

Tighter Hard Instances for PPSZ

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Abstract

We construct uniquely satisfiable k -CNF formulas that are hard for the PPSZ algorithm, the currently best known algorithm solving k -SAT. This algorithm tries to generate a satisfying assignment by picking a random variable at a time and attempting to derive its value using some inference heuristic and otherwise assigning a random value. The “weak PPSZ” checks all subformulas of a given size to derive a value and the “strong PPSZ” runs resolution with width bounded by some given function. Firstly, we construct graph-instances on which “weak PPSZ” has savings of at most $(2 + \epsilon)/k$; the *saving* of an algorithm on an input formula with n variables is the largest γ such that the algorithm succeeds (i.e. finds a satisfying assignment) with probability at least $2^{-(1-\gamma)n}$. Since PPSZ (both weak and strong) is known to have savings of at least $\frac{\pi^2 + o(1)}{6k}$, this is optimal up to the constant factor. In particular, for $k = 3$, our upper bound is $2^{0.333\dots n}$, which is fairly close to the lower bound $2^{0.386\dots n}$ of Hertli [SIAM J. Comput.'14]. We also construct instances based on linear systems over \mathbb{F}_2 for which *strong* PPSZ has savings of at most $O\left(\frac{\log(k)}{k}\right)$. This is only a $\log(k)$ factor away from the optimal bound. Our constructions improve previous savings upper bound of $O\left(\frac{\log^2(k)}{k}\right)$ due to Chen et al. [SODA'13].

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1 Introduction

The k -SAT problem is one of the most fundamental NP-complete problems: given a k -CNF formula decide if there is an assignment to the variables that satisfies all the clauses. While a simple exhaustive search algorithm solves the problem, attempting to beat this trivial approach remains an active direction (see e.g. [11, 12, 4, 10]). Formalizing the true hardness of k -SAT, Impagliazzo and Paturi [7] presented two hypotheses: *Exponential Time Hypothesis (ETH)* which rules out any $2^{o(n)}$ time algorithm for k -SAT where n is the number of variables and *Strong Exponential Time Hypothesis (Strong ETH)* which says that for any $\epsilon > 0$ there exists $k > 0$ such that k -SAT cannot be solved in time $2^{(1-\epsilon)n}$. Both ETH and Strong ETH

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have successfully been used to explain the hardness of many other problems; under ETH one can prove tight lower bounds for many fixed parameter tractable problems (see [8]), and under Strong ETH several lower bounds for polynomial time solvable problems are proved (see e.g. [1]). However the validity of both these hypotheses remains a matter of mystery and in particular regarding Strong ETH no consensus seems to be within reach any time soon.

In this paper we focus on Strong ETH and the problem of constructing hard instances for known classes of algorithms for k -SAT. Paturi, Pudlák, Saks and Zane [10] presented the currently best known randomized algorithm for k -SAT. The algorithm roughly does the following: pick uniformly at random a variable x from the input formula. Try to infer the value of x using some sound heuristic. If this check fails, pick a random value for x . Set x to be this value and repeat. A sound heuristic is an algorithm P that receives a formula F and a variable x such that $P(F, x) = 0$ implies $F \models (x = 0)$ and $P(F, x) = 1$ implies $F \models (x = 1)$. We will consider two heuristics, P_w^{weak} and P_w^{strong} , where P_w^{weak} checks if the value of x can be derived from any set of w clauses of F , and P_w^{strong} checks if the value of x can be derived by a width- w resolution derivation from F . Note that if F is $O(1)$ -CNF then both P_w^{weak} and P_w^{strong} run in subexponential time as long as $w = o(\frac{n}{\log n})$. The first result showing that even simple sound heuristics can yield non-trivial savings over exhaustive search was proved by Paturi, Pudlák and Zane.

► **Theorem 1** ([11]). *Let F be a k -CNF formula on n variables. Then*

$$\Pr[\text{ppsz}(F, P_1^{\text{weak}}) \in \text{sat}(F)] \geq 2^{-(1-\frac{1}{k})n}.$$

Naturally one can ask if stronger heuristics can improve the success probability. It was indeed shown in the following theorem that using $\omega(1)$ -width resolution yields improvements.

► **Theorem 2** ([10]). *Let F be a k -CNF formula on n variables. Then*

$$\Pr[\text{ppsz}(F, P_{\omega(1)}^{\text{strong}}) \in \text{sat}(F)] \geq 2^{-(1-\frac{\pi^2}{6k}-o(1))n}.$$

Later Hertli [6] showed among other things that even $P_{\omega(1)}^{\text{weak}}$ yields the same improvement over the trivial P_1^{weak} .

► **Theorem 3** ([6]). *Let F be a k -CNF formula on n variables. Then*

$$\Pr[\text{ppsz}(F, P_{\omega(1)}^{\text{weak}}) \in \text{sat}(F)] \geq 2^{-(1-\frac{\pi^2}{6k}-o(1))n}.$$

The first construction of hard instances for PPSZ was given by Chen, Scheder, Talebanfard and Tang [3]. These instances are hard even for $P_{\omega(1)}^{\text{strong}}$.

► **Theorem 4** ([3]). *For any large enough $k, n > 0$ there are k -CNF formulas F such that*

$$\Pr[\text{ppsz}(F, P_{n/k}^{\text{strong}}) \in \text{sat}(F)] \leq 2^{-(1-O(\log^2 k/k))n}.$$

In this paper we improve this upper bound. For $P_{\omega(1)}^{\text{weak}}$ we give completely different constructions for which we can show that the success probability of PPSZ is essentially tight. For $P_{\omega(1)}^{\text{strong}}$ we can improve the asymptotics of k from $O(\log^2 k/k)$ to $O(\log k/k)$.

► **Theorem 5.** *For every $k \geq 3$ and every large enough n there exists a uniquely satisfiable k -CNF formula F on n variables such that*

1. $\Pr[\text{ppsz}(F, P_w^{\text{weak}}) \in \text{sat}(F)] \leq 2^{-(1-\frac{2}{k})n}$ for some $w = \Theta(\log n)$,
2. for any $\epsilon > 0$, $\Pr[\text{ppsz}(F, P_w^{\text{weak}}) \in \text{sat}(F)] \leq 2^{-(1-\frac{2(1+\epsilon)}{k})n}$ for some $w = n^{\Theta(\epsilon)}$.

In particular, for $k = 3$, our upper bound is $2^{0.333\dots n}$, which is fairly close to the lower bound $2^{0.386\dots n}$ of [6].

