Covering Lattice Points by Subspaces and Counting Point-Hyperplane Incidences*

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Abstract

Let d and k be integers with $1 \leq k \leq d-1$. Let Λ be a d-dimensional lattice and let K be a d-dimensional compact convex body symmetric about the origin. We provide estimates for the minimum number of k-dimensional linear subspaces needed to cover all points in $\Lambda \cap K$. In particular, our results imply that the minimum number of k-dimensional linear subspaces needed to cover the d-dimensional $n \times \cdots \times n$ grid is at least $\Omega(n^{d(d-k)/(d-1)-\varepsilon})$ and at most $O(n^{d(d-k)/(d-1)})$, where $\varepsilon > 0$ is an arbitrarily small constant. This nearly settles a problem mentioned in the book of Brass, Moser, and Pach [7]. We also find tight bounds for the minimum number of k-dimensional affine subspaces needed to cover $\Lambda \cap K$.

We use these new results to improve the best known lower bound for the maximum number of point-hyperplane incidences by Brass and Knauer [6]. For $d \geq 3$ and $\varepsilon \in (0,1)$, we show that there is an integer $r = r(d,\varepsilon)$ such that for all positive integers n, m the following statement is true. There is a set of n points in $\mathbb{R}^d$ and an arrangement of m hyperplanes in $\mathbb{R}^d$ with no $K_{r,r}$ in their incidence graph and with at least $\Omega\left((mn)^{1-(2d+3)/((d+2)(d+3))-\varepsilon}\right)$ incidences if d is odd and $\Omega\left((mn)^{1-(2d^2+d-2)/((d+2)(d^2+2d-2))-\varepsilon}\right)$ incidences if d is even.

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1 Introduction

In this paper, we study the minimum number of linear or affine subspaces needed to cover points that are contained in the intersection of a given lattice with a given 0-symmetric convex body. We also present an application of our results to the problem of estimating the

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maximum number of incidences between a set of points and an arrangement of hyperplanes. Consequently, this establishes a new lower bound for the time complexity of so-called partitioning algorithms for Hopcroft's problem. Before describing our results in more detail, we first give some preliminaries and introduce necessary definitions.

1.1 Preliminaries

For linearly independent vectors $b_1, \dots, b_d \in \mathbb{R}^d$, the d-dimensional *lattice* $\Lambda = \Lambda(b_1, \dots, b_d)$ with *basis* $\{b_1, \dots, b_d\}$ is the set of all linear combinations of the vectors $b_1, \dots, b_d$ with integer coefficients. We define the *determinant of* Λ as $\det(\Lambda) := |\det(B)|$, where B is the $d \times d$ matrix with the vectors $b_1, \dots, b_d$ as columns. For a positive integer d, we use $\mathcal{L}^d$ to denote the set of d-dimensional lattices Λ, that is, lattices with $\det(\Lambda) \neq 0$.

A convex body K is *symmetric about the origin* 0 if $K = -K$. We let $\mathcal{K}^d$ be the set of d-dimensional compact convex bodies in $\mathbb{R}^d$ that are symmetric about the origin.

For a positive integer n, we use the abbreviation $[n]$ to denote the set $\{1, 2, \dots, n\}$. A point x of a lattice is called *primitive* if whenever its multiple $\lambda \cdot x$ is a lattice point, then λ is an integer. For $K \in \mathcal{K}_d$, let $\mathrm{vol}(K)$ be the d-dimensional Lebesgue measure of K. We say that $\mathrm{vol}(K)$ is the *volume of* K. The closed d-dimensional ball with the radius $r \in \mathbb{R}$, $r \geq 0$, centered in the origin is denoted by $B^d(r)$. If $r = 1$, we simply write B^d instead of $B^d(1)$. For $x \in \mathbb{R}^d$, we use $\|x\|$ to denote the Euclidean norm of x.

Let X be a subset of $\mathbb{R}^d$. We use $\mathrm{aff}(X)$ and $\mathrm{lin}(X)$ to denote the *affine hull of* X and the *linear hull of* X, respectively. The dimension of the affine hull of X is denoted by $\dim(X)$.

For functions $f, g \colon \mathbb{N} \to \mathbb{N}$, we write $f(n) \leq O(g(n))$ if there is a fixed constant c_1 such that $f(n) \leq c_1 \cdot g(n)$ for all $n \in \mathbb{N}$. We write $f(n) \geq \Omega(g(n))$ if there is a fixed constant $c_2 > 0$ such that $f(n) \geq c_2 \cdot g(n)$ for all $n \in \mathbb{N}$. If the constants c_1 and c_2 depend on some parameters $a_1, \dots, a_t$, then we emphasize this by writing $f(n) \leq O_{a_1,\dots,a_t}(g(n))$ and $f(n) \geq \Omega_{a_1,\dots,a_t}(g(n))$, respectively. If $f(n) \leq O_{a_1,\dots,a_t}(n)$ and $f(n) \geq \Omega_{a_1,\dots,a_t}(n)$, then we write $f(n) = \Theta_{a_1,\dots,a_t}(n)$.

1.2 Covering lattice points by subspaces

We say that a collection $\mathcal{S}$ of subsets in $\mathbb{R}^d$ *covers* a set of points P from $\mathbb{R}^d$ if every point from P lies in some set from $\mathcal{S}$.

Let d, k, n, and r be positive integers that satisfy $1 \leq k \leq d-1$. We let $a(d,k,n,r)$ be the maximum size of a set $S \subseteq \mathbb{Z}^d \cap B^d(n)$ such that every k-dimensional *affine* subspace of $\mathbb{R}^d$ contains at most $r-1$ points of S. Similarly, we let $l(d,k,n,r)$ be the maximum size of a set $S \subseteq \mathbb{Z}^d \cap B^d(n)$ such that every k-dimensional *linear* subspace of $\mathbb{R}^d$ contains at most $r-1$ points of S. We also let $g(d,k,n)$ be the minimum number of k-dimensional linear subspaces of $\mathbb{R}^d$ necessary to cover $\mathbb{Z}^d \cap B^d(n)$.

In this paper, we study the functions $a(d,k,n,r)$, $l(d,k,n,r)$, and $g(d,k,n)$ and their generalizations to arbitrary lattices from $\mathcal{L}^d$ and bodies from $\mathcal{K}^d$. We mostly deal with the last two functions, that is, with covering lattice points by linear subspaces. In particular, we obtain new upper bounds on $g(d,k,n)$ (Theorem 4), lower bounds on $l(d,k,n,r)$ (Theorem 5), and we use the estimates for $a(d,k,n,r)$ and $l(d,k,n,r)$ to obtain improved lower bounds for the maximum number of point-hyperplane incidences (Theorem 9). Before doing so, we first give a summary of known results, since many of them are used later in the paper.

The problem of determining $a(d,k,n,r)$ is essentially solved. In general, the set $\mathbb{Z}^d \cap B^d(n)$ can be covered by $(2n+1)^{d-k}$ affine k-dimensional subspaces and thus we have an upper bound $a(d,k,n,r) \leq (r-1)(2n+1)^{d-k}$. This trivial upper bound is asymptotically almost tight for all fixed d, k, and some r, as Brass and Knauer [6] showed with a probabilistic argument that for every $\varepsilon > 0$ there is an $r = r(d, \varepsilon, k) \in \mathbb{N}$ such that for each positive integer n we have

$$a(d, k, n, r) \geq \Omega_{d,\varepsilon,k}\left(n^{d-k-\varepsilon}\right). \tag{1}$$

For fixed d and r, the upper bound is known to be asymptotically tight in the cases $k = 1$ and $k = d - 1$. This is showed by considering points on the modular moment surface for $k = 1$ and the modular moment curve for $k = d - 1$; see [6].

Covering lattice points by linear subspaces seems to be more difficult than covering by affine subspaces. From the definitions we immediately get $l(d, k, n, r) \leq (r - 1)g(d, k, n)$. In the case $k = d - 1$ and d fixed, Bárány, Harcos, Pach, and Tardos [5] obtained the following asymptotically tight estimates for the functions $l(d, d-1, n, d)$ and $g(d, d-1, n)$:

$$l(d, d-1, n, d) = \Theta_d(n^{d/(d-1)}) \qquad \text{and} \qquad g(d, d-1, n) = \Theta_d(n^{d/(d-1)}).$$

In fact, Bárány et al. [5] proved stronger results that estimate the minimum number of $(d-1)$-dimensional linear subspaces necessary to cover the set $\Lambda \cap K$ in terms of so-called successive minima of a given lattice $\Lambda \in \mathcal{L}^d$ and a body $K \in \mathcal{K}^d$.

