Maximum Matching in Two, Three, and a Few More Passes over Graph Stream

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Abstract

We consider the maximum matching problem in the semi-streaming model formalized by Feigenbaum, Kannan, McGregor, Suri, and Zhang [13] that is inspired by giant graphs of today. As our main result, we give a two-pass $(1/2 + 1/16)$-approximation algorithm for triangle-free graphs and a two-pass $(1/2 + 1/32)$-approximation algorithm for general graphs; these improve the approximation ratios of $1/2 + 1/52$ for bipartite graphs and $1/2 + 1/140$ for general graphs by Konrad, Magniez, and Mathieu [20]. In three passes, we are able to achieve approximation ratios of $1/2 + 1/10$ for triangle-free graphs and $1/2 + 1/19.753$ for general graphs. We also give a multi-pass algorithm where we bound the number of passes *precisely* – we give a $(2/3 - \varepsilon)$-approximation algorithm that uses $2/(3\varepsilon)$ passes for triangle-free graphs and $4/(3\varepsilon)$ passes for general graphs. Our algorithms are simple and combinatorial, use $O(n \log n)$ space, and (can be implemented to) have $O(1)$ update time per edge.

For general graphs, our multi-pass algorithm improves the best known *deterministic* algorithms in terms of the number of passes:

- Ahn and Guha [1] give a $(2/3 - \varepsilon)$-approximation algorithm that uses $O(\log(1/\varepsilon)/\varepsilon^2)$ passes, whereas our $(2/3 - \varepsilon)$-approximation algorithm uses $4/(3\varepsilon)$ passes;
- they also give a $(1-\varepsilon)$-approximation algorithm that uses $O(\log n \cdot \mathrm{poly}(1/\varepsilon))$ passes, where n is the number of vertices of the input graph; although our algorithm is $(2/3-\varepsilon)$-approximation, our number of passes do not depend on n.

Earlier multi-pass algorithms either have a large constant inside big-O notation for the number of passes [9] or the constant cannot be determined due to the involved analysis [22, 1], so our multi-pass algorithm should use much fewer passes for approximation ratios bounded slightly below 2/3.

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1 Introduction

Maximum matching is a well-studied problem in a variety of computational models. We consider it in the semi-streaming model formalized by Feigenbaum, Kannan, McGregor, Suri, and Zhang [13] that is inspired by generation of ginormous graphs in recent times. A graph stream is an (adversarial) sequence of the edges of a graph, and a semi-streaming algorithm must access the edges in the given order and use $O(n \,\mathrm{polylog}\, n)$ space only, where n is the number of vertices; note that a matching can have size $\Omega(n)$, so $\Omega(n \log n)$ space is necessary. The number of times an algorithm goes over a stream of edges is called the number of *passes*. A trivial (1/2)-approximation algorithm that can be easily implemented as a one-pass

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semi-streaming algorithm is to output a maximal matching. Since the formalization of the semi-streaming model more than a decade ago, the problem of finding a better than (1/2)-approximation algorithm or proving that one cannot do better has baffled researchers [21]. In a step towards resolving this, Goel, Kapralov, and Khanna [14] proved that for any $\varepsilon > 0$, a one-pass semi-streaming $(2/3 + \varepsilon)$-approximation algorithm does not exist; Kapralov [16], building on those techniques, showed non-existence of one-pass semi-streaming $(1 - 1/e + \varepsilon)$-approximation algorithms for any $\varepsilon > 0$. A natural next question is: Can we do better in, say, two passes or three passes? In answering that, Konrad, Magniez, and Mathieu [20] gave three-pass and two-pass algorithms that output matchings that are better than (1/2)-approximate. In this work, we give algorithms that improve their approximation ratios for two-pass and three-pass algorithms. We also give a multi-pass algorithm that does better than the best known multi-pass algorithms for at least initial few passes. We are able to bound the number of passes precisely: we give a $(2/3 - \varepsilon)$-approximation algorithm that uses $2/(3\varepsilon)$ passes for triangle-free graphs and $4/(3\varepsilon)$ passes for general graphs. Earlier works either have a large constant inside the big-O notation for the number of passes [9] or the constant cannot be determined due to the involved analysis [22, 1]. For example, the $(1 - \varepsilon)$-approximation algorithm by Eggert et al. [9] potentially uses $288/\varepsilon^5$ passes, and for the $(1-\varepsilon)$-approximation algorithms by McGregor [22] and Ahn and Guha [1], the constants inside the big-O bound cannot be determined due to the involved analysis. The $(2/3 - \varepsilon)$-approximation algorithm by Feigenbaum et al. [13] uses $O(\log(1/\varepsilon)/\varepsilon)$ passes, which is $O(\log(1/\varepsilon))$ factor larger than the number of passes we use to get the same approximation ratio. Our algorithms are simple and combinatorial, use $O(n \log n)$ space, and (can be implemented to) have $O(1)$ update time per edge. We also give an explicit and tight analysis of the three-pass algorithm by Konrad et al. [20] that is reminiscent of Feigenbaum et al.'s [13] multi-pass algorithm.