► **Theorem 6.** *For every $k \geq 3$ and every large enough n , there exists a k -CNF formula F on n variables with a unique satisfying assignment such that $\text{ppsz}(F, P_{n/k}^{\text{strong}})$ is successful with probability at most $2^{-(1+\epsilon)n}$, where $\epsilon = O\left(\frac{\log(k)}{k}\right)$.*

Unfortunately we fail to obtain $2^{-(1-O(1/k))n}$ upper bounds for strong PPSZ. Is this possible at all or indeed strong PPSZ succeeds with probability at least $2^{-(1-\omega(1/k))n}$?

The analysis of our hard instances is based on an encoding view of PPSZ. Given a formula F on variables x_1, \dots, x_n and a satisfying assignment \mathbf{b} , PPSZ produces an encoding of the assignment with respect to a given permutation π of the variables in the following way.

```

encode( $\mathbf{b}, \pi, F, P$ )
 $\mathbf{c} :=$  empty string
for  $i = 1, \dots, n$  do
  | if  $P(F, x_{\pi(i)}) \notin \{0, 1\}$  then
  | | append  $\mathbf{b}_{\pi(i)}$  to  $\mathbf{c}$ ;
  | end
  |  $F := F|_{x_{\pi(i)} \rightarrow \mathbf{b}_{\pi(i)}}$ ;
end
    
```

It is not hard to see that we can express the success probability of PPSZ in terms of expected code lengths as follows.

► **Lemma 7** ([10]). *Let F be a k -CNF and let P be a sound heuristic. We have $\Pr[\text{ppsz}(F, P) \in \text{sat}(F)] = \sum_{\mathbf{b} \in \text{sat}(F)} \mathbb{E}_{\pi} 2^{-|\text{encode}(\mathbf{b}, \pi, F, P)|}$.*

Thus our goal is to construct instances having a few satisfying assignments, all of which admitting only long encodings. Defining the optimal encoding length a satisfying assignment \mathbf{b} to be $\text{codelength}(F, P, \mathbf{b}) := \min_{\pi} |\text{encode}(\mathbf{b}, \pi, F, P)|$ we get

$$\Pr[\text{ppsz}(F, P) \in \text{sat}(F)] \leq \sum_{\mathbf{b} \in \text{sat}(F)} 2^{-\text{codelength}(\mathbf{b}, F, P)} .$$

The formulas in Theorem 5 and Theorem 6 have the unique satisfying assignment $\mathbf{0}$. Thus our goal will be to prove a lower bound on $\text{codelength}(F, P, \mathbf{0})$.

2 Notation and Preliminaries

Let F be a CNF formula with variable set V . A restriction (or partial assignment) is a partial function $\rho : V \rightarrow \{0, 1\}$. For $\mathbf{b} \in \{0, 1\}^n$, the notation $S \mapsto \mathbf{b}$ is the restriction that maps $x \in S$ to \mathbf{b}_x and is undefined on $V \setminus S$. By $F|_{\rho}$ we denote the formula arising from fixing the variables according to ρ and then simplifying the resulting formula by removing unsatisfied literals and satisfied clauses. For a matrix $A \in \mathbb{F}_2^{m \times n}$ and $U \subseteq [n]$ we denote by A_U the $(m \times |U|)$ submatrix formed by taking all columns indexed by some $i \in U$. By $\mathbf{0}$ we denote the all-0-assignment as well as the null vector in \mathbb{F}_2^n .

We will identify a vector $\mathbf{a} \in \mathbb{F}_2^n$ with its support $\{i \in [n] \mid a_i = 1\}$. Thus we will liberally write things like $\mathbf{a} \cup \mathbf{b}$, $\mathbf{a} \setminus \mathbf{b}$, $|\mathbf{a}|$, and so on.

We list some key observations relating PPSZ and resolution. The (easy) proofs can be found in the appendix.

► **Definition 8.** Let F be a formula with a unique satisfying assignment, which without loss of generality is $\mathbf{0}$, and let P be a proof heuristic. We say F *collapses under P* if there is an ordering x_1, \dots, x_n of the variables in F such that $F|_{(x_1, \dots, x_{i-1} \mapsto 0)} \vdash_P (x_i = 0)$ for all $1 \leq i \leq n$.

► **Proposition 9.** If $\text{codelength}(F, P, \mathbf{b}) \leq m$ then there is a set S of m variables such that $F|_{S \mapsto \mathbf{b}}$ collapses under P .

The next lemma states that if F collapses “sequentially” under bounded-width resolution, then it collapses “simultaneously” as well.

► **Proposition 10.** Let F be a k -CNF formula with the unique satisfying assignment $\mathbf{0}$, and let $w \geq k$. If F collapses under P_w^{strong} then $F \vdash_w^{\text{strong}} (x = 0)$ for all variables x of F .

The next proposition connects logical implication and collapse under P^{weak} to linear algebra.

► **Proposition 11.** Let $A \in \mathbb{F}_2^{m \times n}$ and F_A be its linear formula. If $\vdash_w^{\text{weak}} (F, x_i) \in \{0, 1\}$ then there is a row vector $\mathbf{r} \in \mathbb{F}_2^m$ of Hamming weight at most w such that $\mathbf{r} \cdot A = \mathbf{e}_i$.

3 Hard Instances for Weak PPSZ: Proof of Theorem 5

The construction in this section is based on a modification of satisfiable Tseitin formulas. Unsatisfiable Tseitin formulas are extensively studied in proof complexity (see e.g. [13, 14]). Given a graph $G = (V, E)$, the girth of G is defined as the size of the shortest cycle in G . We denote this by $g(G)$. For every pair $e, e' \in E(G)$ of edges we define the distance between e and e' by $\min_{u \in e, v \in e'} \{d(u, v)\}$. We will need graphs of bounded degree with large girth. According to a well-known result of Erdős and Sachs [5], for every $k \geq 3$ and every sufficiently large n , there exists a k -regular graph with n vertices and girth $> \log_{k-1} n$. Explicit constructions for infinitely many values of k with a better constant are also known [9].

Given a degree- k graph $G = (V, E)$, the Tseitin formula $T(G)$ is defined as follows. For each edge $e \in E$, there is a propositional variable x_e . For each vertex $v \in V$ we add the constraint $\sum_{e \ni v} x_e = 0 \pmod{2}$, which can be written as a conjunction of 2^{k-1} k -clauses.¹ In our formulas we assume that the girth of the graph is at least $\log_{k-1} n$, where n denotes the number of vertices. Furthermore, we add a clause $\neg x_e \vee \neg x_{e'}$ for each pair of edges e, e' of distance at least $\frac{g(G)}{2} - 1$ (which is $\geq \frac{1}{2} \log_{k-1} n - 1$). We call these clauses *bridges* and we denote the conjunction of all of them by B . Define $F_G := T(G) \wedge B$. Note that F_G has $N = kn/2$ variables.

