicated above. In our construction, we need to force \( g_m \) to be on a line segment \( l_m \) containing \( p \), but we can only force \( g_m \) to be on a rational line. Hence, we require the existence of a rational line that contains \( p \).

As any two rational lines intersect in a rational point, there can be at most one rational line containing the irrational point \( p \). Moreover, there exists a rational line containing \( p \) if and only if \( p = (r_1 + r_2 \alpha, r_3 + r_4 \alpha) \) for some \( r_1, r_2, r_3, r_4 \in \mathbb{Q} \), where \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) is an irrational number. The equation of the rational line containing \( p \) is then \( y = \frac{r_1}{r_2} \cdot x + (r_3 - r_1 \cdot \frac{r_1}{r_2}) \). We say that this line supports \( p \). Therefore, we should not hope that the intersection point of the curves \( c_l \) and \( c_r \) defined by arbitrarily chosen pockets will have a supporting line. Our main idea to overcome this problem has been to reverse-engineer the polygon, after having chosen the positions of the guards. We chose three irrational guards, all with supporting rational lines, and then defined the pockets so that \( g_m \) automatically became the intersection point between the curves \( c_l \) and \( c_r \) associated with the pockets.

We chose all three guards to have coordinates of the form \((r_1 + r_2 \sqrt{2}, r_3 + r_4 \sqrt{2})\) for \( r_1, r_2, r_3, r_4 \in \mathbb{Q} \). Assume, for the ease of presentation, that we already know that we can end up with a polygon described as follows. (In our initial attempts, our polygons were much less regular.) The polygon should consist of the rectangle \( R = [0, 20] \times [0, 4] \) with some pockets added. We would like the pockets to extrude vertically from the horizontal edges of \( R \) such that the pockets meet \( R \) along the segments \((4, 0)(8, 0), (12, 0)(16, 0), (4, 4)(8, 4)\), and \((12, 4)(16, 4)\), respectively.

We now explain the technique for constructing the bottom pocket to the left which should extrude from \( R \) vertically downwards from the corners \((4, 0)\) and \((8, 0)\). We have to define the edge \( e_b \), which is the bottom edge in the pocket. We want \( p_b \) to be a point on \( e_b \) such that \( g_l \) can only see the part of \( e_b \) from \( p_b \) and to the right, whereas \( g_m \) can only see the part of \( e_b \) from \( p_b \) and to the left. Therefore, we define \( p_b \) to be the intersection point between the line containing \( g_l \) and \((4, 0)\) and the line containing \( g_m \) and \((8, 0)\). It follows that \( p_b \) is of the form \((r_1 + r_2 \sqrt{2}, r_3 + r_4 \sqrt{2})\) for some \( r_1, r_2, r_3, r_4 \in \mathbb{Q} \). Hence, there is a unique rational line \( l \) supporting \( p_b \), and \( e_b \) must be a segment on \( l \). We therefore need that both of the points \((4, 0)\) and \((8, 0)\) are above \( l \), since otherwise we do not get a meaningful polygon. However, this is not the case for arbitrary choices of the guards \( g_l \) and \( g_m \). The other pockets add similar restrictions to the positions of the guards.

In the construction we had to take care of other issues as well. In particular, the line \( l_m \) which supports the guard \( g_m \) cannot enter the grey region between the two curves \( c_l \) and \( c_r \), as otherwise the position of \( g_m \) would not be unique, and the guard could be moved to a rational point. Also, the three lines \( l_l, l_m, l_r \) supporting the three guards \( g_l, g_m, g_r \) cannot intersect within the polygon.

4 Proof of Theorems 1 and 2

Basic observations. Recall the construction of the polygon \( \mathcal{P} \) as defined in Section 2, and consider a guard set of \( \mathcal{P} \) of cardinality at most 3. Let \( l_l, l_m, l_r \), respectively, be the restrictions of the following lines to \( \mathcal{P} \):

\[
x = 2, \quad y = 0.3x - 1.05, \quad \text{and} \quad x = 19.
\]

As argued in Section 3, the triangular pockets enforce a guard onto each of these lines.
Lemma 5. Consider any guard set \( S \) for \( \mathcal{P} \) consisting of at most 3 guards. Then (i) \(|S| = 3\), and (ii) there is one guard on each of the lines \( l_\ell, l_m, l_r \).