Technical overview

If we can find a matching M such that there are no augmenting paths of length 3 in $M \cup M^*$, where M^* is a maximum matching, then M is (2/3)-approximate, i.e., $(1/2 + 1/6)$-approximate. This is because, in each connected component of $M \cup M^*$, the ratio of M-edges to M^*-edges is at least 2/3. This is the basis for the $(2/3 - \varepsilon)$-approximation algorithm by Feigenbaum et al. [13] that uses $O(\log(1/\varepsilon)/\varepsilon)$ passes. The same idea is used by Konrad et al. [20] in the analysis of their two-pass algorithms. In the first pass, they find a maximal matching M_0 and some subset of support edges, say S. If M_0 is so bad that $M_0 \cup M^*$ is almost entirely made up of augmenting paths of length 3 (i.e., $|M_0| \approx |M^*|/2$), then by the end of the second pass, they manage to augment (using length-3 augmentations) a constant fraction of M_0 using S and a fresh access to the edges, resulting in a better than (1/2)-approximation. On the other hand, if M_0 is not so bad, then they already have a good matching. One limitation this idea faces is that a fraction of the edges in S may become useless for an augmentation if both its endpoints get matched in M_0 by the end of the first pass. Our main result is a two-pass algorithm (described in Section 5) that differs in two ways from the former approach. Firstly, in the first pass, we only find a maximal matching M_0 so that in the second pass, where we maintain a set S of support edges, S would not contain "useless" edges. Secondly, any augmentation in our algorithm happens immediately when an edge arrives if it forms an augmenting path of length 3 with edges in M_0 and S.

Our results

In light of the discussion so far, one way to evaluate an algorithm is how much advantage it gains over the (1/2)-approximate maximal matching found in the first pass. We summarize

Table 1 Advantages over a maximal matching – advantage α means $(1/2+\alpha)$-approximation.

Problem	Previous work	Advantage	Advantage in this work
Bipartite two-pass	Esfandiari et al. [11]	1/12	Not considered separately
Bipartite three-pass	Esfandiari et al. [11]	1/9.52	
Triangle-free two-pass	Not considered separately		1/16 (in Section 5)
Triangle-free three-pass			1/10 (in Appendix A)
General two-pass	Konrad et al. [20]	1/140	1/32 (in Section 5)
General three-pass	Not considered separately		1/19.753 (in Appendix B)

Table 2 Multi-pass algorithms – see Section 6.

Graph	Results	Approx	# Passes
Bipartite	Feigenbaum et al. [13]	$2/3-\varepsilon$	$O(\log(1/\varepsilon)/\varepsilon)$
	Eggert et al.[9]	$1-\varepsilon$	$288/\varepsilon^5$
	Ahn and Guha [1]	$1-\varepsilon$	$O(\log\log(1/\varepsilon)/\varepsilon^2)$
Triangle free	This work (in Section 6)	$2/3-\varepsilon$	$2/(3\varepsilon)$
General	McGregor [22] *randomized*	$1-\varepsilon$	$O((1/\varepsilon)^{1/\varepsilon})$
	Ahn and Guha [1]	$2/3-\varepsilon$	$O(\log(1/\varepsilon)/\varepsilon^2)$
	Ahn and Guha [1]	$1-\varepsilon$	$O(\log n\cdot \mathrm{poly}(1/\varepsilon))$
	This work (in Section 6)	$2/3-\varepsilon$	$4/(3\varepsilon)$

our two-pass and three-pass results in Table 1 and multi-pass results in Table 2. We stress that we are able to bound the number of passes *precisely*, without big-O notation. For general graphs, our multi-pass algorithm improves the best known *deterministic* algorithms in terms of number of passes – see the third multi-row of Table 2. We note that our multi-pass algorithm is not just a repetition of the second pass of our two-pass algorithm. Such a repetition will give an asymptotically worse number of passes (see, for example, the multi-pass algorithm due to Feigenbaum et al. [13]; the first row of Table 2). We carefully choose the parameters for each pass to get the required number of passes. Also note that Table 1 shows *advantages* over a maximal matching – an algorithm is said to have advantage α if it is a $(1/2+\alpha)$-approximation algorithm (because a maximal matching is $(1/2)$-approximate).

Note of independent work

The work of Esfandiari et al. [11] who claim better approximation ratios for *bipartite graphs* in two passes and three passes is independent and almost concurrent. Our work differs in several aspects. We consider triangle-free graphs (superset of bipartite graphs) and general graphs, and we additionally consider multi-pass algorithms. Also, their algorithm has a post-processing step that uses time $O(\sqrt{n}\cdot|E|)$, whereas our algorithms can be implemented to have $O(1)$ update time per edge. One further detail about this appears in Appendix D.

1.1 Related Work

Karp, Vazirani, and Vazirani [18] gave the celebrated $(1-1/e)$-competitive randomized online algorithm for bipartite graphs in the vertex arrival setting. Goel et al. [14] gave the first one-pass deterministic algorithm with the same approximation ratio, i.e., $1-1/e$, in the semi-streaming model in the vertex arrival setting. For the rest of this section, results involving ε hold for any $\varepsilon>0$. As mentioned earlier, Goel, Kapralov, and Khanna [14] proved nonexistence of one-pass $(2/3+\varepsilon)$-approximation semi-streaming algorithms, which was extended to $(1-1/e+\varepsilon)$-approximation algorithms by Kapralov [16]. On the algorithms side, nothing better than outputting a maximal matching, which is (1/2)-approximate, is known. Closing this gap is considered an outstanding open problem in the streaming community [21].

On the multi-pass front, in the semi-streaming model, Feigenbaum et al. [13] gave a $(2/3-\varepsilon)$-approximation algorithm for bipartite graphs that uses $O(\log(1/\varepsilon)/\varepsilon)$ passes; McGregor [22] improved it to give a $(1-\varepsilon)$-approximation algorithm for general graphs that uses $O((1/\varepsilon)^{1/\varepsilon})$ passes. For bipartite graphs, this was again improved by Eggert et al. [9] who gave a $(1-\varepsilon)$-approximation $O((1/\varepsilon)^5)$-pass algorithm. Ahn and Guha [1] gave a linear-programming based $(1-\varepsilon)$-approximation $O(\log\log(1/\varepsilon)/\varepsilon^2)$-pass algorithm for bipartite graphs. For general graphs, their $(1-\varepsilon)$-approximation algorithm uses number of passes proportional to $\log n$, so it is worse than that of McGregor [22].