For a lattice $\Lambda \in \mathcal{L}^d$, a body $K \in \mathcal{K}^d$, and $i \in [d]$, we let $\lambda_i(\Lambda, K)$ be the *ith successive minimum of* Λ *and* K. That is, $\lambda_i(\Lambda, K) := \inf\{\lambda \in \mathbb{R} \colon \dim(\Lambda \cap (\lambda \cdot K)) \geq i\}$. Since K is compact, it is easy to see that the successive minima are achieved. That is, there are linearly independent vectors $v_1, \ldots, v_d$ from Λ such that $v_i \in \lambda_i(\Lambda, K) \cdot K$ for every $i \in [d]$. Also note that we have $\lambda_1(\Lambda, K) \leq \cdots \leq \lambda_d(\Lambda, K)$ and $\lambda_1(\mathbb{Z}^d, B^d(n)) = \cdots = \lambda_d(\mathbb{Z}^d, B^d(n)) = 1/n$.

▶ **Theorem 1** ([5]). *For an integer $d \geq 2$, a lattice $\Lambda \in \mathcal{L}^d$, and a body $K \in \mathcal{K}^d$, we let $\lambda_i := \lambda_i(\Lambda, K)$ for every $i \in [d]$. If $\lambda_d \leq 1$, then the set $\Lambda \cap K$ can be covered with at most*

$$c2^d d^2 \log_2 d \min_{1 \leq j \leq d-1} (\lambda_j \cdots \lambda_d)^{-1/(d-j)}$$

$(d-1)$-dimensional linear subspaces of $\mathbb{R}^d$, where c is some absolute constant.

On the other hand, if $\lambda_d \leq 1$, then there is a subset S of $\Lambda \cap K$ of size

$$\frac{1-\lambda_d}{16d^2} \min_{1 \leq j \leq d-1} (\lambda_j \cdots \lambda_d)^{-1/(d-j)}$$

such that no $(d-1)$-dimensional linear subspace of $\mathbb{R}^d$ contains d points from S.

We note that the assumption $\lambda_d \leq 1$ is necessary; see the discussion in [5]. Not much is known for linear subspaces of lower dimension. We trivially have $l(d, k, n, r) \geq a(d, k, n, r)$ for all d, k, n, r with $1 \leq k \leq d - 1$. Thus $l(d, k, n, r) \geq \Omega_{d,\varepsilon,k}(n^{d-k-\varepsilon})$ for some $r = r(d, \varepsilon, k)$ by (1). Brass and Knauer [6] conjectured that $l(d, k, n, k + 1) = \Theta_{d,k}(n^{d(d-k)/(d-1)})$ for d fixed. This conjecture was refuted by Lefmann [15] who showed that, for all d and k with $1 \leq k \leq d-1$, there is an absolute constant c such that we have $l(d, k, n, k+1) \leq c \cdot n^{d/\lceil k/2 \rceil}$ for every positive integer n. This bound is asymptotically smaller in n than the growth rate conjectured by Brass and Knauer for sufficiently large d and almost all values of k with $1 \leq k \leq d - 1$.

Covering lattice points by linear subspaces is also mentioned in the book by Brass, Moser, and Pach [7], where the authors pose the following problem.

▶ **Problem 2** ([7, Problem 6 in Chapter 10.2]). *What is the minimum number of k-dimensional linear subspaces necessary to cover the d-dimensional $n \times \cdots \times n$ lattice cube?*

1.3 Point-hyperplane incidences

As we will see later, the problem of determining $a(d,k,n,r)$ and $l(d,n,k,r)$ is related to a problem of bounding the maximum number of point-hyperplane incidences. For an integer $d \geq 2$, let P be a set of n points in $\mathbb{R}^d$ and let $\mathcal{H}$ be an arrangement of m hyperplanes in $\mathbb{R}^d$. An *incidence between* P *and* $\mathcal{H}$ is a pair (p,H) such that $p \in P$, $H \in \mathcal{H}$, and $p \in H$. The number of incidences between P and $\mathcal{H}$ is denoted by $\mathrm{I}(P,\mathcal{H})$.

We are interested in the maximum number of incidences between P and $\mathcal{H}$. In the plane, the famous *Szemerédi–Trotter theorem* [23] says that the maximum number of incidences between a set of n points in $\mathbb{R}^2$ and an arrangement of m lines in $\mathbb{R}^2$ is at most $O((mn)^{2/3} + m + n)$. This is known to be asymptotically tight, as a matching lower bound was found earlier by Erdős [9]. The current best known bounds are $\approx 1.27(mn)^{2/3} + m + n$ [19][1] and $\approx 2.44(mn)^{2/3} + m + n$ [1].

For $d \geq 3$, it is easy to see that there is a set P of n points in $\mathbb{R}^d$ and an arrangement $\mathcal{H}$ of m hyperplanes in $\mathbb{R}^d$ for which the number of incidences is maximum possible, that is $\mathrm{I}(P,\mathcal{H}) = mn$. It suffices to consider the case where all points from P lie in an affine subspace that is contained in every hyperplane from $\mathcal{H}$. In order to avoid this degenerate case, we forbid large complete bipartite graphs in the *incidence graph of* P *and* $\mathcal{H}$, which is denoted by $G(P,\mathcal{H})$. This is the bipartite graph on the vertex set $P \cup \mathcal{H}$ and with edges $\{p,H\}$ where (p,H) is an incidence between P and $\mathcal{H}$.

With this restriction, bounding $\mathrm{I}(P,\mathcal{H})$ becomes more difficult and no tight bounds are known for $d \geq 3$. It follows from the works of Chazelle [8], Brass and Knauer [6], and Apfelbaum and Sharir [2] that the number of incidences between any set P of n points in $\mathbb{R}^d$ and any arrangement $\mathcal{H}$ of m hyperplanes in $\mathbb{R}^d$ with $K_{r,r} \not\subseteq G(P,\mathcal{H})$ satisfies

$$\mathrm{I}(P,\mathcal{H}) \leq O_{d,r}\left((mn)^{1-1/(d+1)} + m + n\right). \tag{2}$$

We note that an upper bound similar to (2) holds in a much more general setting; see the remark in the proof of Theorem 9. The best general lower bound for $\mathrm{I}(P,\mathcal{H})$ is due to a construction of Brass and Knauer [6], which gives the following estimate.

▶ **Theorem 3** ([6]). *Let $d \geq 3$ be an integer. Then for every $\varepsilon > 0$ there is a positive integer $r = r(d,\varepsilon)$ such that for all positive integers n and m there is a set P of n points in $\mathbb{R}^d$ and an arrangement $\mathcal{H}$ of m hyperplanes in $\mathbb{R}^d$ such that $K_{r,r} \not\subseteq G(P,\mathcal{H})$ and*

$$\mathrm{I}(P,\mathcal{H}) \geq \begin{cases} \Omega_{d,\varepsilon}\left((mn)^{1-2/(d+3)-\varepsilon}\right) & \text{if } d \text{ is odd and } d > 3, \\ \Omega_{d,\varepsilon}\left((mn)^{1-2(d+1)/(d+2)^2-\varepsilon}\right) & \text{if } d \text{ is even,} \\ \Omega_{d,\varepsilon}\left((mn)^{7/10}\right) & \text{if } d = 3. \end{cases}$$

For $d \geq 4$, this lower bound has been recently improved by Sheffer [21] in a certain non-diagonal case. Sheffer constructed a set P of n points in $\mathbb{R}^d$, $d \geq 4$, and an arrangement $\mathcal{H}$ of $m = \Theta(n^{(3-3\varepsilon)/(d+1)})$ hyperplanes in $\mathbb{R}^d$ such that $K_{(d-1)/\varepsilon,2} \not\subseteq G(P,\mathcal{H})$ and $\mathrm{I}(P,\mathcal{H}) \geq \Omega\left((mn)^{1-2/(d+4)-\varepsilon}\right)$.

[1] The lower bound claimed by Pach and Tóth [19, Remark 4.2] contains the multiplicative constant ≈ 0.42. This is due to a miscalculation in the last equation in the calculation of the number of incidences. The correct calculation is $I \approx \cdots = 4n\sum_{r=1}^{1/\varepsilon}\phi(r) - 2n\varepsilon^2\sum_{r=1}^{1/\varepsilon} r^2\phi(r) \approx 4n \cdot 3(1/\varepsilon)^2/\pi^2 - 2n\varepsilon^2(3/2)(1/\varepsilon)^4/\pi^2 = 9n/(\varepsilon^2\pi^2)$. This leads to $c \approx 3\sqrt[3]{3/(4\pi^2)} \approx 1.27$.

2 Our results

In this paper, we nearly settle Problem 2 by proving almost tight bounds for the function $g(d,k,n)$ for a fixed d and an arbitrary k from $[d-1]$. For a fixed d, an arbitrary $k \in [d-1]$, and some fixed r, we also provide bounds on the function $l(d,k,n,r)$ that are very close to the bound conjectured by Brass and Knauer [6]. Thus it seems that the conjectured growth rate of $l(d,k,n,r)$ is true if we allow r to be (significantly) larger than $k+1$.

We study these problems in a more general setting where we are given an arbitrary lattice Λ from $\mathcal{L}^d$ and a body K from $\mathcal{K}^d$. Similarly to Theorem 1 by Bárány et al. [5], our bounds are expressed in terms of the successive minima $\lambda_i(\Lambda, K)$, $i \in [d]$.