Now, consider the intervals \( i_1 = [0.5, 0.6) \) and \( i_2 = [1.7, 1.8] \). Similarly as for the case of triangular pockets, we can show that the rectangular pockets \( R_\ell, R_m, R_r \) enforce a guard with an \( x \)-coordinate in \([10.5, 10.6]\), and \( y \)-coordinates in \( i_1 \) and \( i_2 \), respectively.

Lemma 6. Consider any guard set for \( \mathcal{P} \) consisting of 3 guards. Then one of the guards has an \( x \)-coordinate in \([10.5, 10.6]\). For the remaining two guards, one has a \( y \)-coordinate in \( i_1 \) and the other has one in \( i_2 \).

Proof. From Lemma 5, there must be one guard \( g_\ell \) on \( l_\ell \), one guard \( g_m \) on \( l_m \), and the last guard \( g_r \) on \( l_r \). Recall that the rectangular pockets are as follows \( R_\ell : [-10, 0] \times [1.7, 1.8] \), \( R_m : [20, 30] \times [0.5, 0.6] \), and \( R_r : [10.5, 10.6] \times [4, 8] \). It is straightforward to check that none of the guards \( g_\ell, g_r \) can see the two top vertices of the pocket \( R_m \). Therefore, the middle guard \( g_m \) has to see both of these vertices, so it must have an \( x \)-coordinate in \([10.5, 10.6]\).

Then, as \( g_m \in l_m \), the \( y \)-coordinate of \( g_m \) is in \([2.1, 2.13]\). Therefore, \( g_m \) cannot see any of the left vertices of \( R_\ell \) or any of the right vertices of \( R_r \). These four vertices must be seen by the guards \( g_\ell \) and \( g_r \).

As some guard must see the bottom-left corner of the pocket \( R_\ell \), it must be placed at a height of at least 1.7. This guard cannot see any of the right vertices of \( R_r \). Therefore, the last guard must see both right vertices of \( R_r \), and its height must be within \( i_1 = [0.5, 0.6] \). Then, this guard cannot see any left vertex of the pocket \( R_\ell \), and the second guard must see both left vertices of the pocket, so its height must be within \( i_2 = [1.7, 1.8] \).

Dependences between guard positions. Let \( \{g_\ell, g_m, g_r\} \) be a guard set of \( \mathcal{P} \) with \( g_\ell \in l_\ell \), \( g_m \in l_m \), and \( g_r \in l_r \). We will now analyze dependencies between the positions of the guards that are caused by the quadrilateral pockets of \( \mathcal{P} \). Recall that the non-axis-parallel edges of these pockets are denoted by \( e_1^\ell, e_2^\ell, e_1^b, \) and \( e_2^b \).

We will first prove two technical lemmas.

Lemma 7. Let \( h \in [0, 4] \) be the height of the guard \( g_\ell \). If \( h > \frac{1481}{180} \approx 2.87 \) then \( g_\ell \) cannot see any point on \( e_1^\ell \), and otherwise it can see a part of \( e_1^\ell \) starting from the \( x \)-coordinate \( \frac{908-188h}{181-47h} \) and to the right of it. If \( h < \frac{9}{19} \approx 0.47 \) then \( g_\ell \) cannot see any point on \( e_2^b \), and otherwise it can see a part of \( e_2^b \) starting from the \( x \)-coordinate \( \frac{76h+12}{19h-3} \) and to the right of it.

Proof. Consider the guard \( g_\ell \) and the top-left pocket. The left-most point on \( e_1^\ell \) that \( g_\ell \) can see is at the intersection of the following two lines: the line containing \( g_\ell \) and the bottom-left corner of the pocket (i.e., the point \((4, 4)\)), and the line containing \( e_1^\ell \). If \( g_\ell = (2, h) \), then the equation of the first line is \( y = \frac{h}{2}x + (2h - 4) \). The second contains points \((4, \frac{200}{19})\) and \((8, \frac{294}{19})\), and its equation is \( y = \frac{4}{19}x + \frac{266}{19} \). The \( x \)-coordinate of the intersection is \( \frac{908-188h}{181-47h} \).