For the problem of one-pass weighted matching, there is a line of work starting with Feigenbaum et al. [13] giving a 6-approximation semi-streaming algorithm. Subsequent results improved this approximation ratio: see McGregor [22], Zelke [24], Epstein et al. [10], Crouch and Stubbs [8], Grigorescu et al.[15], and most recently in a breakthrough, giving a $(2+\varepsilon)$-approximation semi-streaming algorithm, Paz and Schwartzman [23]. The multi-pass version of the problem was considered first by McGregor [22], then by Ahn and Guha [1]. Chakrabarti and Kale [5] and Chekuri et al. [6] consider a more general version of the matching problem where a submodular function is defined on the edges of the input graph.

The problem of estimating the *size* of a maximum matching (instead of outputting the actual matching) has also been considered. We mention Kapralov et al. [17], Esfandiari et al. [12], Bury and Schwiegelshohn [4], and Assadi et al. [2].

In the dynamic streams, edges of the input graph can be removed as well. The works of Konrad [19], Assadi et al. [3], and Chitnis et al. [7] consider the maximum matching problem in dynamic streams.

1.2 Organization of the Paper

After setting up notation in Section 2, we give a tight analysis of the three-pass algorithm for bipartite graphs by Konrad et al. [20] in Section 3. In Section 4, we see our simple two-pass algorithm for triangle-free graphs. Then in Section 5, we see our main result – the improved two-pass algorithm, and then we see the multi-pass algorithm in Section 6. The results that are not covered in the main sections are covered in the appendix.

2 Preliminaries

We work on graph *streams*. The input is a sequence of edges (stream) of a graph $G=(V,E)$, where V is the set of vertices and E is the set of edges; a bipartite graph is denoted as $G=(A,B,E)$. A streaming algorithm may go over the stream a few times (multi-pass) and use space $O(n \operatorname{polylog} n)$, where $n=|V|$. In this paper, we give algorithms that make two, three, or a few more passes over the input graph stream. A matching M is a subset of edges such that each vertex has at most one edge in M incident to it. The maximum cardinality matching problem, or maximum matching, for short, is to find a largest matching in the given graph. Our goal is to design streaming algorithms for maximum matching.

For a subset F of edges and a subset U of vertices, we denote by $U(F)\subseteq U$ the set of vertices in U that have an edge in F incident on them. Conversely, we denote by $F(U)\subseteq F$ the set of edges in F that have an endpoint in U. For a subset F of edges and a vertex

$v \in V(F)$, we denote by $N_F(v)$ the set of v's neighbors in the graph $(V(F), F)$, and we define $\deg_F(v) := |N_F(v)|$.

In the first pass, our algorithms compute a *maximal* matching which we denote by M_0. We use M^* to indicate a matching of maximum cardinality. Assume that M_0 and M^* are given. For $i \in \{3, 5, 7, \ldots\}$, a connected component of $M_0 \cup M^*$ that is a path of length i is called an i-augmenting path (nonaugmenting otherwise). We say that an edge in M_0 is 3-augmentable if it belongs to a 3-augmenting path, otherwise we say that it is non-3-augmentable.

▶ **Lemma 1** (Lemma 1 in [20]). *Let $\alpha \geqslant 0$, M_0 be a maximal matching in G, and M^* be a maximum matching in G such that $|M_0| \leqslant (1/2+\alpha)|M^*|$. Then the number of 3-augmentable edges in M_0 is at least $(1/2 - 3\alpha)|M^*|$, and the number of non-3-augmentable edges in M_0 is at most $4\alpha|M^*|$.*

Proof. Let the number of 3-augmentable edges in M_0 be k. For each 3-augmentable edge in M_0, there are two edges in M^* incident on it. Also, each non-3-augmentable edge in M_0 lies in a connected component of $M_0 \cup M^*$ in which the ratio of the number of M^*-edges to the number of M_0-edges is at most $3/2$. Hence,

$$\begin{aligned} |M^*| &\leqslant 2k + \frac{3}{2}(|M_0| - k) && \text{since \# non-3-augmentable edges} = |M_0| - k\,, \\ &\leqslant 2k + \frac{3}{2}\left(\left(\frac{1}{2} + \alpha\right)|M^*| - k\right) && \text{because } |M_0| \leqslant (1/2+\alpha)|M^*|\,, \\ &= \frac{1}{2}k + \left(\frac{3}{4} + \frac{3}{2}\alpha\right)|M^*|\,, \end{aligned}$$

which, after simplification, gives $k \geqslant (1/2 - 3\alpha)|M^*|$. And the number of non-3-augmentable edges in M_0 is $|M_0| - k \leqslant |M_0| - (1/2-3\alpha)|M^*| \leqslant (1/2+\alpha-1/2+3\alpha)|M^*| = 4\alpha|M^*|$. ◀

We make the following simple, yet crucial, observation.

▶ **Observation 2.** *Let M_0 be a maximal matching. Then $V(M_0)$ is a vertex cover, and there is no edge between any two vertices in $V \setminus V(M_0)$. Therefore, even if the input graph is not a bipartite graph, the set of edges incident on $V \setminus V(M_0)$, i.e., $E(V \setminus V(M_0))$ give rise to a bipartite graph with bipartition $(V \setminus V(M_0), V(M_0))$.*

For all the algorithms in this paper, it can be verified that their space complexity is $O(n \log n)$ and update time per edge is $O(1)$. We also ignore floors and ceilings for the sake of exposition.

3 Analyzing the Three Pass Algorithm for Bipartite Graphs

We analyze the three-pass algorithm for bipartite graphs given by Konrad et al. [20], i.e., Algorithm 1 by considering the distribution of lengths of augmenting paths. We also give a tight example.