2.1 Covering lattice points by linear subspaces

First, we prove a new upper bound on the minimum number of k-dimensional linear subspaces that are necessary to cover points in the intersection of a given lattice with a body from $\mathcal{K}^d$.

▶ **Theorem 4.** *For integers d and k with $1 \leq k \leq d-1$, a lattice $\Lambda \in \mathcal{L}^d$, and a body $K \in \mathcal{K}^d$, we let $\lambda_i := \lambda_i(\Lambda, K)$ for $i = 1, \dots, d$. If $\lambda_d \leq 1$, then we can cover $\Lambda \cap K$ with $O_{d,k}(\alpha^{d-k})$ k-dimensional linear subspaces of $\mathbb{R}^d$, where*

$$\alpha := \min_{1 \leq j \leq k} (\lambda_j \cdots \lambda_d)^{-1/(d-j)}.$$

We also prove the following lower bound.

▶ **Theorem 5.** *For integers d and k with $1 \leq k \leq d-1$, a lattice $\Lambda \in \mathcal{L}^d$, and a body $K \in \mathcal{K}^d$, we let $\lambda_i := \lambda_i(\Lambda, K)$ for $i = 1, \dots, d$. If $\lambda_d \leq 1$, then, for every $\varepsilon \in (0,1)$, there is a positive integer $r = r(d, \varepsilon, k)$ and a set $S \subseteq \Lambda \cap K$ of size at least $\Omega_{d,\varepsilon,k}(((1-\lambda_d)\beta)^{d-k-\varepsilon})$, where*

$$\beta := \min_{1 \leq j \leq d-1} (\lambda_j \cdots \lambda_d)^{-1/(d-j)},$$

such that every k-dimensional linear subspace of $\mathbb{R}^d$ contains at most $r-1$ points from S.

We remark that we can get rid of the ε in the exponent if $k=1$ or $k=d-1$; for details, see Theorem 1 for the case $k=d-1$ and the proof in Section 4 for the case $k=1$. Also note that in the definition of α in Theorem 4 the minimum is taken over the set $\{1,\dots,k\}$, while in the definition of β in Theorem 5 the minimum is taken over $\{1,\dots,d-1\}$. There are examples, which show that α cannot be replaced by β in Theorem 4. It suffices to consider $d=3$, $k=1$, and let Λ be the lattice $\{(x_1/n, x_2/2, x_3/2) \in \mathbb{R}^3 \colon x_1, x_2, x_3 \in \mathbb{Z}\}$ for some large positive integer n. Then $\lambda_1(\Lambda, B^3) = 1/n$, $\lambda_2(\Lambda, B^3) = 1/2$, $\lambda_3(\Lambda, B^3) = 1/2$, and thus $\beta = (\lambda_2\lambda_3)^{-1} = 4$. However, it is not difficult to see that we need at least $\Omega(n)$ 1-dimensional linear subspaces to cover $\Lambda \cap B^3$, which is asymptotically larger than $\beta^2 = O(1)$. On the other hand, $\alpha = (\lambda_1\lambda_2\lambda_3)^{-1/2}$ and $O(\alpha^2) = O(n)$ 1-dimensional linear subspaces suffice to cover $\Lambda \cap B^3$. We thus suspect that the lower bound can be improved.

Since $\lambda_i(\mathbb{Z}^d, B^d(n)) = 1/n$ for every $i \in [d]$, we can apply Theorem 5 with $\Lambda = \mathbb{Z}^d$ and $K = B^d(n)$ and obtain the following lower bound on $l(d,k,n,r)$.

▶ **Corollary 6.** *Let d and k be integers with $1 \leq k \leq d-1$. Then, for every $\varepsilon \in (0,1)$, there is an $r = r(d,\varepsilon,k) \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ we have*

$$l(d,k,n,r) \geq \Omega_{d,\varepsilon,k}(n^{d(d-k)/(d-1)-\varepsilon}).$$

The existence of the set S from Theorem 5 is showed by a probabilistic argument. It would be interesting to find, at least for some value $1 < k < d-1$, some fixed $r \in \mathbb{N}$, and arbitrarily large $n \in \mathbb{N}$, a construction of a subset R of $\mathbb{Z}^d \cap B^d(n)$ of size $\Omega_{d,k}(n^{d(d-k)/(d-1)})$ such that every k-dimensional linear subspace contains at most $r-1$ points from R. Such constructions are known for $k=1$ and $k=d-1$; see [6, 20].

Since we have $l(d,k,n,r) \leq (r-1)g(d,k,n)$ for every $r \in \mathbb{N}$, Theorem 4 and Corollary 6 give the following almost tight estimates on $g(d,k,n)$. This nearly settles Problem 2.

▶ **Corollary 7.** *Let d, k, and n be integers with $1 \leq k \leq d-1$. Then, for every $\varepsilon \in (0,1)$, we have*

$$\Omega_{d,\varepsilon,k}(n^{d(d-k)/(d-1)-\varepsilon}) \leq g(d,k,n) \leq O_{d,k}(n^{d(d-k)/(d-1)}).$$

2.2 Covering lattice points by affine subspaces

For *affine* subspaces, Brass and Knauer [6] considered only the case of covering the d-dimensional $n \times \cdots \times n$ lattice cube by k-dimensional affine subspaces. To our knowledge, the case for general $\Lambda \in \mathcal{L}^d$ and $K \in \mathcal{K}^d$ was not considered in the literature. We extend the results of Brass and Knauer to covering $\Lambda \cap K$.

▶ **Theorem 8.** *For integers d and k with $1 \leq k \leq d-1$, a lattice $\Lambda \in \mathcal{L}^d$, and a body $K \in \mathcal{K}^d$, we let $\lambda_i := \lambda_i(\Lambda, K)$ for $i = 1, \dots, d$. If $\lambda_d \leq 1$, then the set $\Lambda \cap K$ can be covered with $O_{d,k}((\lambda_{k+1} \cdots \lambda_d)^{-1})$ k-dimensional affine subspaces of $\mathbb{R}^d$.*

On the other hand, at least $\Omega_{d,k}((\lambda_{k+1} \cdots \lambda_d)^{-1})$ k-dimensional affine subspaces of $\mathbb{R}^d$ are necessary to cover $\Lambda \cap K$.

2.3 Point-hyperplane incidences

As an application of Corollary 6, we improve the best known lower bounds on the maximum number of point-hyperplane incidences in $\mathbb{R}^d$ for $d \geq 4$. That is, we improve the bounds from Theorem 3. To our knowledge, this is the first improvement on the estimates for $\mathrm{I}(P, \mathcal{H})$ in the general case during the last 13 years.

▶ **Theorem 9.** *For every integer $d \geq 2$ and $\varepsilon \in (0,1)$, there is an $r = r(d,\varepsilon) \in \mathbb{N}$ such that for all positive integers n and m the following statement is true. There is a set P of n points in $\mathbb{R}^d$ and an arrangement $\mathcal{H}$ of m hyperplanes in $\mathbb{R}^d$ such that $K_{r,r} \not\subseteq G(P, \mathcal{H})$ and*

$$\mathrm{I}(P,\mathcal{H}) \geq \begin{cases} \Omega_{d,\varepsilon}\left((mn)^{1-(2d+3)/((d+2)(d+3))-\varepsilon}\right) & \text{if } d \text{ is odd,} \\ \Omega_{d,\varepsilon}\left((mn)^{1-(2d^2+d-2)/((d+2)(d^2+2d-2))-\varepsilon}\right) & \text{if } d \text{ is even.} \end{cases}$$

We can get rid of the ε in the exponent for $d \leq 3$. That is, we have the bounds $\Omega((mn)^{2/3})$ for $d = 2$ and $\Omega((mn)^{7/10})$ for $d = 3$. For $d = 3$, our bound is the same as the bound from Theorem 3. For larger d, our bounds become stronger. In particular, the exponents in the lower bounds from Theorem 9 exceed the exponents from Theorem 3 by $1/((d+2)(d+3))$ for $d > 3$ odd and by $d^2/((d+2)^2(d^2+2d-2))$ for d even. However, the bounds are not tight.

In the non-diagonal case, when one of n and m is significantly larger that the other, the proof of Theorem 9 yields the following stronger bound.