It reaches a value of 8 (i.e., the point coincides with the right endpoint of \( e_1^\ell \)) when \( h = \frac{1481}{180} \).

Now, consider the guard \( g_\ell \) and the bottom-left pocket. The leftmost point on \( e_2^b \) that \( g_\ell \) can see is at the intersection of the following two lines: the line containing \( g_\ell \) and the top-left corner of the pocket (i.e., the point \((4, 0)\)), and the line containing \( e_2^b \). The first of these lines has equation \( y = -\frac{h}{2}x + 2h \). The second line contains points \((4, -\frac{12}{19})\), \((8, -\frac{18}{19})\), and its equation is \( y = -\frac{3}{10}x - \frac{6}{19} \). The \( x \)-coordinate of the intersection is \( \frac{76h+12}{19h-3} \), which reaches 8 when \( h = \frac{9}{19} \).
Lemma 8. Let \( h \in [0, 4] \) be the height of the guard \( g_r \). If \( h > \frac{507}{200} = 2.028 \) then \( g_r \) cannot see any point on \( e^r_5 \), and otherwise it can see a part of \( e^r_5 \) starting from the \( x \)-coordinate \( \frac{4000h - 6768}{245h + 54} \) and to the left of it. If \( h < \frac{17}{12} \approx 1.417 \) then \( g_r \) cannot see any point on \( e^r_5 \), and otherwise it can see a part of \( e^r_5 \) starting from the \( x \)-coordinate \( \frac{224h - 56}{14h + 1} \) and to the left of it.

Proof. Consider the guard \( g_r \) and the top-right pocket. The rightmost point on \( e^r_5 \) that \( g_r \) can see is at the intersection of the following two lines: the line containing \( g_r \) and the bottom-right corner of the pocket (i.e., the point \((16, 4)\)), and the line containing \( e^r_5 \). If \( g_r = (19, h) \), then the equation of the first line is \( y = \frac{h - 2}{3}x + \frac{76 - 16h}{3} \). The second contains points \((12, \frac{2486}{275})\) and \((16, \frac{1776}{275})\), and its equation is \( y = -\frac{71}{150}x + \frac{3616}{375} \). The \( x \)-coordinate of the intersection is \( \frac{4000h - 6768}{245h + 54} \). It reaches a value of 12 (i.e., the point coincides with the left endpoint of \( e^r_5 \)) when \( h = \frac{507}{200} = 2.028 \).

Now, consider the guard \( g_r \) and the bottom-right pocket. The rightmost point on \( e^r_5 \) that \( g_r \) can see is at the intersection of the following two lines: the line containing \( g_r \) and the top-right corner of the pocket (i.e., the point \((16, 0)\)), and the line containing \( e^r_5 \). The first of these lines has equation \( y = \frac{h}{3}x - \frac{16h}{3} \). The second line contains points \((12, -\frac{340}{27})\), \((16, -\frac{26}{27})\), and its equation is \( y = -\frac{1}{12}x - \frac{4}{3} \). The \( x \)-coordinate of the intersection is \( \frac{224h - 56}{14h + 1} \), which reaches 12 when \( h = \frac{17}{12} \approx 1.21 \).

We will now further restrict possible positions of the guards.

Lemma 9. The \( y \)-coordinate of the guard \( g_r \) is in the interval \( i_1 = [0.5, 0.6] \), and the \( y \)-coordinate of the guard \( g_r \) is in the interval \( i_2 = [1.7, 1.8] \).

Proof. As the guards \( g_l \) and \( g_r \) lie on line segments \( l_t \) and \( l_r \), their \( x \)-coordinates are 2 and 19, respectively. From Lemma 6, the \( x \)-coordinate of \( g_m \) is in the interval \([10.5, 10.6]\). Also, one of the guards \( g_r, g_m \) has a \( y \)-coordinate in \( i_1 \), and the other one in \( i_2 \).