The following proposition follows readily.

► **Proposition 12.** F_G has the unique satisfying assignment $\mathbf{0}$.

Proof. For an assignment α let G_α denote the spanning subgraph of G containing the edges e with $\alpha(e) = 1$. Note that α satisfies $T(G)$ if and only if G_α is *even*, i.e., every vertex has even degree. There are two cases: either G_α is the empty graph, in which case $\alpha = \mathbf{0}$ and satisfies F_G , too. Or G_α contains a cycle C , which has length at least $g(G)$ and therefore contains a bridge. In this case, α violates B . ◀

¹ The original Tseitin tautologies express the fact that the system $\sum_{e \ni v} x_e = a_v \pmod{2}$ is unsatisfiable if $\sum_v a_v = 1 \pmod{2}$.

We will consider PPSZ with P_w^{weak} when $w = O(\log n)$

► **Lemma 13.** *In F_G any encoding of the all-0 assignment has length at least $(1 - \frac{2}{k})N$ under $w \leq \frac{1}{2} \log_{k-1} n - 1$.*

Proof. Suppose for contradiction that $\text{codelength}(F, P_w^{\text{weak}}) \leq (1 - \frac{2}{k})N$ collapses. Then there is a restriction ρ that sets some $(1 - \frac{2}{k})N$ variables to 0 such that $F_G|_\rho$ collapses under P_w^{weak} . Note that ρ leaves at least $\frac{2}{k}N = n$ variables (i.e., edge) unset. Obviously this set of edges contains a cycle C . Let $\sigma := (E \setminus C \mapsto \mathbf{0})$. Clearly $F_G|_\sigma$ also collapses. This contradicts the next lemma:

► **Lemma 14.** *Let $C \subseteq E$ be a cycle and $\sigma := (E \setminus C \mapsto \mathbf{0})$. Then $F_G|_\sigma$ does not collapse under P_w^{weak} .*

Proof. Suppose it does collapse. Then there exists an edge $e \in C$ and a set $F' \subseteq F|_\sigma$ of at most w clauses such that $F' \models \neg x_e$.

The clauses in F' are either coming from the Tseitin part or from the bridges. Consider a path $P = v_1, \dots, v_s$ of maximum length on which e appears and the vertices of P are mentioned by Tseitin clauses in F' . Note that P cannot contain the whole cycle, since otherwise there would be too many clauses in F' . Let v_0 and v_{s+1} be vertices on $C \setminus P$ connected to v_1 and v_s , respectively. We extend P by v_0 and v_{s+1} . Since $w \leq \frac{1}{2} \log_{k-1} n - 1$, there is no bridge between any pair of edges appearing on P . We can now simply set all the variables in P to 1 and all other variables to 0. This would satisfy F' and yet it sets x_e to 1, contradicting that $F' \models \neg x_e$. ◀

This concludes the proof of Lemma 13. ◀

Below we show that it is possible to obtain similar lower bounds even when w is some function in $n^{O(\epsilon)}$.

► **Lemma 15.** *For every $\epsilon > 0$ and every sufficiently large n , any encoding of the all-0 assignment with $w < n^{\frac{\epsilon}{8(k-1)}}$ has length at least $(1 - \frac{2(1+\epsilon)}{k})N$.*

Proof. Let S be the set of edges appearing in any encoding of the all-0 assignment. We will show that $|E \setminus S| < (1 + \epsilon)n$. Assume for a contradiction that $|E \setminus S| \geq (1 + \epsilon)n$. We will show that $E \setminus S$ contains a large subgraph which is expanding in a certain sense.

► **Definition 16.** In a graph G we say that a path $P = v_1, \dots, v_t$ is *slender* if for all $1 \leq i \leq t$ we have $d(v_i) \leq 2$.

► **Lemma 17.** *Let $G = (V, E)$ be a graph on n vertices such that $|E| \geq (1 + \epsilon)n$ for some $\epsilon > 0$. There exists an induced subgraph $H \subseteq G$ on at least $\Omega(\epsilon^{3/4}n^{1/4})$ vertices with $\delta(G) \geq 2$ with no slender path of length $\geq 2/\epsilon$.*

Proof. Let $r = 2/\epsilon$. We first find a subgraph of minimum degree at least 2 on at least $\Omega(\sqrt{n/r})$ vertices with many edges. To do this we can remove vertices of degree at most 1 at a time. Having removed t vertices we are left with a graph on $n - t$ vertices and at least $(1 + \frac{2}{r})n - t$ edges. It holds that $(1 + \frac{2}{r})n - t \geq (1 + \frac{2}{r})(n - t)$. As the remaining graph has at most $\binom{n-t}{2}$ edges we have $(1 + 2/r)n \leq \binom{n-t}{2} + t$. This implies $t \leq n - \Omega(\sqrt{n/r})$. Let $n' = n - t$. We thus have $n' \geq \Omega(\sqrt{n/r})$.

If the remaining graph has no slender path of length r we are done. Otherwise let v_1, \dots, v_{t_1} be a maximal slender path, i.e., $d(v_i) = 2$ for all $1 \leq i \leq t_1$ and v_1 and v_{t_1} have a neighbor (possibly the same) outside P of degree at least 3. We remove v_1, \dots, v_{t_1} from the

graph. If there are any vertices of degree 1 we remove them one at a time until there are no more such vertices. Let the total number removed vertices be t'_1 . We repeat this for d rounds until there are no more slender paths of length r and all vertices have degree at least 2. Let t_i and t'_i be defined similarly for the i th iteration. We have $t'_i \geq r$ and thus $d \leq n'/r$. Note that the total number of removed edges is $t'_1 + \dots + t'_d + d$ and hence at most $n' + n'/r$. We are left with a graph with at least n'/r edges and hence at least $\Omega(\sqrt{n'/r}) = \Omega(\epsilon^{3/4}n^{1/4})$ vertices. \blacktriangleleft