▶ **Theorem 3.** *Algorithm 1 is a three-pass, semi-streaming, $(1/2 + 1/10)$-approximation algorithm for maximum matching in bipartite graphs.*

Proof. Without loss of generality, let M^* be a maximum matching such that all nonaugmenting connected components of $M_0 \cup M^*$ are single edges. For $i = \{3, 5, 7, \ldots\}$, let k_i denote the number of i-augmenting paths in $M_0 \cup M^*$, and let $k = |M_0 \cap M^*|$. Then

$$|M_0| = k + \sum_i \frac{i-1}{2}k_i \quad \text{and} \quad |M^*| = k + \sum_i \frac{i+1}{2}k_i\,. \tag{1}$$

Algorithm 1 Three-pass algorithm for bipartite graphs due to Konrad et al. [20]

1: In the first pass, find a maximal matching M_0.
2: In the second pass, find a maximal matching
 - M_A in $F_2 := \{ab : a \in A(M_0), b \in B \setminus B(M_0)\}$ (see Figure 1).
3: In the third pass, find a maximal matching
 - M_B in $F_3 := \{ab : a \in A \setminus A(M_0) \text{ and } \exists a' \in A(M_A) \text{ such that } a'b \in M_0\}$.
4: Augment M_0 using edges in M_A and M_B and return the resulting matching M.

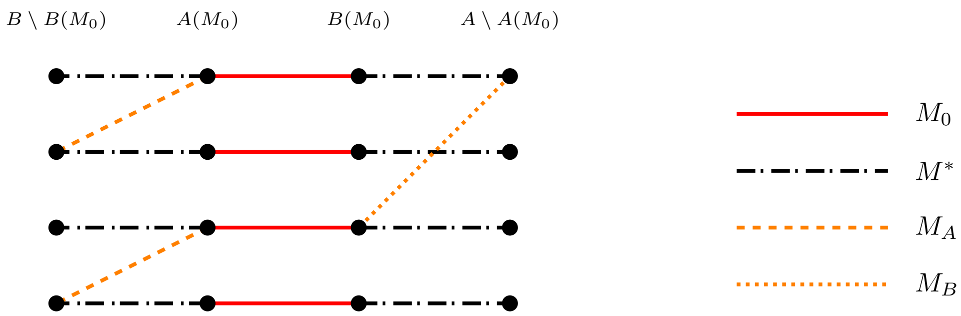

Figure 1 Example: state of variables in an execution of Algorithm 1.

Consider an i-augmenting path $b_1a_1b_2a_2b_3 \cdots b_{(i+1)/2}a_{(i+1)/2}$ in $M_0 \cup M^*$, where for each j, we have $a_j \in A$ and $b_j \in B$. We call the vertex $a_{(i-1)/2}$ a good vertex, because an edge in M_A incident to $a_{(i-1)/2}$ can potentially be augmented using the edge $b_{(i+1)/2}a_{(i+1)/2}$. To elaborate, consider the set of all edges in M_A incident on good vertices; call it M'_A. Consider the set of edges of the type $b_{(i+1)/2}a_{(i+1)/2}$ from each i-augmenting path; call it M_F. Note that M_F is a matching. Then we can augment M_0 using M'_A and M_F by as much as $|M'_A|$.

There is a matching of size $\sum_i k_i$ in F_2 formed by edges of the type b_1a_1 from each i-augmenting path. Since M_A is maximal in F_2, we have $|M_A| \geqslant (\sum_i k_i)/2$. Now, the number of good vertices is $\sum_i k_i$; therefore, the number of bad (i.e., not good) vertices is $|M_0| - \sum_i k_i$. So the number of edges in M_A incident on good vertices (see Figure 2)

$$|M'_A| \geqslant \frac{\sum_i k_i}{2} - \left(|M_0| - \sum_i k_i\right) = \frac{3}{2}\sum_i k_i - |M_0|\,.$$

Let $B_G := \{b \in B : \exists a \in A(M'_A) \text{ such that } ab \in M_0\}$. Let $M'_F \subseteq M_F$ be defined as $M'_F := \{ba \in M_F : b \in B_G\}$. Then we know that $|M'_F| = |M'_A|$ and $M'_F \subseteq M_F \subseteq F_3$. Since we select a maximal matching in F_3 in the third pass,

$$|M_B| \geqslant \frac{|M'_F|}{2} = \frac{|M'_A|}{2} = \frac{3}{4}\sum_i k_i - \frac{|M_0|}{2}\,. \tag{2}$$

So the output size

$$\begin{aligned} |M| &= |M_0| + |M_B| \\ &\geqslant |M_0| + \frac{3}{4}\sum_i k_i - \frac{|M_0|}{2} && \text{by (1) and (2),} \\ &= \frac{|M_0|}{2} + \frac{3}{4}(|M^*| - |M_0|) && \text{by (1), } \sum_i k_i = |M^*| - |M_0|\,, \end{aligned}$$

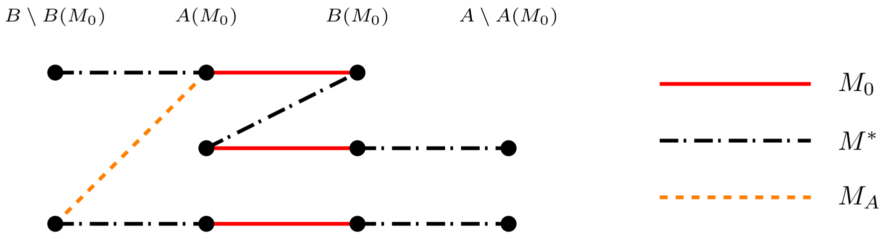

Figure 2 Tight example for Algorithm 1: M_A has only one edge that lands on a bad vertex and cannot be augmented in the third pass. So $|M| = |M_0| = 3$ and $|M^*| = 5$.