Suppose that the \( y \)-coordinate of \( g_r \) is in \( i_1 \), i.e., it is at most 0.6. Let \( v = (12, -\frac{340}{27}) \) be the left endpoint of the edge \( e^r_5 \). We will show that none of the guards can see \( v \). Clearly, as the \( x \)-coordinates of \( g_l \) and \( g_m \) are smaller than 12, neither of them can see \( v \). From Lemma 8, \( g_r \) cannot see \( v \). Therefore, the \( y \)-coordinate of \( g_r \) must be in \( i_1 \), and the \( y \)-coordinate of \( g_r \) in \( i_2 \).

Lemma 10. The guards \( g_l \) and \( g_m \) must together see all of \( e^l_1 \) and \( e^l_2 \), and the guards \( g_m \) and \( g_r \) must together see all of \( e^r_5 \) and \( e^r_6 \).

Proof. By the construction of \( P \), it holds that if a guard sees a point on one of the edges \( e^l_1, e^l_2, e^r_5, \) and \( e^r_6 \), then the guard sees an interval of the edge containing an endpoint of the edge. It now follows that if three guards together see one of these edges, then two do as well. In order to prove the lemma, it thus suffices to prove that

- \( g_l \) and \( g_r \) cannot together see any of the edges \( e^l_1, e^l_2, e^r_5, \) and \( e^r_6 \),
- \( g_l \) and \( g_m \) cannot together see any of the right edges \( e^r_6 \),
- \( g_m \) and \( g_r \) cannot together see any of the left edges \( e^l_1 \) and \( e^l_2 \).

We now prove that \( g_l \) and \( g_r \) cannot together see any of the right edges \( e^r_5 \) and \( e^r_6 \) (see Figure 6a). Since \( h \in i_2 \), Lemma 8 gives that \( g_r \) cannot see \( e^r_5 \) to the right of the point \((\frac{742}{275} \frac{1629}{275})\), and \( e^r_6 \) to the right of the point \((\frac{1736}{1411}, -\frac{216}{1411})\). It is now easy to verify that no point on \( l_t \) can see any of these two points. Hence, \( g_l \) and \( g_r \) cannot together see any of the edges \( e^r_5 \) and \( e^r_6 \).

We now prove that \( g_l \) and \( g_r \) cannot together see \( e^l_1 \) (see Figure 6b). Since the \( y \)-coordinate of \( g_r \) is in \( i_2 \), it follows that \( g_r \) does not see any point on \( e^l_1 \). Since the \( x \)-coordinate of \( g_r \) is less than 4, neither \( g_l \) nor \( g_r \) can see the left endpoint of \( e^l_1 \).
The guard \( e \) together see any of the right pockets.

To show that \( g_r \) and \( g_r \) cannot together see the edge \( e^b_r \), we argue as follows (see Figure 6b). The guard \( g_r \) is placed at a height of at most 0.6, and \( g_r \) at a height of at most 1.8. It follows from Lemma 7 and from elementary computations that neither of the guards can see the interval of \( e^b_r \) with \( x \)-coordinates between \( \frac{2075}{507} < 4.1 \) and \( \frac{52}{9} > 6.8 \).

As the \( x \)-coordinate of both \( g_r \) and \( g_m \) is smaller than 12, none of these guards can see the left endpoint of the edges \( e_r^r \). Therefore, \( g_r \) and \( g_m \) cannot together see any of the edges \( e^r_r \). Similarly, as the \( x \)-coordinates of \( g_m \) and \( g_r \) are greater than 8, \( g_m \) and \( g_r \) cannot together see \( e^r_r \) or \( e^b_r \). This completes our proof. □

Computing the unique solution. We can now show that there is only one guard set for \( P \) consisting of three guards. Let us start by computing the right-most possible position of \( g_m \) such that \( g_r \) and \( g_m \) can see together both left pockets.

**Lemma 11.** The maximum \( x \)-coordinate of \( g_m \) such that \( g_r \) and \( g_m \) can together see \( e^r_r \) and \( e^b_r \) is \( x = 3.5 + 5\sqrt{2} \). The corresponding position of \( g_r \) is \((2, 2 - \sqrt{2})\).