Applying Lemma 17 on $E \setminus S$ we obtain a subgraph H with minimum degree at least 2 which does not contain any slender path of length $\geq 2/\epsilon$. Setting all edges outside of H to 0, we obtain that there exists a set of at most w clauses F' in the restricted formula which implies $x_e = 0$ for some $e \in H$. Let $e = (u, v)$. We will construct a tree T_{uv} in H by growing two disjoint rooted trees T_u and T_v , starting at u and v , respectively. The crucial requirement is that in both T_u and T_v any path of length $\geq 2/\epsilon$ that goes downwards in the rooted tree there exists a vertex of degree ≥ 3 . We call such a vertex a *branching* vertex. Furthermore, in T_{uv} the distance between the first branching vertices in T_u and T_v is at most $2/\epsilon$. Using the fact that the minimum degree in H is at least 2 and it does not contain any slender path of length $2/\epsilon$ and that the girth is at least $\log_{k-1} n$, we can easily construct T_{uv} so that each root to leaf path in both T_u and T_v has $\frac{\epsilon}{8} \log_{k-1} n$ branching vertices. Since the horizon $w < n^{\frac{\epsilon}{8(k-1)}}$, there are vertices u' and v' in T_u and T_v , respectively, that are not mentioned in F' . Consider the unique path between u' and v' in T_{uv} . Note that this path has length at most $\frac{1}{2} \log_{k-1} n$. However, since we put bridges only between edges of distance more than $\frac{1}{2} \log_{k-1} n$, there is no bridge between any pair of edges on this path. Setting all edges on the path including e to 1 and everything else to 0 satisfies F' , contradicting to $F' \models \neg x_e$. \blacktriangleleft

Lemma 13 implies that $\text{codelength}(F_G, P_w^{\text{weak}}) \geq (1 - \frac{2}{k})N$ for $w \leq \frac{1}{2} \log_{k-1} n - 1$. Similarly, Lemma 15 implies that $\text{codelength}(F_G, P_w^{\text{weak}}) \geq (1 - \frac{2(1+\epsilon)}{k})N$ for $w < n^{\frac{\epsilon}{8(k-1)}}$. This completes the proof of Theorem 5.

4 Hard Linear Formulas for Strong PPSZ: Proof of Theorem 6

Suppose $A \in \mathbf{F}^{m \times n}$ is a matrix in which every row has Hamming weight at most k . Then the system $A \cdot \mathbf{x} = \mathbf{0}$ consists of m linear equations over n variables, each of which involves at most k variables. One can encode it as a k -CNF formula with $2^{k-1} \cdot m$ clauses. Let us denote this formula by F_A . A CNF formula which in this way encodes a system of linear equations will be called a *linear CNF* formula.

4.1 Robust Expanding Matrices

As often in the realm of resolution, our proof of hardness relies on a certain notion of expansion. Loosely speaking, a matrix A is a *robust expander* if for every “sufficiently large” submatrix A_U and every “sufficiently diverse” set of row vectors $\mathbf{u}_1, \dots, \mathbf{u}_\ell$ at least one of the vectors $\mathbf{u}_i \cdot A_U$ has “large” Hamming weight. We will now define this notion formally. Throughout this section, let $k \in \mathbb{N}$ be arbitrary but fixed (this is the k for which we want to construct hard k -CNF formulas). A sequence $\mathbf{u}_1, \dots, \mathbf{u}_\ell \in \mathbb{F}_2^n$ is *well-increasing* if $n/k \leq |\mathbf{u}_i \setminus (\mathbf{u}_1 \cup \dots \cup \mathbf{u}_{i-1})| \leq 4n/k$ for every $1 \leq i \leq \ell$. This is what we mean by “sufficiently diverse”.

► **Definition 18** (Robust Expanders). A matrix $A \in \mathbb{F}_2^{n \times n}$ is called a t -robust (ℓ, w) -expander if for every $U \subseteq [n]$ of size t and well-increasing sequence $\mathbf{u}_1, \dots, \mathbf{u}_\ell$, there is some $1 \leq i \leq \ell$ such that $|\mathbf{u}_i \cdot A_U| > w$.

► **Theorem 19** (Robust Expanders Are Hard). Let $t, w \in \mathbb{N}$, $w \geq 2n/k$, and $\ell := \lfloor \frac{k \cdot t}{4n} \rfloor$. If A is a t -robust (ℓ, w) -expander, then $\text{codelength}(F, P_w^{\text{strong}}, \mathbf{0}) \geq n - t$.

► **Theorem 20** (Robust Expanders Exist). For every sufficiently large n , there is a matrix $A \in \mathbb{F}_2^{n \times n}$ such that (1) every row of A has Hamming weight at most $k + 1$; (2) the rank of A is at least $n - 2 \log(n)$; (3) A is a t -robust (ℓ, w) -expander for $t = \frac{60 \cdot \log(k)}{k} \cdot n$, $\ell = \lfloor \frac{k \cdot t}{4n} \rfloor$, and $w = 2n/k$.

With these theorems we can prove Theorem 6 for strong PPSZ. Write $t = \frac{60 \cdot \log(k)}{k} \cdot n$ and let A be a matrix as promised by Theorem 20. By Theorem 19 we know that $\text{codelength}(F_A, P_{2n/k}^{\text{strong}}, \mathbf{0}) \geq n - t$. The Steinitz exchange lemma from linear algebra gives us $2 \log(n)$ unit row vectors that we can add to A to obtain a matrix $A' \in \mathbb{F}_2^{(n+2 \log(n)) \times n}$ of row rank n . This means that $F_{A'}$ has the unique satisfying assignment $\mathbf{0}$. Each added unit row vector in A' is a unit clause in $F_{A'}$. It can easily be verified that adding a unit clause reduces codelength by at most 1. Therefore $\text{codelength}(F_{A'}, P_{n/k}^{\text{weak}}, \mathbf{0}) \geq \text{codelength}(F_A, P_{n/k}^{\text{weak}}, \mathbf{0}) - 2 \log(n) \geq n - t - 2 \log(n)$. This proves Theorem 6.

Proof of Theorem 19. Let P be the strong proof heuristic which performs resolution of width up to w . We assume that $\text{codelength}(F_A, P, \mathbf{0}) \leq n - t$ and will derive a contradiction to the assumption that A is a robust expander.

By Proposition 9 and 10, $\text{codelength}(F_A, P, \mathbf{0}) \leq n - t$ means that there is a partition $[n] = U \uplus S$ with $|U| = t$ such that $F' \vdash_P (x_i = 0)$ for every $i \in U$, where $F' := F_A|_{S \rightarrow \mathbf{0}}$ is the formula obtained from F by setting every variable in S to 0. For notational simplicity assume $U = \{1, \dots, t\}$. By a connection between resolution and linear algebra which is folklore by now (see e.g. [2]), the fact that $F' \vdash_P (x_i = 0)$ means the following:

► **Proposition 21** (Connection Between Resolution and Linear Algebra). For every $i \in U$ there exists a binary tree T_i in which every node v is labeled with a row vector $\mathbf{r}_v \in \mathbb{F}_2^n$ such that:

1. for a leaf v , the label \mathbf{r}_v is a unit vector,
2. if v is an inner node and v_0, v_1 are its children then $\mathbf{r}_v = \mathbf{r}_{v_0} + \mathbf{r}_{v_1}$.
3. $|\mathbf{r}_v \cdot A_U| \leq w$ for every node v of T_i ,
4. $\mathbf{r}_{\text{root}} \cdot A_U = \mathbf{e}_i$.