▶ **Theorem 10.** *For all integers d and k with $0 \leq k \leq d-2$ and for $\varepsilon \in (0,1)$, there is an $r = r(d,\varepsilon,k) \in \mathbb{N}$ such that for all positive integers n and m the following statement is true.*

There is a set P of n points in $\mathbb{R}^d$ and an arrangement $\mathcal{H}$ of m hyperplanes in $\mathbb{R}^d$ such that $K_{r,r} \not\subseteq G(P,\mathcal{H})$ and

$$\mathrm{I}(P,\mathcal{H}) \geq \Omega_{d,\varepsilon,k}\left(n^{1-(k+1)/((k+2-1/d)(d-k))-\varepsilon} m^{1-(d-1)/(dk+2d-1)-\varepsilon}\right).$$

For example, in the case $m = \Theta(n^{(3-3\varepsilon)/(d+1)})$ considered by Sheffer [21], Theorem 10 gives a slightly better bound than $I(P,\mathcal{H}) \geq \Omega((mn)^{1-2/(d+4)-\varepsilon})$ if we set, for example, $k = \lfloor (d-1)/4 \rfloor$. However, the forbidden complete bipartite subgraph in the incidence graph is larger than $K_{(d-1)/\varepsilon,2}$.

The following problem is known as the counting version of *Hopcroft's problem* [6, 10]: given n points in $\mathbb{R}^d$ and m hyperplanes in $\mathbb{R}^d$, how fast can we count the incidences between them? We note that the lower bounds from Theorem 9 also establish the best known lower bounds for the time complexity of so-called *partitioning algorithms* [10] for the counting version of Hopcroft's problem; see [6] for more details.

In the proofs of our results, we make no serious effort to optimize the constants. We also omit floor and ceiling signs whenever they are not crucial.

3 Proof of Theorem 4

Here we sketch the proof of the upper bound on the minimum number of k-dimensional linear subspaces needed to cover points from a given d-dimensional lattice that are contained in a body K from $\mathcal{K}^d$. We first prove Theorem 4 in the special case $K = B^d$ (Theorem 14) and then we extend the result to arbitrary $K \in \mathcal{K}^d$. Since the proof is rather long and complicated, we only prove a weaker bound (Corollary 16) and then we give a high-level overview of the main ideas of the full proof, which can be found in the full version of the paper [3].

3.1 Sketch of the proof for balls

We first introduce some auxiliary results that are used later. The following classical result is due to Minkowski [18] and shows a relation between $\mathrm{vol}(K)$, $\det(\Lambda)$, and the successive minima of $\Lambda \in \mathcal{L}^d$ and $K \in \mathcal{K}^d$.

▶ **Theorem 11** (Minkowski's second theorem [18]). *Let d be a positive integer. For every $\Lambda \in \mathcal{L}^d$ and every $K \in \mathcal{K}^d$, we have*

$$\frac{1}{2^d} \cdot \frac{\mathrm{vol}(K)}{\det(\Lambda)} \leq \frac{1}{\lambda_1(\Lambda,K)\cdots\lambda_d(\Lambda,K)} \leq \frac{d!}{2^d} \cdot \frac{\mathrm{vol}(K)}{\det(\Lambda)}.$$

A result similar to the first bound from Theorem 11 can be obtained if the volume is replaced by the point enumerator; see Henk [13].

▶ **Theorem 12** ([13, Theorem 1.5]). *Let d be a positive integer. For every $\Lambda \in \mathcal{L}^d$ and every $K \in \mathcal{K}^d$, we have*

$$|\Lambda \cap K| \leq 2^{d-1} \prod_{i=1}^{d} \left\lfloor \frac{2}{\lambda_i(\Lambda,K)} + 1 \right\rfloor.$$

For $\Lambda \in \mathcal{L}^d$ and $K \in \mathcal{K}^d$, let $v_1,\dots,v_d$ be linearly independent vectors such that $v_i \in \Lambda \cap (\lambda_i(\Lambda,K) \cdot K)$ for every $i \in [d]$. For $d > 2$, the vectors $v_1,\dots,v_d$ do not necessarily form a basis of Λ [22, see Section X.5]. However, the following theorem shows that there exists a basis with vectors of lengths not much larger than the lengths of $v_1,\dots,v_d$.

▶ **Theorem 13** (First finiteness theorem [22, see Lemma 2 in Section X.6]). *Let d be a positive integer. For every $\Lambda \in \mathcal{L}^d$ and every $K \in \mathcal{K}^d$, there is a basis $\{b_1, \dots, b_d\}$ of Λ with $b_i \in (3/2)^{i-1}\lambda_i(\Lambda, K) \cdot K$ for every $i \in [d]$.*

Now, let Λ be a d-dimensional lattice with $\lambda_d(\Lambda, B^d) \leq 1$. Throughout this section, we use λ_i to denote the ith successive minimum $\lambda_i(\Lambda, B^d)$ for $i = 1, \dots, d$. Let k be an integer with $1 \leq k \leq d-1$. We show the following result.

▶ **Theorem 14.** *There is a constant $C = C(d,k)$ such that the set $\Lambda \cap B^d$ can be covered with $C \cdot \alpha^{d-k}$ k-dimensional linear subspaces of $\mathbb{R}^d$, where*

$$\alpha := \min_{1 \leq j \leq k} (\lambda_j \cdots \lambda_d)^{-1/(d-j)}.$$

As the first step towards the proof of Theorem 14, we show a weaker bound on the number of k-dimensional linear subspaces needed to cover $\Lambda \cap B^d$; see Corollary 16. To do so, we prove the following lemma that is also used later in the proof of Theorem 8.

▶ **Lemma 15.** *Let d and s be integers with $0 \leq s \leq d-1$. There is a positive integer $r = r(d,s)$ and a projection p of $\mathbb{R}^d$ along s vectors of Λ onto a $(d-s)$-dimensional linear subspace N of $\mathbb{R}^d$ such that $\Lambda \cap B^d$ is mapped to $\Lambda \cap N \cap B^d(r)$ and such that $\lambda_i(\Lambda \cap N, B^d(r) \cap N) = \Theta_{d,s}(\lambda_{i+s})$ for every $i \in [d-s]$.*

Proof. If $s = 0$, then we set p to be the identity on $\mathbb{R}^d$ and $r := 1$. Thus we assume $s \geq 1$.

For $j = 0, \dots, d-1$, we set $r_j := (2^{d^2}+1)^j$. For $j = 0, \dots, d-1$ and a lattice $\Lambda_j \in \mathcal{L}^{d-j}$, we show that there is a projection p_j of $\mathbb{R}^{d-j}$ along a vector $v_j \in \Lambda_j$ onto a $(d-j-1)$-dimensional linear subspace N of $\mathbb{R}^{d-j}$ such that $\Lambda_j \cap B^{d-j}(r_j)$ is mapped to $\Lambda_j \cap N \cap B^{d-j}(r_{j+1})$ by p_j and such that

$$\lambda_{i+1}(\Lambda_j, B^{d-j}(r_j))/(2^{d^2}+1) \leq \lambda_i(\Lambda_j \cap N, B^{d-j}(r_{j+1}) \cap N) \leq \lambda_{i+1}(\Lambda_j, B^{d-j}(r_j))$$

for every $i \in [d-j-1]$. We let p_j be the projection for $\Lambda_j := p_{j-1}(\Lambda_{j-1})$ for every $j = 1, \dots, s-1$, where $\Lambda_0 := \Lambda$ and p_0 is the projection for Λ_0. The statement of the lemma is then obtained by setting $p := p_{s-1} \circ \cdots \circ p_0$.

Let $B = \{b_1, \dots, b_{d-j}\}$ be a basis of Λ_j such that $b_i \in (3/2)^{i-1}\lambda_i(\Lambda_j, B^{d-j}(r_j)) \cdot B^{d-j}(r_j)$ for every $i \in [d-j]$. Such basis exists by the First finiteness theorem (Theorem 13). In particular, b_1 is the shortest vector from $\Lambda_j \cap B^{d-j}(r_j)$. Let $v_j := b_1$ and let N be the linear subspace generated by $b_2, \dots, b_{d-j}$. Let Λ_N be the set $\Lambda_j \cap N$. Note that Λ_N is a $(d-j-1)$-dimensional lattice with the basis $\{b_2, \dots, b_{d-j}\}$.

We consider the projection p_j onto N along v_j. That is, every $x \in \mathbb{R}^{d-j}$ is mapped to $p_j(x) = \sum_{i=2}^{d-j} t_i b_i$, where $x = \sum_{i=1}^{d-j} t_i b_i$, $t_i \in \mathbb{R}$, is the expression of x with respect to the basis B.

We show that $p_j(z) \in \Lambda_N \cap B^{d-j}(r_{j+1})$ for every $z \in \Lambda_j \cap B^{d-j}(r_j)$. We have $p_j(z) \in \Lambda_N$, since B is a basis of Λ_j and $B \setminus \{b_1\}$ is a basis of Λ_N. Let $z = \sum_{i=1}^{d-j} t_i b_i$, $t_i \in \mathbb{Z}$, be the expression of z with respect to B and let v be the Euclidean distance between b_1 and N.