**Proof.** Consider the guard \( g_r \) at position \((2, h)\). From Lemma 9, we know that \( h \in [0.5, 0.6] \). If \( g_m \) and \( g_r \) together see \( e^r_r \), we know from Lemma 7 that \( g_m \) has to be on or below the line containing the vertices \((8, 4)\) and \((\frac{908-188h}{181-47h}, \frac{2}{54}, \frac{908-188h}{181-47h} + \frac{206}{47})\), i.e., the line with equation \( y = \frac{92 - 23h}{135 + 47h} x + \frac{-1276 + 372h}{135 + 47h} \). As \( g_m \) is at the line \( y = 0.3x - 1.05 \), its \( x \)-coordinate satisfies \( 0.3x - 1.05 \leq \frac{92 - 23h}{135 + 47h} x + \frac{-1276 + 372h}{135 + 47h} \), i.e., \( x \leq \frac{28355 - 8427h}{2650 - 742h} \).

If \( g_m \) and \( g_r \) together see \( e^b_r \), then \( g_m \) has to be on or above the line containing the vertices \((8, 0)\) and \((\frac{76h + 12}{196 - 3} - \frac{4}{196 - 3}, \frac{76h + 12}{196 - 3} - \frac{6}{176})\), i.e., the line with equation \( y = \frac{36h}{196 - 9} x - \frac{24h}{196 - 9} \), i.e., \( x(1 - h) \leq \frac{81h + 189}{54} \). Therefore, since \( h < 1 \), we must have \( x \leq \frac{81h + 189}{54 - 54h} \). The first of the two values decreases with \( h \), and the second one increases with \( h \). Therefore the maximum is obtained when \( \frac{81h + 189}{54 - 54h} = \frac{81h + 189}{54} = 1.5 \), i.e., for \( h = 2 - \sqrt{2} \). The value of \( x \) is then \( 3.5 + 5\sqrt{2} \). The corresponding position of the guard \( g_r \) is \((2, h) = (2, 2 - \sqrt{2})\). □

Similarly, we can compute the left-most possible position of \( g_m \) such that \( g_m \) and \( g_r \) can see together both right pockets.

**Lemma 12.** The minimum \( x \)-coordinate of \( g_m \) such that \( g_r \) and \( g_m \) can see both \( e^r_r \) and \( e^b_r \) is \( x = 3.5 + 5\sqrt{2} \). The corresponding position of \( g_r \) is \((19, 1 + \sqrt{2})\).

**Proof.** Consider the guard \( g_r \) at position \((19, h)\). From Lemma 9, we know that \( h \in [1.7, 1.8] \). If \( g_m \) and \( g_r \) together see \( e^r_r \), we know from Lemma 8 that \( g_m \) has to be on or below the
Irrational Guards are Sometimes Needed

Figure 7 A sketch of a polygon that can be guarded by 6 guards when irrational coordinates are allowed, but needs 8 guards when only rational coordinates are allowed.

line containing the vertices (12, 4) and \( \left( \frac{4000h - 9768}{250h - 645}, \frac{-71}{190} \right) \), i.e., the line with equation \( y = \frac{46h - 184}{250h - 507}x + \frac{44h + 180}{250h - 507} \). As \( g_m \) is at the line \( y = 0.3x - 1.05 \), its \( x \) coordinate satisfies: \( 0.3x - 1.05 \leq \frac{46h - 184}{250h - 507}x + \frac{44h + 180}{250h - 507} \), i.e., \( x \geq \frac{490h - 243}{200h + 22} \).

If \( g_m \) and \( g_r \) together see \( \ell_r \) then \( g_m \) has to be on or above the line containing the vertices (12, 0) and \( \left( \frac{224h - 56}{14h + 1}, \frac{-1}{42} \right) \), i.e., the line with equation \( y = \frac{6h}{17 - 14h}x - \frac{72h}{17 - 14h} \). Hence, the \( x \)-coordinate of \( g_r \) must satisfy \( 0.3x - 1.05 \geq \frac{6h}{17 - 14h}x - \frac{72h}{17 - 14h} \), i.e., \( x \geq \frac{34h - 7}{14h - 2} \).