We call T_i the resolution tree of x_i .

For $i \in \{1, \dots, t\}$ let \mathbf{r}_i be the root labels of the tree T_i . Since $\mathbf{r}_i \cdot A_U = \mathbf{e}_i$ we conclude that the vectors $\mathbf{r}_1, \dots, \mathbf{r}_t$ are linearly independent. In particular this means that $|\mathbf{r}_1 \cup \dots \cup \mathbf{r}_t| \geq t$. Equipped with these observations and the previous proposition, we can now construct a well-increasing sequence $\mathbf{u}_1, \dots, \mathbf{u}_{\ell^*}$ with $\ell^* := \lfloor \frac{k \cdot t}{4n} \rfloor$ and $|\mathbf{u}_i \cdot A_U| \leq w$ for all $1 \leq i \leq \ell^*$. This will be a contradiction to the assumption that A is a robust expander.

Start with the empty sequence and $\ell = 0$. While $\ell < \ell^*$, we try to extend the current well-increasing sequence $\mathbf{u}_1, \dots, \mathbf{u}_\ell$ by considering two cases. For convenience let $\mathbf{u} = \mathbf{u}_1 \cup \dots \cup \mathbf{u}_\ell$. Note that $\frac{\ell \cdot n}{k} \leq |\mathbf{u}| \leq \frac{4\ell \cdot n}{k}$.

Case 1. Suppose some vector \mathbf{r}_i among $\mathbf{r}_1, \dots, \mathbf{r}_t$ satisfies $|\mathbf{r}_i \setminus \mathbf{u}| > 2n/k$. Recall that \mathbf{r}_i is the root label of the tree T_i . We walk from the root of T_i to a leaf by always choosing the child v for which the “weight” $|\mathbf{r}_v \setminus \mathbf{u}|$ is largest. Note that this weight is more than $2n/k$ at

the root and at most 1 at a leaf. Also, in every step the weight decreases by at most a factor of 2. Thus we find a node v on the path for which $n/k \leq |\mathbf{r}_v \setminus \mathbf{u}| \leq 2n/k$. We set $\mathbf{u}_{\ell+1} := \mathbf{r}_v$ and see that the sequence $\mathbf{u}_1, \dots, \mathbf{u}_{\ell+1}$ is well-increasing by the choice of $\mathbf{u}_{\ell+1}$. Also, since $\mathbf{u}_{\ell+1}$ is the label of a node in a resolution tree, it holds that $|\mathbf{u}_{\ell+1} \cdot A_U| \leq w$.

Case 2. Suppose $|\mathbf{r}_i \setminus \mathbf{u}| \leq 2n/k$ for all $1 \leq i \leq t$. Since $|(\mathbf{r}_1 \cup \dots \cup \mathbf{r}_t) \setminus \mathbf{u}| \geq t - |\mathbf{u}| \geq t - \frac{4\ell \cdot n}{k} \geq t - \frac{4\ell^* \cdot n}{k} + \frac{4n}{k} \geq \frac{4n}{k}$, we can find a subset $I \subseteq [t]$ with $|I| \leq 2n/k$ such that $2n/k \leq |\bigcup_{i \in I} \mathbf{r}_i \setminus \mathbf{u}| \leq 4n/k$. If we let v be a random linear combination of the $\mathbf{r}_i, i \in I$, we see that $\mathbb{E}[|\mathbf{v} \setminus \mathbf{u}|] \geq n/k$. Thus, there is some vector \mathbf{v} which is a linear combination of the $\mathbf{r}_i, i \in I$ and $n/k \leq |\mathbf{v} \setminus \mathbf{u}| \leq 4n/k$. Furthermore, since $|\mathbf{r}_i \cdot A_U| = |\mathbf{e}_i| = 1$ we get $|\mathbf{v} \cdot A_U| \leq \sum_{i \in I} |\mathbf{r}_i \cdot A_U| = |I| \leq 2n/k \leq w$. We can extend the sequence $\mathbf{u}_1, \dots, \mathbf{u}_\ell$ by setting $\mathbf{u}_{\ell+1} = \mathbf{v}$.

To summarize, this iteratively constructs a well-increasing sequence $\mathbf{u}_1, \dots, \mathbf{u}_{\ell^*}$ with $|\mathbf{u}_i \cdot A_U| \leq w$. We obtain a contradiction to the assumption that A is a robust expander, which completes the proof. ◀

4.2 Robust Kernel Expanders Exist – Proof of Theorem 20

Proof of Theorem 20. We will show that a matrix A sampled from a suitable probability distribution is a t -robust (ℓ, w) expander with high probability, for $t = \frac{60 \cdot \log(k)}{k} \cdot n$, $\ell = 5 \log(k)$, and $w = 2n/k$. Note that by definition, this will also be a t -robust (ℓ', w) -expander for every $\ell' \geq \ell$, thus also for $\ell' = \lfloor \frac{k \cdot t}{4n} \rfloor = \lfloor \frac{60}{4} \cdot \log(k) \rfloor \geq 5 \log(k) = \ell$.

Take a step k random walk in the Hamming cube $\{0, 1\}^n$ and let \mathbf{X} be its endpoint. We view \mathbf{X} as a row vector in \mathbb{F}_2^n . Repeating this experiment n times independently gives n row vectors that form a matrix $B \in \mathbb{F}_2^{n \times n}$. Surely each row of B has Hamming weight at most k , and B turns out to be a robust expander. Unfortunately its kernel will have dimension $\Theta\left(\frac{\log^2(k)n}{k}\right)$ on expectation—too large for our purposes. We introduce a nice trick that boosts the rank of B .