From the definitions of Λ_N and B, we have

$$\lambda_{i+1}(\Lambda_j, B^{d-j}(r_j)) \leq \lambda_i(\Lambda_N, B^{d-j}(r_j) \cap N) \leq (3/2)^i \lambda_{i+1}(\Lambda_j, B^{d-j}(r_j)) \tag{3}$$

for every $i \in [d-j-1]$. Using Minkowski's second theorem (Theorem 11) twice, the upper bound in (3), and the choice of b_1, we obtain

$$\begin{aligned}\frac{\mathrm{vol}(B^{d-j}(r_j))}{2^{d-j}\det(\Lambda_j)} &\leq \frac{1}{\lambda_1(\Lambda_j, B^{d-j}(r_j))\cdots\lambda_{d-j}(\Lambda_j, B^{d-j}(r_j))} \\ &\leq \frac{r_j}{\|b_1\|}\cdot\frac{(3/2)^{(d-j)(d-j-1)/2}}{\lambda_1(\Lambda_N, B^{d-j}(r_j)\cap N)\cdots\lambda_{d-j-1}(\Lambda_N, B^{d-j}(r_j)\cap N)} \\ &\leq \frac{r_j}{\|b_1\|}\cdot\frac{(3/2)^{(d-j)(d-j-1)/2}\cdot(d-j-1)!\cdot\mathrm{vol}(B^{d-j}(r_j)\cap N)}{2^{d-j-1}\cdot\det(\Lambda_N)}.\end{aligned}$$

Since $\det(\Lambda_j) = v \cdot \det(\Lambda_N)$, we can rewrite this expression as

$$\|b_1\| \leq \frac{r_j\cdot(3/2)^{(d-j)(d-j-1)/2}\cdot(d-j-1)!\cdot 2^{d-j}\cdot\mathrm{vol}(B^{d-j}(r_j)\cap N)\cdot\det(\Lambda_j)}{2^{d-j-1}\cdot\mathrm{vol}(B^{d-j}(r_j))\cdot\det(\Lambda_N)} \leq 2^{d^2}\cdot v.$$

To derive the last inequality, we use the well-known formula

$$\mathrm{vol}(B^m(r)) = \begin{cases}\frac{2((m-1)/2)!(4\pi)^{(m-1)/2}}{m!}\cdot r^m & \text{if } m \text{ is odd,}\\ \frac{\pi^{m/2}}{(m/2)!}\cdot r^m & \text{if } m \text{ is even}\end{cases}$$

for the volume of $B^m(r)$, $m, r \in \mathbb{N}$. Since $\mathrm{vol}(B^{d-j}(r_j)\cap N) = \mathrm{vol}(B^{d-j-1}(r_j))$, we have $\mathrm{vol}(B^{d-j}(r_j)\cap N)/\,\mathrm{vol}(B^{d-j}(r_j)) \leq 2^{d-j}/r_j$. The Euclidean distance between z and N equals $|t_1|\cdot v$, which is at most r_j, as $z \in B^{d-j}(r_j)$. Thus, since $|t_1| \leq r_j/v$ and $1/v \leq 2^{d^2}/\|b_1\|$, we obtain $|t_1| \leq 2^{d^2}\cdot r_j/\|b_1\|$. This implies

$$\|p_j(z)\| = \|z - t_1 b_1\| \leq \|z\| + |t_1|\cdot\|b_1\| \leq r_j + 2^{d^2} r_j = r_{j+1}$$

and we see that $p_j(z)$ lies in $\Lambda_N \cap B^{d-j}(r_{j+1})$.

Note that $\lambda_i(\Lambda_N, B^{d-j}(r_{j+1})\cap N) = (2^{d^2}+1)^{-1}\cdot\lambda_i(\Lambda_N, B^{d-j}(r_j)\cap N)$ for every $i \in [d-j-1]$. Using this fact together with the bounds in (3), we obtain

$$\frac{\lambda_{i+1}(\Lambda_j, B^{d-j}(r_j))}{2^{d^2}+1} \leq \lambda_i(\Lambda_N, B^{d-j}(r_{j+1})\cap N) \leq \frac{(3/2)^{d-j}\lambda_{i+1}(\Lambda_j, B^{d-j}(r_j))}{2^{d^2}+1}$$

for every $i \in [d-j-1]$. ◀

▶ **Corollary 16.** *The set $\Lambda \cap B^d$ can be covered with $O_{d,k}((\lambda_k\cdots\lambda_d)^{-1})$ k-dimensional linear subspaces of $\mathbb{R}^d$.*

Proof. By Lemma 15, there is a positive integer $r = r(d, k-1)$ and a projection p of $\mathbb{R}^d$ along $k-1$ vectors $b_1, \ldots, b_{k-1} \in \Lambda$ onto a $(d-k+1)$-dimensional linear subspace N of $\mathbb{R}^d$ such that $\Lambda \cap B^d$ is mapped to $\Lambda \cap N \cap B^d(r)$ and such that $\lambda'_i := \lambda_i(\Lambda \cap N, B^d(r)\cap N) = \Theta_{d,k}(\lambda_{i+k-1})$ for every $i \in [d-k+1]$. We use Λ_N to denote the $(d-k+1)$-dimensional sublattice $\Lambda \cap N$ of Λ.

We consider the set $\mathcal{S} := \{\mathrm{lin}(\{y, b_1, \ldots, b_{k-1}\}) : y \in (\Lambda_N \setminus \{0\}) \cap B^d(r)\}$. Then $\mathcal{S}$ consists of k-dimensional linear subspaces and its projection $p(\mathcal{S})$ covers $\Lambda_N \cap B^d(r)$. By Theorem 12, the size of $\mathcal{S}$ is at most

$$|\Lambda_N \cap B^d(r)| \leq 2^{d-k}\prod_{i=1}^{d-k+1}\left\lfloor\frac{2}{\lambda'_i}+1\right\rfloor \leq O_{d,k}\left(\prod_{i=1}^{d-k+1}\frac{1}{\lambda'_i}\right) \leq O_{d,k}((\lambda_k\cdots\lambda_d)^{-1}),$$

where the second inequality follows from the assumption $\lambda_d \leq 1$, as then $\lambda'_{d-k+1} \leq O_{d,k}(\lambda_d)$ implies $\lambda'_1 \leq \cdots \leq \lambda'_{d-k+1} \leq O_{d,k}(1)$. The last inequality is obtained from $\lambda'_i \geq \Omega_{d,k}(\lambda_{i+k-1})$ for every $i \in [d-k+1]$. Moreover, $\mathcal{S}$ covers $\Lambda \cap K$, since for every $y \in \Lambda_N \cap B^d(r)$ there is $S \in \mathcal{S}$ with $y \in p(S)$ and $p(z) \in \Lambda_N \cap B^d(r)$ for every $z \in \Lambda \cap B^d$. ◀

Let q be an integer from $\{d-k+1,\dots,d\}$ such that $\alpha = (\lambda_{d-q+1}\cdots\lambda_d)^{-1/(q-1)}$. The bound from Corollary 16 matches the bound from Theorem 14 in the case $k=1$. The case $k=d-1$ was shown by Bárány et al. [5]; see Theorem 1. Thus we may assume $d \geq 4$. Corollary 16 also provides the same bound as Theorem 14 if $q = d-k+1$, so we assume $q \geq d-k+2$.

We now sketch the proof of the upper bound $O_{d,k}(\alpha^{d-k})$ if $q \geq d-k+2$. Let Λ^* be the dual lattice of Λ. That is, Λ^* is the set of vectors y from $\mathbb{R}^d$ that satisfy $\langle x,y\rangle \in \mathbb{Z}$ for every $x \in \Lambda$. In the rest of the section, we use μ_i to denote $\lambda_i(\Lambda^*, B^d)$ for every $i \in [d]$. It follows from the results of Mahler [16] and Banaszczyk [4] that $1 \leq \lambda_i \cdot \mu_{d-i+1} \leq d$ holds for every $i \in [d]$. This together with the assumption $\lambda_d \leq 1$ implies $\mu_1 \geq 1$ and $\alpha = \Theta_{d,k}((\mu_1\cdots\mu_q)^{1/(q-1)})$.

We now proceed by induction on $d-k$. The case $d-k=1$ is treated similarly as in the proof of Theorem 1 by Bárány et al. [5]. Using the pigeonhole principle, we can construct a set D' of primitive points from $\Lambda^* \setminus \{0\}$ such that $|D'| \leq O_d(\alpha)$ and such that for every $x \in \Lambda \cap B^d$ there is $z \in D'$ with $\langle x,z\rangle = 0$. We let $\mathcal{S}$ to be the set of hyperplanes that contain the origin and have normal vectors from D'. Observe that $\mathcal{S}$ is a set of $O_d(\alpha) = O_d(\alpha^{d-k})$ $(d-1)$-dimensional linear subspaces that cover $\Lambda \cap B^d$.