We have to minimize the value of \( \max \left\{ \frac{490h - 243}{200h + 22}, \frac{34h - 7}{14h - 2} \right\} \). When the value of \( h \) increases, the first of these two values increases, and the second one decreases. The minimum value is therefore obtained when \( \frac{490h - 243}{200h + 22} \approx \frac{34h - 7}{14h - 2} \), i.e., for \( h = 1 + \sqrt{2} \). The value of \( x \) is then 3.5 + 5\( \sqrt{2} \).

We are now ready to prove our main theorems.

Proof of Theorem 1. Let \( \mathcal{P} \) be the polygon constructed as in Section 2, and let \( S \) be a guard set for \( \mathcal{P} \) consisting of at most 3 guards. From Lemma 5 we have \( |S| = 3 \), and there is one guard at each of the lines \( l_t, l_m, l_r \). Denote these guards by \( g_t, g_m, g_r \), respectively. From Lemma 10 we know that if \( g_t, g_m, \) and \( g_r \) together see all of \( \mathcal{P} \), then \( g_t \) and \( g_m \) must see all of \( \ell_t \) and \( \ell_m \), and \( g_t \) and \( g_r \) must see all of \( \ell_t \) and \( \ell_r \). It then follows from Lemmas 11 and 12 that \( g_m \) must have coordinates \( (3.5 + 5\sqrt{2}, 1.5\sqrt{2}) \approx (10.57, 2.12) \), \( g_t = (2, 2 - \sqrt{2}) \approx (2, 0.59) \), and \( g_r = (19.1 + 2\sqrt{2}) \approx (19.1, 7.1) \). Thus, indeed, the guards \( g_t, g_m, \) and \( g_r \) see the entire polygon \( \mathcal{P} \) and are the only three guards doing so.

By scaling \( \mathcal{P} \) up by the least common multiple of the denominators in the coordinates of the corners of \( \mathcal{P} \), we obtain a polygon with integer coordinates. This does not affect the number of guards required to see all of \( \mathcal{P} \).

In order to guard \( \mathcal{P} \) using four guards with rational coordinates, we choose two rational guards \( g_m' \) and \( g_m'' \) on \( l_m \) a little bit to the left and to the right of \( g_m \), respectively. The guard \( g_m' \) sees a little more of both of the edges \( \ell_t \) and \( \ell_r \) than does \( g_m \), whereas \( g_m'' \) sees a little more of \( \ell_t \) and \( \ell_r \). Therefore, we can choose a rational guard \( g_t' \) on \( l_t \) close to \( g_t \) such that \( g_t' \) and \( g_m' \) together see \( \ell_t \) and \( \ell_r \), and a rational guard \( g_r' \) on \( l_r \) with analogous properties. Thus, \( g_t', g_m', g_m'', g_r' \) guard \( \mathcal{P} \).

Proof of Theorem 2. We will now construct a polygon \( \mathcal{P}_n \) that can be guarded by \( 3n \) guards placed at points with irrational coordinates, but such that when we restrict guard positions
to points with rational coordinates, the minimum number of guards becomes $4n$. We start by making $n$ copies of the polygon $\mathcal{P}$ described above, which we denote by $\mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(n)}$. We connect the copies into one polygon $\mathcal{P}_n$ as follows. Each consecutive pair $\mathcal{P}^{(i)}, \mathcal{P}^{(i+1)}$ is connected by a thin corridor consisting of a horizontal piece $H^{(i)}$ visible by the rightmost guard in $\mathcal{P}^{(i)}$, and a vertical piece $V^{(i)}$ visible to the middle guard in $\mathcal{P}^{(i+1)}$ (see Figure 7 for the case $n = 2$). We can then guard $\mathcal{P}_n$ using $3n$ guards, by placing three guards within each polygon $\mathcal{P}^{(i)}$ in the same way as for $\mathcal{P}$, i.e., at irrational points.