► **Lemma 22.** *Let $B \in \mathbb{F}_2^{n \times n}$ be a matrix and let P be a random permutation matrix. Then $\mathbb{E}[|\ker(B + P)|] \leq n + 1$.*

Proof. The kernel of a matrix $A \in \mathbb{F}_2^{n \times n}$ is the set $\{\mathbf{x} \in \mathbb{F}_2^n \mid A \cdot \mathbf{x} = \mathbf{0}\}$. With linearity of expectation we calculate:

$$\begin{aligned} \mathbb{E}[|\ker(B + P)|] &= \sum_{\mathbf{x} \in \mathbb{F}_2^n} \Pr[(B + P) \cdot \mathbf{x} = \mathbf{0}] \\ &= \sum_{\mathbf{x} \in \mathbb{F}_2^n} \Pr[B \cdot \mathbf{x} = P \cdot \mathbf{x}] \end{aligned}$$

Note that $B \cdot \mathbf{x}$ is a fixed vector whereas $P \cdot \mathbf{x}$ is a uniformly distributed over all vectors of weight $|\mathbf{x}|$. Thus, the probability that this happens to be $B \cdot \mathbf{x}$ is exactly $\binom{n}{|\mathbf{x}|}^{-1}$ if $|B \cdot \mathbf{x}| = |\mathbf{x}|$ and 0 otherwise. Thus the above is at most

$$\sum_{w=0}^n \sum_{\mathbf{x} \in \mathbb{F}_2^n; |\mathbf{x}|=w} \binom{n}{w}^{-1} = n + 1. \quad \blacktriangleleft$$

We set $A := B + P$. By Markov's inequality, $|\ker(A)| \leq n^2$ with high probability, and therefore also $\text{rank}(A) \geq n - 2 \log(n)$ with high probability. Also, each row of A has Hamming weight at most $k + 1$.

It remains to show that A has the desired expansion properties. First we fix a set U of size t and a well-increasing sequence $\mathbf{u}_1, \dots, \mathbf{u}_\ell$ and estimate the probability that $|\mathbf{u}_i \cdot A_U| \leq w$ for all i . For this we need the following fact about random walks in the Hamming cube which we will prove later.

► **Lemma 23** (Hamming Cube Mixing Lemma). *Let $U \subseteq [n]$ and $\mathbf{z} \in \{0, 1\}^U$. Let \mathbf{x} be the endpoint of a length d random walk in $\{0, 1\}^n$ starting at $\mathbf{0}$. Then*

$$\Pr[\mathbf{x}_U = \mathbf{z}] \leq 2 \left(\frac{1 + (1 - 2/n)^d}{2} \right)^{|U|}.$$

In particular if $d \geq n$ and $|U| = t$ is sufficiently large, then this probability is at most $2^{-2t/3}$. From this lemma it is easy to show the following:

► **Lemma 24.** *Let $\mathbf{u}_1, \dots, \mathbf{u}_\ell$ be a well-increasing sequence. Then the probability that $|\mathbf{u}_i \cdot A_U| \leq w$ for all i is at most $2^{-\frac{2t}{3}} \cdot \binom{t}{\leq w}^\ell$.*

Proof. Let \mathcal{E}_j be the event that $|\mathbf{u}_i \cdot A_U| \leq 2n/k$ for all $1 \leq i \leq j$. We want to bound $\Pr[\mathcal{E}_\ell] = \prod_{j=1}^\ell \Pr[|\mathbf{u}_j \cdot A_U| \leq 2n/k \mid \mathcal{E}_{j-1}]$. We claim that for each $1 \leq j \leq \ell$ the probability $\Pr[|\mathbf{u}_j \cdot A_U| \leq 2n/k \mid \mathcal{E}_{j-1}]$ is at most $2^{-2t/3} \cdot \binom{t}{\leq 2n/k}$.

We divide \mathbf{u}_i into an “old part” v_i and a “new part” w_i . Formally, we write $\mathbf{v}_i = \mathbf{u}_i \cap (\mathbf{u}_1 \cup \dots \cup \mathbf{u}_{i-1})$ and $\mathbf{w}_i = \mathbf{u}_i \setminus (\mathbf{u}_1 \cup \dots \cup \mathbf{u}_{i-1})$. We know that $|\mathbf{w}_i| \geq n/k$ since the sequence is well-increasing. Also, $\mathbf{u}_i = \mathbf{v}_i + \mathbf{w}_i$. Let $\mathbf{y} \in \mathbb{F}_2^t$ be a fixed vector. Note that

$$\Pr[\mathbf{u}_j \cdot A_U = \mathbf{y} \mid \mathcal{E}_{j-1}] = \Pr[\mathbf{w}_j \cdot A_U = \mathbf{v}_j \cdot A_U + \mathbf{y} \mid \mathcal{E}_{j-1}]$$

Now $\mathbf{v}_j \cdot A_U$ and \mathcal{E}_{j-1} both only depend on the rows \mathbf{a}_h of A with $h \in \mathbf{v}_j$, and $\mathbf{w}_j \cdot A_U$ is independent these. Thus, it suffices to bound $\Pr[\mathbf{w}_j \cdot A_U = \mathbf{z}]$ for some unknown but fixed vector \mathbf{z} . Remember that $A = B + P$ where P is a random $n \times n$ permutation matrix.

$$\Pr[\mathbf{w}_j \cdot A_U = \mathbf{z}] = \Pr[\mathbf{w}_j \cdot B_U = \mathbf{z} + P_U \cdot \mathbf{w}_j]$$

What is the distribution of $\mathbf{w}_j \cdot B_U$? It is the sum of $|\mathbf{w}_j|$ rows of B_U and thus distributed like the endpoint of a $|\mathbf{w}_j| \cdot k \geq n$ step random walk in $\{0, 1\}^n$ starting at $\mathbf{0}$ and then projected to the coordinates in U . By the Hamming Cube Mixing Lemma (Lemma 23) with $d = n$ we get

$$\Pr[\mathbf{w}_j \cdot A_U = \mathbf{z}] \leq 2 \left(\frac{1 + (1 - \frac{2}{n})^n}{2} \right)^t \leq 2^{-2t/3}.$$

We conclude that $\Pr[\mathbf{u}_j \cdot A_U = \mathbf{y} \mid \mathcal{E}_{j-1}] \leq 2^{-2t/3}$ for every fixed $\mathbf{y} \in \mathbb{F}_2^t$. Therefore

$$\Pr[|\mathbf{u}_j \cdot A_U| \leq 2n/k \mid \mathcal{E}_{j-1}] \leq 2^{-2t/3} \cdot \binom{t}{\leq 2n/k}.$$

This proves the claim. Via the chain rule, the claim immediately implies the lemma. ◀

To prove the theorem, it remains to do a union bound over the choices of $U \subseteq [n]$ and the well-increasing sequence. The number of ways to choose $U \subseteq [n]$ of size t is $\binom{n}{t} \leq \left(\frac{en}{t}\right)^t \leq k^t$. Bounding the number of well-increasing sequences is more subtle.