For the inductive step, assume that $d-k \geq 2$. We consider the set $\mathcal{S}$ of hyperplanes in $\mathbb{R}^d$ that has been constructed in the base of the induction. For every hyperplane $H(z) \in \mathcal{S}$ with the normal vector $z \in D'$, we let $\Lambda_{H(z)}$ be the set $\Lambda \cap H(z)$. Note that $\Lambda_{H(z)}$ is a lattice of dimension at most $d-1$. We now proceed inductively and cover each set $\Lambda_{H(z)} \cap B^d$ using the inductive hypothesis for $\Lambda_{H(z)}$ and k. To do so, we employ the fact that, for every $z \in D'$, the larger $\|z\|$ is, the fewer k-dimensional linear subspaces we need to cover $\Lambda_{H(z)} \cap B^d$. In particular, we prove that if z is a point from D' and $q \geq d-k+2$, then $\Lambda_{H(z)} \cap B^d$ can be covered with $O_{d,k}\left(((\mu_1\cdots\mu_q)/\|z\|)^{(d-k-1)/(q-2)}\right)$ k-dimensional linear subspaces.

Then we partition D' into subsets $S_1,\dots,S_q$ such that all vectors from S_i have approximately the same Euclidean norm. Then, for every $i \in [q]$, we sum the number c_i of k-dimensional linear subspaces needed to cover $\Lambda_{H(z)} \cap B^d$ for $z \in S_i$ and show that $c_1 + \cdots + c_q \leq O_{d,k}(\alpha^{d-k})$.

3.2 The general case

Here, we finish the proof of Theorem 4 by extending Theorem 14 to arbitrary convex bodies from $\mathcal{K}^d$. This is done by approximating a given body K from $\mathcal{K}^d$ with ellipsoids. A *d-dimensional ellipsoid* in $\mathbb{R}^d$ is an image of B^d under a nonsingular affine map. Such approximation exists by the following classical result, called *John's lemma* [14].

▶ **Lemma 17** (John's lemma [17, see Theorem 13.4.1]). *For every positive integer d and every $K \in \mathcal{K}^d$, there is a d-dimensional ellipsoid E with the center in the origin that satisfies*

$$E/\sqrt{d} \subseteq K \subseteq E.$$

Let $\Lambda \in \mathcal{L}^d$ be a given lattice and let $\lambda_i := \lambda_i(\Lambda, K)$ for every $i \in [d]$. From our assumptions, we know that $\lambda_d \leq 1$. Let E be the ellipsoid from Lemma 17. Since E is an ellipsoid, there is a nonsingular affine map $h\colon \mathbb{R}^d \to \mathbb{R}^d$ such that $E = h(B^d)$. Since E is centered in the origin, we see that h is in fact a linear map. Thus $\Lambda' := h^{-1}(\Lambda) \in \mathcal{L}^d$. Observe that we have $\lambda_i = \lambda_i(\Lambda', h^{-1}(K))$ for every $i \in [d]$.

For every $i \in [d]$, we use λ'_i to denote the ith successive minimum $\lambda_i(\Lambda', B^d) = \lambda_i(\Lambda, E)$. From the choice of E, we have $\lambda_i/\sqrt{d} \leq \lambda'_i \leq \lambda_i$. In particular, $\lambda'_d \leq 1$. Thus, by Theorem 14, the set $\Lambda' \cap B^d$ can be covered with $O_{d,k}((\alpha')^{d-k})$ k-dimensional linear subspaces, where $\alpha' := \min_{1\leq j\leq k}(\lambda'_j \cdots \lambda'_d)^{-1/(d-j)}$.

Since $\lambda_i = \Theta_d(\lambda'_i)$ for every $i \in [d]$, we see that the set $\Lambda' \cap h^{-1}(K)$ can be covered with $O_{d,k}(\alpha^{d-k})$ k-dimensional linear subspaces, where $\alpha := \min_{1\leq j\leq k}(\lambda_j \cdots \lambda_d)^{-1/(d-j)}$. Since every nonsingular linear transformation preserves incidences and successive minima and maps a k-dimensional linear subspace to a k-dimensional linear subspace, the set $\Lambda \cap K$ can be covered with $O_{d,k}(\alpha^{d-k})$ k-dimensional linear subspaces.

4 Proof of Theorem 5

Let d and k be positive integers satisfying $1 \leq k \leq d-1$ and let K be a body from $\mathcal{K}^d$ with $\lambda_d(\mathbb{Z}^d, K) \leq 1$. For every $i \in [d]$, we let λ_i be the ith successive minimum $\lambda_i(\mathbb{Z}^d, K)$. Let ε be a number from $(0,1)$. We use a probabilistic approach to show that there is a set $S \subseteq \mathbb{Z}^d \cap K$ of size at least $\Omega_{d,\varepsilon,k}(((1-\lambda_d)\beta)^{d-k-\varepsilon})$, where $\beta := \min_{1\leq j\leq d-1}(\lambda_j \cdots \lambda_d)^{-1/(d-j)}$, such that every k-dimensional linear subspace contains at most $r-1$ points from S.

Note that it is sufficient to prove the statement only for the lattice $\mathbb{Z}^d$. For a general lattice $\Lambda \in \mathcal{L}^d$ we can apply a linear transformation h such that $h(\Lambda) = \mathbb{Z}^d$ and then use the result for $\mathbb{Z}^d$ and $h(K)$, since $\lambda_i(\Lambda, K) = \lambda_i(\mathbb{Z}^d, h(K))$ for every $i \in [d]$. We also remark that in the case $k = d-1$ the stronger lower bound $\Omega_d((1-\lambda_d)\beta)$ from Theorem 1 by Bárány et al. [5] applies.

The proof is based on the following two results, first of which is by Bárány et al. [5].

▶ **Lemma 18** ([5]). *For an integer $d \geq 2$ and $K \in \mathcal{K}^d$, if $\lambda_d < 1$ and p is an integer satisfying $1 < p < (1-\lambda_d)\beta/(8d^2)$, then, for every $v \in \mathbb{R}^d$, there exist an integer $1 \leq j < p$ and a point $w \in \mathbb{Z}^d$ with $jv + pw \in K$.*

For a prime number p, let $\mathbb{F}_p$ be the finite field of size p. The second main ingredient in the proof of Theorem 5 is the following lemma.

▶ **Lemma 19.** *Let d and k be integers satisfying $2 \leq k \leq d-2$ and let $\varepsilon \in (0,1)$. Then there is a positive integer $p_0 = p_0(d,\varepsilon,k)$ such that for every prime number $p \geq p_0$ there exists a subset R of $\mathbb{F}_p^{d-1}$ of size at least $p^{d-k-\varepsilon}/2$ such that every $(k-1)$-dimensional affine subspace of $\mathbb{F}_p^{d-1}$ contains at most $r-1$ points from R for $r := \lceil k(d-k+1)/\varepsilon \rceil$.*

Proof. We assume that p is large enough with respect to d, ε, and k so that $p^{k-1} > r$. We set $P := p^{1-k-\varepsilon}$ and we let X be a subset of $\mathbb{F}_p^{d-1}$ obtained by choosing every point from $\mathbb{F}_p^{d-1}$ independently at random with the probability P.

Let A be a $(k-1)$-dimensional affine subspace of $\mathbb{F}_p^{d-1}$. Then $|A| = p^{k-1}$. It is well-known that the number of $(k-1)$-dimensional linear subspaces of $\mathbb{F}_p^{d-1}$ is exactly the *Gaussian binomial coefficient*

$$\begin{bmatrix} d-1 \\ k-1 \end{bmatrix}_p := \frac{(p^{d-1}-1)(p^{d-1}-p)\cdots(p^{d-1}-p^{k-2})}{(p^{k-1}-1)(p^{k-1}-p)\cdots(p^{k-1}-p^{k-2})}$$
$$\leq \frac{p^{d-1}\cdot p^{d-2}\cdots p^{d-k+1}}{(p^{k-1}-1)(p^{k-2}-1)\cdots(p-1)} \leq p^{(k-1)d-(k-1)k/2-(k-1)(k-2)/2} = p^{(k-1)(d-k+1)}. \quad (4)$$

We used the fact $p^{k-i}-1 \geq p^{k-i-1}$ for $k > i$ in the last inequality.

Since every $(k-1)$-dimensional affine subspace A of $\mathbb{F}_p^{d-1}$ is of the form $A = x + L$ for some $x \in \mathbb{F}_p^{d-1}$ and a $(k-1)$-dimensional linear subspace L of $\mathbb{F}_p^{d-1}$ and $x + L = y + L$ if and only if $x - y \in L$, the total number of $(k-1)$-dimensional affine subspaces of $\mathbb{F}_p^{d-1}$

is $p^{d-k}\begin{bmatrix} d-1 \\ k-1 \end{bmatrix}_p$. This is because by considering pairs (x, L), where $x \in \mathbb{F}_p^{d-1}$ and L is a $(k-1)$-dimensional linear subspace of $\mathbb{F}_p^{d-1}$, every $(k-1)$-dimensional affine subspace A is counted p^{k-1} times.