Algorithm 2 Two-pass algorithm for triangle-free graphs

In the first pass: $M_0 \leftarrow$ maximal matching
In the second pass: $S \leftarrow \textsc{Semi}(\lambda, V(M_0), V \setminus V(M_0))$ (see Figure 3).
After the second pass, augment M_0 greedily using edges in S to get M; output M.
function $\textsc{Semi}(\lambda, X, Y)$ ▷ based on Algorithm 7 in Konrad et al. [20]
    $S \leftarrow \emptyset$
    foreach edge xy such that $x \in X$, $y \in Y$ **do**
        if $\deg_S(x) = 0$ and $\deg_S(y) \leqslant \lambda - 1$ **then**
            $S \leftarrow S \cup \{xy\}$

i.e., $|M| \geqslant 3|M^*|/4 - |M_0|/4$, but we also have $|M| \geqslant |M_0|$, hence

$$|M| \geqslant \max\left\{|M_0|, \frac{3}{4}|M^*| - \frac{1}{4}|M_0|\right\} .$$

So the bound is minimized when $|M_0| = 3|M^*|/4 - |M_0|/4 = 3|M^*|/5 = (1/2+1/10)|M^*|$. ◀

As we can see in the proof above, the worst case happens when $|M| = |M_0| = 3|M^*|/5$. Setting $k_3 = k_5 \geqslant 1$, $k = 0$, and $k_i = 0$ for $i > 5$ gives us the tight example shown in Figure 2.

4 A Simple Two Pass Algorithm for Triangle Free Graphs

Before seeing our main result, we see a simple two pass algorithm for triangle-free graphs. The function Semi() in Algorithm 2 greedily computes a subset of edges such that each vertex in X has degree at most one and each vertex in Y has degree at most λ; we call such a subset a (λ, X, Y)-semi-matching (Konrad et al. [20] call this a λ-bounded semi-matching). In Algorithm 2, we find a maximal matching M_0 in the first pass, and, in the second pass, we find a $(\lambda, V(M_0), V \setminus V(M_0))$-semi-matching S. After the second pass, we greedily augment edges in M_0 one by one using edges in S.

▶ **Theorem 4.** *Algorithm 2 is a two-pass, semi-streaming,* $(1/2 + 1/20)$*-approximation algorithm for maximum matching in triangle-free graphs.*

Proof. As in the proof of Theorem 3, let M^* be a maximum matching such that all nonaugmenting connected components of $M_0 \cup M^*$ are single edges. For $i = \{3, 5, 7, \ldots\}$, let k_i denote the number of i-augmenting paths in $M_0 \cup M^*$, and let k denote the number of edges in $M^* \cap M_0$.

Consider an i-augmenting path $x_1 y_1 x_2 y_2 x_3 \cdots x_{(i+1)/2} y_{(i+1)/2}$ in $M_0 \cup M^*$. We call the vertices $y_1 \in V(M_0)$ and $x_{(i+1)/2} \in V(M_0)$ good vertices, because the edges $x_1 y_1 \in M^*$ and

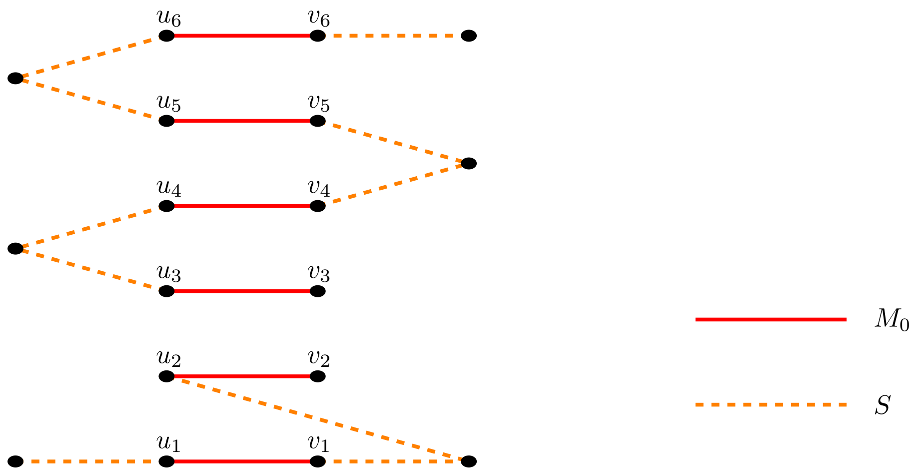

Figure 3 Example showing M_0 and S at the end of the second pass of Algorithm 2 with $\lambda = 2$. When we greedily augment M_0 after the second pass, we may choose to augment u_5v_5 and lose two possible augmentations of edges u_4v_4 and u_6v_6.

$x_{(i+1)/2}y_{(i+1)/2} \in M^*$ can potentially be added to S by our algorithm. Denote by V_G the set of good vertices and by $V_B := V(M_0) \setminus V_G$ the set of bad vertices. Then $|V_G| = 2\sum_i k_i$. Note that $V_G \cap V_B = \emptyset$ and $V_G \cup V_B = V(M_0)$ by definition.