► **Lemma 25.** *The number of well-increasing sequences is at most $k^{\frac{4\ell n}{k}} \cdot 2^{\frac{4\ell^2}{k} \cdot n}$.*

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Proof. First, write $\mathbf{u} := \mathbf{u}_1 \cup \dots \cup \mathbf{u}_\ell$ and note that $|\mathbf{u}| \leq \frac{4\ell n}{k}$. Thus, the number of possible \mathbf{u} is at most $\binom{n}{\leq \frac{4\ell n}{k}} \leq k^{\frac{4\ell n}{k}}$. Once we have chosen \mathbf{u} , there are at most $2^{|\mathbf{u}|}$ choices for each individual \mathbf{u}_i and at most $2^{\ell \cdot |\mathbf{u}|} \leq 2^{\frac{4\ell^2}{k} \cdot n}$ well-increasing sequences. \blacktriangleleft

Let us now multiply (1) the number of choices for U , (2) the number of well-increasing sequences, and (3) for a fixed U and well-increasing sequence $\mathbf{u}_1, \dots, \mathbf{u}_\ell$, the probability that $|\mathbf{u}_i \cdot A_U| \leq w$. We see that this is at most

$$\begin{aligned} k^t \cdot k^{\frac{4\ell n}{k}} \cdot 2^{\frac{4\ell^2}{k} \cdot n} \cdot 2^{-\frac{2 \cdot \ell \cdot t}{3}} \cdot \binom{t}{\leq w}^\ell \\ = 2^{\frac{\log^2(k) \cdot n}{k} \cdot (60 + 4 \cdot 5 + 4 \cdot 5^2 - \frac{2}{3} \cdot 5 \cdot 60 + 2 \cdot 5)} = o(1). \end{aligned}$$

Here we used $\binom{t}{\leq w} \leq \left(\frac{et}{w}\right)^w \leq k^{2n/k}$. We conclude that A has the desired expansion properties with high probability. In addition, it has rank at least $n - 2 \log(n)$, and every row has Hamming weight at most $k + 1$. This concludes the proof. \blacktriangleleft

5 Proof of Lemma 23

Let Q be the random walk matrix of the n -dimensional Hamming cube. That is, $Q_{\mathbf{x}, \mathbf{y}} = 1/n$ if \mathbf{x} and \mathbf{y} have Hamming distance 1, and 0 otherwise. Note that Q is a $(2^n \times 2^n)$ -matrix, i.e., it takes as input vectors of dimension 2^n , or equivalently, functions from \mathbb{F}_2^n to \mathbf{R} . If $f : \mathbb{F}_2^n \rightarrow [0, 1]$ is a probability distribution over \mathbb{F}_2^n , then $Q^d f$ is the distribution that we get when sampling $\mathbf{x} \sim f$ and performing a random walk of length d . Let f be the function that is 1 at $\mathbf{0}$ and 0 elsewhere. For \mathbf{X} being the endpoint of an d -step random walk starting at $\mathbf{0}$, it holds that

$$\Pr[\mathbf{X} = \mathbf{y}] = (Q^d f)(\mathbf{y}).$$

Fortunately, we can understand $Q^d f$, since we know the eigenvalues of Q : The Hamming cube is the Cayley graph of the additive group of \mathbb{F}_2^n with generating set $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. The reader who could not make sense of this last sentence may read the next couple of paragraphs. The reader who is familiar with Cayley graphs and the discrete Fourier transform can skip them.

► **Definition 26.** For $S \subseteq [n]$, define $\chi_S : \mathbb{F}_2^n \rightarrow \mathbf{R}$ by

$$\chi_S(\mathbf{x}) := (-1)^{\sum_{i \in S} x_i}.$$

One checks that the χ_S form an orthonormal basis of the space of functions $\mathbb{F}_2^n \rightarrow \mathbf{R}$ when we choose the following inner product:

$$\langle f, g \rangle := \mathbb{E}_{\mathbf{x} \in \mathbb{F}_2^n} [f(\mathbf{x})g(\mathbf{x})].$$

Each χ_S is an eigenvector of Q :

$$\begin{aligned} (Q \cdot \chi_S)(\mathbf{x}) &= \sum_{\mathbf{y}} Q_{\mathbf{x}, \mathbf{y}} \chi_S(\mathbf{y}) = \sum_{\mathbf{y}: d_H(\mathbf{x}, \mathbf{y})=1} \frac{1}{n} \chi_S(\mathbf{y}) \\ &= \sum_{i=1}^n \frac{1}{n} \chi_S(\mathbf{x} + \mathbf{e}_i) = \sum_{i=1}^n \frac{1}{n} \chi_S(\mathbf{x}) \chi_S(\mathbf{e}_i) \\ &= \chi_S(\mathbf{x}) \frac{1}{n} \sum_{i=1}^n \chi_S(\mathbf{e}_i). \end{aligned}$$

So $\lambda_S := \frac{1}{n} \sum_{i=1}^n \chi_S(\mathbf{e}_i)$ is the eigenvector of χ_S . Let us evaluate λ_S :

$$\lambda_S = \frac{1}{n} \sum_{i=1}^n \chi_S(\mathbf{e}_i) = \frac{1}{n} \sum_{i=1}^n (-1)^{[i \in S]} = \frac{1}{n} (n - 2|S|) = 1 - \frac{2|S|}{n}.$$

Let $f : \mathbb{F}_2^n \rightarrow \mathbb{R}$ be the function that is 1 on $\mathbf{0}$ and 0 otherwise. To understand $Q^t f$, we write f in the basis of the eigenvectors of Q . Since the χ_S are orthonormal under the scalar product $\langle \cdot, \cdot \rangle$, we can write

$$f = \sum_{S \subseteq [n]} \hat{f}_S \chi_S,$$

where the coefficients \hat{f}_S are

$$\hat{f}_S := \langle f, \chi_S \rangle = \mathbb{E}_{\mathbf{x} \in \mathbb{F}_2^n} [f(\mathbf{x}) \chi_S(\mathbf{x})] = 2^{-n},$$

since $\mathbf{x} = \mathbf{0}$ is the only element that contributes to the expectation. Thus,

$$Q^t f = Q^t \left(\sum_S \hat{f}_S \chi_S \right) = Q^t \left(\sum_S 2^{-n} \chi_S \right) = 2^{-n} \sum_S \lambda_S^t \chi_S.$$