Now, assume that $\mathcal{P}_n$ can be guarded by at most $4n - 1$ guards. We will show that at least one guard must be irrational. For formal reasons, we define $H^{(0)} = V^{(0)} = H^{(n)} = V^{(n)} = \emptyset$. The horizontal and vertical corridors $H^{(i)}$ and $V^{(i)}$, for $i \in \{0, \ldots, n\}$, intersect at a rectangular area $B^{(i)} = H^{(i)} \cap V^{(i)}$ which we call a bend. For $i \in \{1, \ldots, n - 1\}$, the bend $B^{(i)}$ is non-empty and visible from both polygons $\mathcal{P}^{(i)}$ and $\mathcal{P}^{(i+1)}$. Define the extension of $\mathcal{P}^{(i)}$, denoted by $E(\mathcal{P}^{(i)})$, to be the union of $\mathcal{P}^{(i)}$ and the adjacent corridors excluding the bends, i.e., $E(\mathcal{P}^{(i)}) = \mathcal{P}^{(i)} \cup (V^{(i-1)} \setminus B^{(i-1)}) \cup (H^{(i)} \setminus B^{(i)})$. Since the extensions are pairwise disjoint, there is an extension $E(\mathcal{P}^{(i)})$ containing at most three guards. If there are no guards in any of the bends $B^{(i-1)}, B^{(i)}$ it follows from Theorem 1 that three guards must be placed inside $\mathcal{P}^{(i)}$ at irrational coordinates, so assume that there is a guard in one or both of the bends. If the adjacent corridors $V^{(i-1)}$ and $H^{(i)}$ are long enough and thin enough, a guard in the bends $B^{(i-1)}$ and $B^{(i)}$ cannot see any of the convex corners of $\mathcal{P}^{(i)}$ in the rectangular pockets, any point in a triangular pocket, or any point in a quadrilateral pocket. Hence, all the features of $\mathcal{P}^{(i)}$ that enforce the irrationality of the guards are unseen by the guards in the bends and it follows that there must be irrational guards in $\mathcal{P}^{(i)}$. Therefore, at least $4n$ guards are needed if we require them to be rational. Similarly as in the proof of Theorem 1, we can show that $4n$ rational guards are enough to guard $\mathcal{P}_n$. ▶

5 Rectilinear Polygon

Figure 8 depicts a rectilinear polygon $\mathcal{P}_R$ with corners at rational coordinates that can be guarded by 9 guards, but requires 10 guards if we restrict the guards to points with rational coordinates. The construction of $\mathcal{P}_R$ starts with the polygon $\mathcal{P}$ from Theorem 1. We extend the non-rectilinear parts by “equivalent” rectilinear parts, colored gray in the figure. The rectilinear pockets are constructed in such a way that each of them requires at least one guard in the interior. Additionally, if the interior of each pocket contains only one guard, then these guards must be placed at specific positions, making the area not seen by these six additional guards exactly the polygon $\mathcal{P}$ described in Section 2 (the white area in Figure 8). Thus, the remaining 3 guards must be placed at three irrational points by Theorem 1.

6 Future Work

One of the most prominent open questions related to the art gallery problem is whether the problem is in NP. Recently, some researchers popularized an interesting complexity class, called $\exists \mathbb{R}$, being somewhere between NP and PSPACE [8, 24, 7, 19]. Many geometric problems for which membership in NP is uncertain have been shown to be complete for the complexity class $\exists \mathbb{R}$. Famous examples are: order type realizability, pseudoline stretchability, recognition of segment intersection graphs, recognition of unit disk intersection graphs, recognition of point visibility graphs, minimizing rectilinear crossing number, linkage realizability. This suggests that there might indeed be no polynomial sized witness for any of these problems as this would imply NP = $\exists \mathbb{R}$. It is an interesting open problem whether the art gallery problem is $\exists \mathbb{R}$-complete or not.
The irrational coordinates of the guards in our examples are all of degree 2, i.e., they are roots in second-degree polynomials with integer coefficients. We would like to know if polygons exist where irrational numbers of higher degree are needed in the coordinates of an optimal solution.

We show that there exist polygons for which $|\text{OPT}_{Q}| \geq \frac{4}{3}|\text{OPT}|$. It follows from the work by Bonnet and Miltzow [4] that it always holds that $|\text{OPT}_{Q}| \leq 9|\text{OPT}|$. It is interesting to see if any of these bounds can be improved.

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