We use the following Chernoff-type bound (see the last bound of [12]) to estimate the probability that A contains at least r points of X. Let $q \in [0,1]$ and let $Y_1, \dots, Y_m$ be independent 0-1 random variables with $\Pr[Y_i = 1] = q$ for every $i \in [m]$. Then, for $mq \leq s < m$, we have

$$\Pr[Y_1 + \dots + Y_m \geq s] \leq \left(\frac{mq}{s}\right)^s e^{s-mq}. \tag{5}$$

Choosing Y_x as the indicator variable for the event $x \in A \cap X$ for each $x \in A$, we have $m = |A| = p^{k-1}$ and $q = P$. Since $p, r \geq 1$ and $p^{k-1} > r$, we have $p^{-\varepsilon} = mq \leq r < m = p^{k-1}$ and thus the bound (5) implies

$$\Pr[|A \cap X| \geq r] \leq \left(\frac{p^{k-1}P}{r}\right)^r e^{r-p^{k-1}P} = \left(\frac{p^{-\varepsilon}}{r}\right)^r e^{r-p^{-\varepsilon}} = p^{-\varepsilon r} e^{r(1-\ln r)-p^{-\varepsilon}} < p^{-\varepsilon r},$$

where the last inequality follows from $r \geq e$, as then $1 - \ln r \leq 0$.

By the Union bound, the probability that there is a $(k-1)$-dimensional affine subspace A of $\mathbb{F}_p^{d-1}$ with $|A \cap X| \geq r$ is less than

$$p^{d-k}\begin{bmatrix} d-1 \\ k-1 \end{bmatrix}_p \cdot p^{-\varepsilon r} \leq p^{(d-k)+(k-1)(d-k+1)-\varepsilon r} \leq p^{k(d-k+1)-1-k(d-k+1)} = p^{-1},$$

where the first inequality follows from (4) and the second inequality is due to the choice of r. From $p \geq 2$, we see that this probability is less than $1/2$.

The expected size of X is $\mathbb{E}[|X|] = |\mathbb{F}_p^{d-1}| \cdot P = p^{d-1}p^{1-k-\varepsilon} = p^{d-k-\varepsilon}$. Since $|X| \sim \mathrm{Bi}(p^{d-1}, P)$, the variance of $|X|$ is $p^{d-1}P(1-P) < p^{d-k-\varepsilon}$ and Chebyshev's inequality implies $\Pr[||X| - \mathbb{E}[|X|]| \geq \sqrt{2p^{d-k-\varepsilon}}] < p^{d-k-\varepsilon}/(2p^{d-k-\varepsilon}) = 1/2$.

Thus there is a set R of size at least $p^{d-k-\varepsilon} - \sqrt{2p^{d-k-\varepsilon}} \geq p^{d-k-\varepsilon}/2$ such that every $(k-1)$-dimensional affine subspace of $\mathbb{F}_p^{d-1}$ contains at most $r-1$ points from R. ◀

Let $\varepsilon \in (0,1)$ be given. To derive Theorem 5, we combine Lemma 18 with Lemma 19. This is a similar approach as in [5], where the authors derive a lower bound for the case $k = d-1$ by combining Lemma 18 with a construction found by Erdős in connection with Heilbronn's triangle problem [20].

Let p be the largest prime number that satisfies the assumptions of Lemma 18. If such p does not exist, then the statement of the theorem is trivial. By Bertrand's postulate, we have $p > (1-\lambda_d)\beta/(16d^2)$. We may assume that $p \geq p_0$, where $p_0 = p_0(d, \varepsilon, k)$ is the constant from Lemma 19, since otherwise the statement of Theorem 5 is trivial.

For $k \geq 2$ and $t := \lceil p^{d-k-\varepsilon}/2 \rceil$, let $R = \{v_1, \dots, v_t\} \subseteq \mathbb{F}_p^{d-1}$ be the set of points from Lemma 19. That is, every $(k-1)$-dimensional affine subspace of $\mathbb{F}_p^{d-1}$ contains at most $r-1$ points from R for $r := \lceil k(d-k+1)/\varepsilon \rceil$. In particular, every r-tuple of points from R contains $k+1$ affinely independent points over the field $\mathbb{F}_p$. For $k = 1$, we can set $r := 2$ and let R be the whole set $\mathbb{F}_p^{d-1}$ of size $t := p^{d-k} = p^{d-1}$. Then every r-tuple of points from R contains two affinely independent points over the field $\mathbb{F}_p$.

For $i = 1, \dots, t$, let $u_i \in \mathbb{Z}^d$ be the vector obtained from v_i by adding 1 as the last coordinate. From the choice of R, every r-tuple of points from $\{u_1, \dots, u_t\}$ contains $k+1$ points that are linearly independent over the field $\mathbb{F}_p$.

By Lemma 18, there exist an integer $1 \leq j_i < p$ and a point $w_i \in \mathbb{Z}^d$ for every $i \in [t]$ such that $u_i' := j_i u_i + p w_i$ lies in K. We have $u_i' \equiv j_i u_i \pmod{p}$ for every $i \in [t]$ and thus every

r-tuple of vectors from $S := \{u'_1, \ldots, u'_t\} \subseteq \mathbb{Z}^d$ contains $k+1$ linearly independent vectors over the field $\mathbb{F}_p$, and hence over $\mathbb{R}$. In other words, every k-dimensional linear subspace of $\mathbb{R}^d$ contains at most $r-1$ points from S. Since $|S| = t = \lceil p^{d-k-\varepsilon}/2 \rceil$ and $p > (1-\lambda_d)\beta/(16d^2)$, we have $l(d,k,n,r) \geq \Omega_{d,k}(((1-\lambda_d)\beta)^{d-k-\varepsilon})$. This completes the proof of Theorem 5.

5 Proof of Theorem 8

Let d and k be integers with $1 \leq k \leq d-1$ and let $\Lambda \in \mathcal{L}^d$ and $K \in \mathcal{K}^d$. We let $\lambda_i := \lambda_i(\Lambda, K)$ for every $i \in [d]$ and assume that $\lambda_d \leq 1$. First, we observe that it is sufficient to prove the statement only for $K = B^d$, as we can then strengthen the statement to an arbitrary $K \in \mathcal{K}^d$ using John's lemma (Lemma 17) analogously as in the proof of Theorem 4.

First, we prove the upper bound. That is, we show that $\Lambda \cap B^d$ can be covered with $O_{d,k}((\lambda_{k+1} \cdots \lambda_d)^{-1})$ k-dimensional affine subspaces of $\mathbb{R}^d$. By Lemma 15, there is a positive integer $r = r(d,k)$ and a projection p of $\mathbb{R}^d$ along k vectors $b_1, \ldots, b_k$ from Λ onto a $(d-k)$-dimensional linear subspace N of $\mathbb{R}^d$ such that $\Lambda \cap B^d$ is mapped to $\Lambda \cap N \cap B^d(r)$ and such that $\lambda'_i := \lambda_i(\Lambda \cap N, B^d(r) \cap N) = \Theta_{d,k}(\lambda_{i+k})$ for every $i \in [d-k]$.

For each point z of $\Lambda \cap N \cap B^d(r)$, we define $A(z)$ to be the affine hull of the set $\{z, b_1 + z, \ldots, b_k + z\}$. Every $A(z)$ is then a k-dimensional affine subspace of $\mathbb{R}^d$ and the set $\mathcal{A} := \{A(z) \colon z \in \Lambda \cap N \cap B^d(r)\}$ covers $\Lambda \cap B^d$, since $p(z) \in \Lambda \cap N \cap B^d(r)$ for every $z \in \Lambda \cap B^d$. We have $|\mathcal{A}| = |\Lambda \cap N \cap B^d(r)|$ and, since $\lambda_d \leq 1$ and $\lambda'_1 \leq \cdots \leq \lambda'_{d-k} \leq O_{d,k}(\lambda_d)$, Theorem 12 implies $|\Lambda \cap N \cap B^d(r)| \leq O_{d,k}((\lambda'_1 \cdots \lambda'_{d-k})^{-1})$. The bound $\lambda'_i \geq \Omega_{d,k}(\lambda_{i+k})$ for every $i \in [d-k]$ then gives $|\mathcal{A}| \leq O_{d,k}((\lambda_{k+1} \cdots \lambda_d)^{-1})$.

To show the lower bound, we prove that we need at least $\Omega_{d,k}((\lambda_{k+1} \cdots \lambda_d)^{-1})$ k-dimensional affine subspaces of $\mathbb{R}^d$ to cover $\Lambda \cap B^d$.