Let $V_{\mathrm{NC}} := V_G \setminus V(S)$ be the set of good vertices *not covered* by S. An edge $uv \in M^*$ with $u \in V \setminus V(M_0)$ and $v \in V_{\mathrm{NC}}$ was not added to S, because $\deg_S(u) = \lambda$. Hence

$$\lambda |V_{\mathrm{NC}}| \leqslant |V(M_0)| - |V_{\mathrm{NC}}| \quad \text{i.e.,} \quad |V_{\mathrm{NC}}| \leqslant \frac{2}{\lambda+1}|M_0|\,, \tag{3}$$

because at most $|V(M_0)| - |V_{\mathrm{NC}}|$ vertices in $V(M_0)$ are covered by S. Now,

$$\begin{aligned}
|V(M_0) \setminus V(S)| &= |V_G \setminus V(S)| + |V_B \setminus V(S)| && \because V_G \cap V_B = \emptyset \text{ and } V_G \cup V_B = V(M_0),\\
&\leqslant |V_{\mathrm{NC}}| + |V_B| && \because V_{\mathrm{NC}} = V_G \setminus V(S), |V_B \setminus V(S)| \leqslant |V_B|,\\
&\leqslant \frac{2}{\lambda+1}|M_0| + |V(M_0)| - |V_G| && \text{by (3) and } \because |V_B| = |V(M_0)| - |V_G|,\\
&= \frac{2}{\lambda+1}|M_0| + |V(M_0)| - 2\sum_i k_i && \text{because } |V_G| = 2\sum_i k_i\,.
\end{aligned}$$

Using $|V(M_0)| = |V(M_0) \setminus V(S)| + |V(M_0) \cap V(S)|$ and the above, we get

$$\begin{aligned}
|V(M_0) \cap V(S)| &\geqslant |V(M_0)| - \left(\frac{2}{\lambda+1}|M_0| + |V(M_0)| - 2\sum_i k_i\right)\\
&= 2\left(\sum_i k_i - \frac{1}{\lambda+1}|M_0|\right)\,.
\end{aligned} \tag{4}$$

We observe that at most $|M_0|$ vertices in $V(M_0)$ (one endpoint of each edge) can be covered by S without having both endpoints of an edge in M_0 covered. Hence, at least $|V(M_0) \cap V(S)| - |M_0|$ edges in M_0 have both their endpoints covered by S, which, by (4), is at least

$$2\left(\sum_i k_i - \frac{1}{\lambda+1}|M_0|\right) - |M_0| = 2\sum_i k_i - \frac{\lambda+3}{\lambda+1}|M_0|\,. \tag{5}$$

After the second pass, when we greedily augment an edge from the above edges, i.e., edges whose both endpoints are covered by S, we may potentially lose $2(\lambda-1)$ other augmentations (see Figure 3). To see this, consider $uv \in M_0$ such that $u, v \in V(S)$ and $au \in S$ and $vb \in S$. The graph is triangle free, so we know that $a \neq b$, and we *can* augment M_0 using the 3-augmenting path $auvb$; but we may lose at most $\lambda - 1$ edges incident to a in S and at most $\lambda - 1$ edges incident to b in S. Therefore the number of augmentations c we get after the second pass is at least $1/(2\lambda - 1)$ times the right hand side of (5), i.e.,

$$c \geqslant \frac{2}{2\lambda - 1} \sum_i k_i - \frac{\lambda + 3}{(2\lambda - 1)(\lambda + 1)} |M_0| \, .$$

So the output size $|M| = |M_0| + c$, and using the above bound on c and simplifying we get:

$$|M| \geqslant \frac{2}{2\lambda - 1} \sum_i k_i + \frac{2(\lambda^2 - 2)}{(2\lambda - 1)(\lambda + 1)} |M_0| \, ;$$

substituting $\sum_i k_i = |M^*| - |M_0|$, by (1), in the above,

$$|M| \geqslant \frac{2}{2\lambda - 1} |M^*| + \frac{2(\lambda^2 - \lambda - 3)}{(2\lambda - 1)(\lambda + 1)} |M_0| \, .$$

Using $\lambda = 3$ and the fact that M_0 is 2-approximate, we get

$$|M| \geqslant \frac{2}{5}|M^*| + \frac{3}{10}|M_0| \geqslant \frac{2}{5}|M^*| + \frac{3}{20}|M^*| = \frac{11}{20}|M^*| = \left(\frac{1}{2} + \frac{1}{20}\right)|M^*| \, . \qquad ◀$$

5 Improved Two Pass Algorithm

We present our main result that is a two pass algorithm in this section. In the first pass, we find a maximal matching M_0. In the second pass, we maintain a set S of support edges xy, such that $x \in V \setminus V(M_0)$, $y \in V(M_0)$, and $\deg_S(y) \leqslant \lambda_{\mathrm{M}}$ and $\deg_S(x) \leqslant \lambda_{\mathrm{U}}$, where $\lambda_{\mathrm{M}} \geqslant 1$ and $\lambda_{\mathrm{U}} \geqslant 1$ are parameters denoting maximum degree allowed in S for matched and unmatched vertices (with respect to M_0), respectively. Whenever a new edge forms a 3-augmenting path with an edge in M_0 and an edge in S, we augment. We store the vertices involved in a 3-augmentation in the variable I. We ignore a new edge if it is incident to a vertex in I. Unused support edges that are incident to a vertex in I become "useless"; hence to address this, we store the endpoints of M_0 edges that share an endpoint with such useless edges in the variable I_B, and we ignore a new edge if it is incident to a vertex in I_B. Algorithm 3 gives a formal description.

Setting up a charging scheme to lower bound the number of augmentations

We first lay the groundwork and give a charging scheme.

▶ **Observation 5.** *For general graphs (that are possibly not triangle-free), we need to set* $\lambda_{\mathrm{M}} \geqslant 2$.