For $\mathbf{y} \in \mathbb{F}_2^n$, let us bound the probability $\Pr[\mathbf{X} = \mathbf{y}]$: With the above equation, we get

$$\begin{aligned} (Q^t f)(\mathbf{y}) &= 2^{-n} \sum_S \lambda_S^t \chi_S(\mathbf{y}) = 2^{-n} \sum_S \left(1 - \frac{2|S|}{n}\right)^t \chi_S(\mathbf{y}) \\ &\leq 2^{-n} \sum_{s=0}^n \binom{n}{s} \left|1 - \frac{2s}{n}\right|^t \quad (\text{since } |\chi_S(\mathbf{y})| = 1) \\ &\leq 2^{-n} \sum_{s=0}^{\lfloor n/2 \rfloor} \binom{n}{s} \left|1 - \frac{2s}{n}\right|^t + 2^{-n} \sum_{s=\lceil n/2 \rceil}^n \binom{n}{s} \left|1 - \frac{2s}{n}\right|^t \\ &= 2^{-n} \sum_{s=0}^{\lfloor n/2 \rfloor} \binom{n}{s} \left|1 - \frac{2s}{n}\right|^t + 2^{-n} \sum_{r=0}^{\lfloor n/2 \rfloor} \binom{n}{n-r} \left|1 - \frac{2(n-r)}{n}\right|^t \\ &= 2 \cdot 2^{-n} \sum_{s=0}^{\lfloor n/2 \rfloor} \binom{n}{s} \left(1 - \frac{2s}{n}\right)^t \\ &\leq 2 \cdot 2^{-n} \sum_{s=0}^{\lfloor n/2 \rfloor} \binom{n}{s} \left(1 - \frac{2}{n}\right)^{st} \leq 2 \cdot 2^{-n} \sum_{s=0}^n \binom{n}{s} \left(1 - \frac{2}{n}\right)^{st} \\ &= 2 \left(\frac{1 + \left(1 - \frac{2}{n}\right)^t}{2} \right)^n. \end{aligned}$$

This proves the lemma for $U = [n]$. In general, however, we are interested in the distribution of \mathbf{X}_U , i.e., \mathbf{X} projected to the coordinates in U .

► **Observation 27.** Perform a “lazy” random walk on $\{0, 1\}^{|U|}$ as follows: Start at $\mathbf{0}$. At each step, take each edge with probability $1/n$. With the remaining probability $1 - |U|/n$, don’t move in this step. Then the end point of this walk after t steps has distribution \mathbf{X}_U .

Let Q be transition matrix of the random walk on $\{0, 1\}^{|U|}$. Then

$$\tilde{Q} := \frac{|U|}{n} Q + \frac{n - |U|}{n} I$$

is the transition matrix of the lazy random walk described above. For each $S \subseteq U$, χ_S is an eigenvector of Q , and the corresponding eigenvalue is $\lambda_S = 1 - \frac{2|S|}{|U|}$. The matrix \tilde{Q} has the same eigenvectors as Q , and its eigenvalues are

$$\tilde{\lambda}_S = \frac{|U|}{n} \lambda_S + \frac{n - |U|}{n} \cdot 1 = 1 - \frac{2|S|}{n}.$$

Let $f : \{0, 1\}^{|U|} \rightarrow \mathbb{R}$ be the function that is 1 at $\mathbf{0}$ and 0 elsewhere. We write $f = \sum_{S \subseteq U} \hat{f}_S \chi_S$. Let $u := |U|$. By the same calculation as above, $\hat{f}_S = 2^{-u}$. Thus, for $\mathbf{y} \in \{0, 1\}^u$ we get

$$\begin{aligned} (\tilde{Q}^t f)(\mathbf{y}) &= 2^{-u} \sum_S \lambda_S^t \chi_S(\mathbf{y}) = 2^{-u} \sum_S \left(1 - \frac{2|S|}{n}\right)^t \chi_S(\mathbf{y}) \\ &\leq 2^{-u} \sum_{s=0}^u \binom{u}{s} \left|1 - \frac{2s}{n}\right|^t \quad (\text{since } |\chi_S(\mathbf{y})| = 1). \end{aligned}$$

If $u \leq n/2$, we observe that all eigenvalues $1 - 2s/n$ are non-negative.² In this case we continue:

$$2^{-u} \sum_{s=0}^u \binom{u}{s} \left(1 - \frac{2s}{n}\right)^t \leq 2^{-u} \sum_{s=0}^u \binom{u}{s} \left(1 - \frac{2}{n}\right)^{st} = \left(\frac{1 + (1 - \frac{2}{n})^t}{2}\right)^u \quad (1)$$

and we are done. If $u > n/2$, things get more tricky. We split the sum in two parts:

$$2^{-u} \sum_{s=0}^{\lfloor n/2 \rfloor} \binom{u}{s} \left(1 - \frac{2s}{n}\right)^t + 2^{-u} \sum_{s=\lfloor n/2 \rfloor + 1}^u \binom{u}{s} \left(\frac{2s}{n} - 1\right)^t. \quad (2)$$

We can bound the first sum exactly similar as in (1):

$$\begin{aligned} 2^{-u} \sum_{s=0}^{\lfloor n/2 \rfloor} \binom{u}{s} \left(1 - \frac{2s}{n}\right)^t &\leq 2^{-u} \sum_{s=0}^{\lfloor n/2 \rfloor} \binom{u}{s} \left(1 - \frac{2}{n}\right)^{st} \leq 2^{-u} \sum_{s=0}^u \binom{u}{s} \left(1 - \frac{2}{n}\right)^{st} \\ &= \left(\frac{1 + (1 - \frac{2}{n})^t}{2}\right)^u. \end{aligned}$$

Let us bound the second sum in (2). For notational convenience, we let it run from $\lceil n/2 \rceil$ to u , only making it larger. We change the parameter s to $r := u - s$. Thus

$$\begin{aligned} 2^{-u} \sum_{s=\lceil n/2 \rceil}^u \binom{u}{s} \left(\frac{2s}{n} - 1\right)^t &= 2^{-u} \sum_{r=0}^{u-\lceil n/2 \rceil} \binom{u}{u-r} \left(\frac{2(u-r)}{n} - 1\right)^t \\ &= 2^{-u} \sum_{r=0}^{u-\lceil n/2 \rceil} \binom{u}{r} \left(\frac{2u-n}{n} - \frac{2r}{n}\right)^t \\ &\leq 2^{-u} \sum_{r=0}^{u-\lceil n/2 \rceil} \binom{u}{r} \left(1 - \frac{2r}{n}\right)^t \quad (\text{since } u \leq n) \\ &\leq 2^{-u} \sum_{r=0}^{u-\lceil n/2 \rceil} \binom{u}{r} \left(1 - \frac{2}{n}\right)^{rt} \\ &\leq 2^{-u} \sum_{r=0}^u \binom{u}{r} \left(1 - \frac{2}{n}\right)^{rt} = \left(\frac{1 + (1 - \frac{2}{n})^t}{2}\right)^u. \end{aligned}$$

Thus, both sums in (2) are bounded by (1) and the lemma follows.

² The reader might observe that in our application indeed $|U| \ll n/2$.

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