Let A be a k-dimensional affine subspace of $\mathbb{R}^d$. We show that A contains at most $O_{d,k}((\lambda_1 \cdots \lambda_k)^{-1})$ points from $\Lambda \cap B^d$. Let y be an arbitrary point from $\Lambda \cap A \cap B^d$. Then $A = L + y$, where L is a k-dimensional linear subspace of $\mathbb{R}^d$, and $(\Lambda \cap A) - y = \Lambda \cap L$. For every $i \in [k]$, we let $\lambda'_i := \lambda_i(\Lambda \cap L, B^d(2))$ and we observe that $\lambda'_i \geq \lambda_i/2$. By Theorem 12, we have $|\Lambda \cap L \cap B^d(2)| \leq O_{d,k}((\lambda'_1 \cdots \lambda'_s)^{-1})$, where s is the maximum integer j from $[k]$ with $\lambda'_j \leq 1$. Since $\lambda'_i \geq \lambda_i/2$ for every $i \in [k]$, we have $|\Lambda \cap L \cap B^d(2)| \leq O_{d,k}((\lambda_1 \cdots \lambda_k)^{-1})$. For every $x \in A \cap B^d$, we have $\|x - y\| \leq \|x\| + \|y\| \leq 2$ and thus $x - y \in L \cap B^d(2)$. It follows that $(\Lambda \cap A \cap B^d) - y \subseteq \Lambda \cap L \cap B^d(2)$ and thus $|\Lambda \cap A \cap B^d| \leq O_{d,k}((\lambda_1 \cdots \lambda_k)^{-1})$.

Let $\mathcal{A}$ be a collection of k-dimensional affine subspaces of $\mathbb{R}^d$ that covers $\Lambda \cap B^d$. We have $|\mathcal{A}| \geq |\Lambda \cap B^d|/m$, where m is the maximum of $|\Lambda \cap A \cap B^d|$ taken over all subspaces A from $\mathcal{A}$. We know that $m \leq O_{d,k}((\lambda_1 \cdots \lambda_k)^{-1})$. It is a well-known fact that follows from Minkowski's second theorem (Theorem 11) that $|\Lambda \cap B^d| \geq \Omega_{d,k}((\lambda_1 \cdots \lambda_d)^{-1})$. Thus we obtain

$$|\mathcal{A}| \geq \frac{|\Lambda \cap B^d|}{m} \geq \frac{\Omega_{d,k}((\lambda_1 \cdots \lambda_d)^{-1})}{O_{d,k}((\lambda_1 \cdots \lambda_k)^{-1})} \geq \Omega_{d,k}((\lambda_{k+1} \cdots \lambda_d)^{-1}),$$

which finishes the proof of Theorem 8.

6 Proofs of Theorems 9 and 10

We now improve the lower bounds from Theorem 3 on the number of point-hyperplane incidences. We use essentially the same construction as Brass and Knauer [6].

Assume that we are given integers d and k with $0 \leq k \leq d-2$ and let ε be a real number in $(0,1)$. Let $\delta = \delta(d, \varepsilon, k) \in (0,1)$ be a sufficiently small constant. By (1), there is a positive integer $r_1 = r_1(d, \delta, k)$ and a constant $c_1 = c_1(d, \delta, k)$ such that for every $s \in \mathbb{N}$ there is a subset P of $\mathbb{Z}^d \cap B^d(s)$ of size $c_1 \cdot s^{d-k-\delta}$ such that every k-dimensional affine subspace of $\mathbb{R}^d$ contains at most $r_1 - 1$ points from P. In the case $k = 0$, we can clearly obtain the stronger bound $c_1 \cdot s^d$.

By Corollary 6, there is a positive integer $r_2 = r_2(d, \delta, k)$ and a constant $c_2 = c_2(d, \delta, k)$ such that for every $t \in \mathbb{N}$ there is a subset N' of $\mathbb{Z}^d \cap B^d(t)$ of size $c_2 \cdot t^{d(k+1-\delta)/(d-1)}$ such that every $(d-k-1)$-dimensional linear subspace contains at most $r_2 - 1$ points from N'. In particular, every 1-dimensional linear subspace contains at most $r_2 - 1$ points from N' and thus there is a set $N \subseteq N'$ of size $|N| = |N'|/(r_2 - 1) = c_2 \cdot t^{d(k+1-\delta)/(d-1)}/(r_2 - 1)$ containing only primitive vectors. We note that for $k = 0$ we can apply Theorem 1 instead of Corollary 6 and obtain the stronger bound $|N| = c_2 \cdot t^{d/(d-1)}/(r_2 - 1)$. We let $\mathcal{H}$ be the set of hyperplanes in $\mathbb{R}^d$ with normal vectors from N such that every hyperplane from $\mathcal{H}$ contains at least one point of P.

We show that the graph $G(P, \mathcal{H})$ does not contain K_{r_1,r_2}. If there is an r_2-tuple of hyperplanes from $\mathcal{H}$ with a nonempty intersection, then these hyperplanes have distinct normal vectors that span a linear subspace of dimension at least $d - k$ by the choice of N. The intersection of these hyperplanes is thus an affine subspace of dimension at most k. From the definition of P, it contains at most $r_1 - 1$ points from P.

We set $n := c_1 \cdot s^{d-k-\delta}$ and $m := \frac{3c_2}{r_2-1} \cdot s \cdot t^{d(k+2-1/d-\delta)/(d-1)}$. Then we have $|P| = n$. For every $p \in P$ and $z \in N$, we have $\langle p, z\rangle \in \mathbb{Z}$ and $|\langle p, z\rangle| \leq \|p\|\|z\| \leq st$ by the Cauchy–Schwarz inequality. Thus every point z from N is the normal vector of at most $2st + 1 \leq 3st$ hyperplanes from $\mathcal{H}$. It follows that

$$|\mathcal{H}| \leq 3st|N| = 3st\frac{c_2 \cdot t^{d(k+1-\delta)/(d-1)}}{r_2 - 1} = \frac{3c_2}{r_2 - 1} \cdot s \cdot t^{d(k+2-1/d-\delta)/(d-1)} = m.$$

From the definition of $\mathcal{H}$, the number of incidences between P and $\mathcal{H}$ is at least

$$\begin{aligned}|P||N| = n \cdot \frac{c_2 \cdot t^{d(k+1-\delta)/(d-1)}}{r_2 - 1} &= \Omega_{d,\varepsilon,k}\left(n \cdot (m/s)^{(k+1-\delta)/(k+2-1/d-\delta)}\right) \\ &= \Omega_{d,\varepsilon,k}\left(n^{1-(k+1-\delta)/((k+2-1/d-\delta)(d-k-\delta))} m^{(k+1-\delta)/(k+2-1/d-\delta)}\right) \\ &\geq \Omega_{d,\varepsilon,k}\left(n^{1-(k+1)/((k+2-1/d)(d-k))-\varepsilon} m^{(k+1)/(k+2-1/d)-\varepsilon}\right), \end{aligned} \tag{6}$$

where the last inequality holds for δ sufficiently small with respect to d, ε, and k. This finishes the proof of Theorem 10.

To maximize the number of incidences in the diagonal case, we choose $k := \lfloor \frac{d-2}{2} \rfloor$. For d odd, we then have at least

$$\Omega_{d,\varepsilon}\left(n^{1-2(d-1)/((d+1-2/d)(d+3))-\varepsilon} m^{(d-1)/(d+1-2/d)-\varepsilon}\right)$$

incidences by (6). By duality, we may obtain a symmetrical expression by averaging the exponents. Then we obtain

$$\mathrm{I}(P, \mathcal{H}) \geq \Omega_{d,\varepsilon}\left((mn)^{(d^2+3d+3)/(d^2+5d+6)-\varepsilon}\right) = \Omega_{d,\varepsilon}\left((mn)^{1-(2d+3)/((d+2)(d+3))-\varepsilon}\right).$$

For d even, the choice of k implies that the number of incidences is at least

$$\Omega_{d,\varepsilon}\left(n^{1-2d/((d+2-2/d)(d+2))-\varepsilon} m^{d/(d+2-2/d)-\varepsilon}\right)$$

by (6). Using the averaging argument, we obtain

$$\mathrm{I}(P,\mathcal{H}) \geq \Omega_{d,\varepsilon}\left((mn)^{(d^3+2d^2+d-2)/((d+2)(d^2+2d-2))-\varepsilon}\right)$$
$$= \Omega_{d,\varepsilon}\left((mn)^{1-(2d^2+d-2)/((d+2)(d^2+2d-2))-\varepsilon}\right).$$

This completes the proof of Theorem 9. For $d \leq 3$, we have $k = 0$ and thus we can get rid of the ε in the exponent by applying the stronger bounds on m and n.

▶ Remark. An upper bound similar to (2) holds in a much more general setting, where we bound the maximum number of edges in $K_{r,r}$-free *semi-algebraic bipartite graphs* $G = (P \cup Q, E)$ *in* $(\mathbb{R}^d, \mathbb{R}^d)$ *with bounded description complexity* t (see [11] for definitions). Fox et al. [11] showed that the maximum number of edges in such graphs with $|P| = n$ and $|Q| = m$ is at most $O_{d,\varepsilon,r,t}((mn)^{1-1/(d+1)+\varepsilon} + m + n)$ for any $\varepsilon > 0$. Theorem 9 provides the best known lower bound for this problem, as every incidence graph $G(P,\mathcal{H})$ of P and $\mathcal{H}$ in $\mathbb{R}^d$ is a semi-algebraic graph in $(\mathbb{R}^d, \mathbb{R}^d)$ with bounded description complexity.

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