To see why, suppose $\lambda_{\mathrm{M}} = 1$. Let uv be a 3-augmentable edge in M_0. Then, for the edge uv, we might end up storing the edges ub and vb in S, and the edge uv would not get augmented. If $\lambda_{\mathrm{M}} \geqslant 2$, and we store at least λ_{M} edges incident to u, then an edge incident to v will

Algorithm 3 Improved two-pass algorithm: input graph G

In the first pass, find a maximal matching M_0.
if G is triangle-free **then**
  Return IMPROVE-MATCHING($M_0, 2, 1$)
else
  Return IMPROVE-MATCHING($M_0, 4, 2$)
function IMPROVE-MATCHING($M_0, \lambda_{\mathrm{U}}, \lambda_{\mathrm{M}}$)
  $M \leftarrow M_0$, $S \leftarrow \emptyset$, $I \leftarrow \emptyset$ and $I_B \leftarrow \emptyset$
  foreach edge xy in the stream **do**
    if x or $y \in I \cup I_B$ **then**
      Continue, i.e., ignore xy.
    else if $x \in V(M_0)$ and $y \in V(M_0)$ **then**
      Continue, i.e., ignore xy.
    else if there exist v and b such that $yv \in M_0$ and $vb \in S$ **then**
      $M \leftarrow M \setminus \{yv\} \cup \{xy, vb\}$ ▷ a 3-augmentation
      Let $I_x \leftarrow \{u_x, v_x : xu_x \in S$ and $u_x v_x \in M_0\}$.
      Let $I_b \leftarrow \{u_b, v_b : u_b v_b \in M_0$ and $v_b b \in S\}$.
      Then $I \leftarrow I \cup \{x, y, v, b\}$ and $I_B \leftarrow I_B \cup I_x \cup I_b$.
    else
      Without loss of generality, assume that $x \in V \setminus V(M_0)$ and $y \in V(M_0)$.
      if $\deg_S(x) < \lambda_{\mathrm{U}}$ and $\deg_S(y) < \lambda_{\mathrm{M}}$ **then** ▷ See Figure 4.
        $S \leftarrow S \cup \{xy\}$ ▷ **Note:** Once an edge is added to S, it is never removed from it.
  Return M.

not form a triangle with at least one of those and uv would get augmented. So, for general graphs, we need to set $\lambda_{\mathrm{M}} \geqslant 2$.

Let $|M_0| = (1/2 + \alpha)|M^*|$. For a 3-augmentable edge $uv \in M_0$, let $auvb$ be the 3-augmenting path such that $au, vb \in M^*$. Without loss of generality, assume that au arrived before vb. Then we make the following observation.

▶ **Observation 6.** *When au arrived, it may not be added to S for one of the following reasons:*

- *The vertex a was already matched.*
- *There were λ_{M} edges incident to u in S.*
- *There were λ_{U} edges incident to a in S.*

We call some edges in M_0 *good*, some *partially* good, and some *bad*. An edge is good if it got augmented. An edge $uv \in M_0$ is bad if it is 3-augmentable, not good, and vertex a or b had λ_{U} edges incident to them in S when edge au or vb arrived. An edge $uv \in M_0$ is partially good if it is 3-augmentable, but neither good nor bad ("partially" good because, as we will see later, we can hold some good edge $u'v' \in M_0$ responsible for uv not getting augmented). Note that all 3-augmentable edges get some label according to our labeling. We require the following lemma to describe the charging scheme.

▶ **Lemma 7.** *Suppose au was not added to S because there were already λ_{M} edges incident to u in S. If, later, uv did not get augmented when vb arrived, then*

- *b was already matched via augmenting path $a''u''v''b$, or*
- *there exists $a'u \in S$ and $u'v' \in M_0$ such that a' was matched via augmenting path $a'u'v'b'$.*

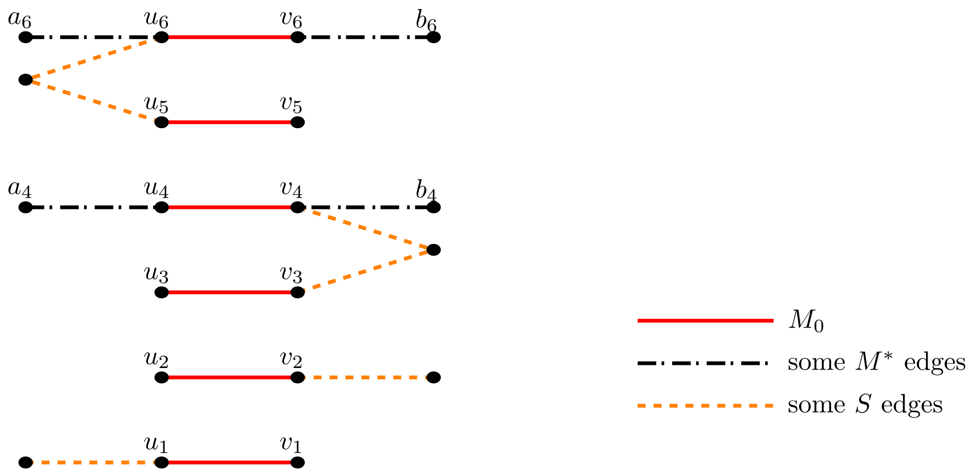

Figure 4 Example showing M_0 and some of the edges in M^* and S during the second pass of Algorithm 3 for triangle-free graphs with $\lambda_U = 2$ and $\lambda_M = 1$. At most one of u_i and v_i can have positive degree in S, because we would rather augment $u_i v_i$ instead of adding the latter edge to S. By our convention, $a_4 u_4$ arrived before $v_4 b_4$, and $a_6 u_6$ arrived before $v_6 b_6$. Since $a_4 u_4$ was not added to S, we have $\deg_S(a_4) = \lambda_U$ (S edges incident to a_4 are not shown).

Proof. When au arrived, $|N_S(u)| \geqslant \lambda_M$. If b was unmatched when vb arrived, then some $a' \in N_S(u) \setminus \{b\}$ must have been matched, otherwise we would have augmented uv. Now for triangle-free graphs $b \notin N_S(u)$, so $|N_S(u) \setminus \{b\}| = |N_S(u)| \geqslant 1$, and for general graphs, by Observation 5, $\lambda_M \geqslant 2$, so $|N_S(u) \setminus \{b\}| \geqslant \lambda_M - 1 \geqslant 1$. ◀

Charging Scheme

As alluded to earlier, we charge a partially good edge to some good edge. Recall that for a 3-augmentable edge $uv \in M_0$, we denote by $au, vb \in M^*$ the edges that form the 3-augmenting path with uv such that au arrived before vb. We use Observation 6 and consider the following cases. See Figure 5.

- Suppose au was not added to S because a was already matched. Then, let $u'v' \in M_0$ was augmented using $au'v'b'$. If $\deg_S(a) \leqslant \lambda_U - 1$, then we charge uv to $u'v'$. Otherwise, uv is bad.
- Suppose au was not added to S because $\deg_S(u) = \lambda_M$. Then we use Lemma 7. We either charge uv to $u'v'$, or if $\deg_S(b) \leqslant \lambda_U - 1$, then we charge uv to $u''v''$. Otherwise, uv is bad.
- Suppose au was not added to S because $\deg_S(a) = \lambda_U$, then uv is bad.
- Otherwise, au was added to S, but uv did not get augmented when vb arrived. Then:
  - Either there exists $a' \in N_S(u)$ that was matched via augmenting path $a'u'v'b'$ (note that a' may be same as a), then we charge uv to $u'v'$;
  - or b was already matched via augmenting path $a''u''v''b$, and vb was ignored; in this case, if $\deg_S(b) \leqslant \lambda_U - 1$, then we charge uv to $u''v''$, otherwise, uv is bad.

We now bound the number of bad edges in M_0 from above.

▶ **Lemma 8.** *The number of bad edges is at most $\lambda_M |M_0| / \lambda_U$.*

Proof. We claim that for any $uv \in M_0$, $\deg_S(u) + \deg_S(v) \leqslant \lambda_M$, hence $|S| \leqslant \lambda_M |M_0|$. A short argument is that the $(\lambda_M + 1)$th edge would cause an augmentation and will not

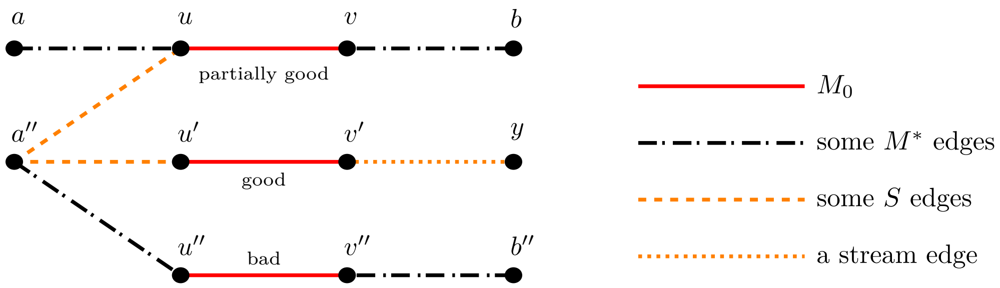

Figure 5 Example showing a good edge, a bad edge, and a partially good edge. We use parameters $\lambda_{\mathrm{U}} = 2$ and $\lambda_{\mathrm{M}} = 1$, so we are in the triangle-free case. The edge $u'v'$ is not 3-augmentable but was augmented using $a''u'v'y$, so $u'v'$ is a good edge. The edge $u''v''$ is a 3-augmentable edge that was not augmented and when $a''u''$ arrived, $\deg_S(a'') = 2$, so $u''v''$ is a bad edge. For uv, we did not take au in S, because $\deg_S(u) = 1$, so uv is a partially good edge, and we can charge uv to $u'v'$ using Lemma 7.

be added to S. Let us assume the claim. By the definition of a bad edge, λ_{U} edges in S are "responsible" for one bad edge in M_0. Also, an edge au' (or $v''b$, resp.) in S can be responsible for at most one bad edge that can only be uv if $au \notin S$ (or if $vb \notin S$, resp.; considering the 3-augmenting path $auvb$). Hence, the total number of bad edges is at most $|S|/\lambda_{\mathrm{U}} \leqslant \lambda_{\mathrm{M}}|M_0|/\lambda_{\mathrm{U}}$. Now we prove the claim.

We first prove for triangle-free graphs by contradiction. Let $\deg_S(u) + \deg_S(v) > \lambda_{\mathrm{M}}$, and let $vy \in S$ be the $(\lambda_{\mathrm{M}} + 1)$th edge incident to one of u and v that was added to S. Since $\lambda_{\mathrm{M}} \geqslant 1$ and $\deg_S(v) \leqslant \lambda_{\mathrm{M}}$, we have $\deg_S(u) \geqslant 1$, i.e. $N_S(u) \neq \emptyset$. Now when vy arrived:

- the vertex y was unmatched, otherwise vy would not be added to S;
- no vertex $x \in N_S(u)$ was matched, otherwise $u, v \in I_B$, and vy would not be added to S.

The above implies that when vy arrived, due to some $x \in N_S(u)$ the if condition on Line 14 became true, and we augmented uv via $xuvy$ instead of adding vy to S. This is a contradiction.

For general graphs, we argue by contradiction slightly informally for the sake of brevity. By Observation 5, for general graphs, $\lambda_{\mathrm{M}} \geqslant 2$. Let $\deg_S(u) + \deg_S(v) > \lambda_{\mathrm{M}} \geqslant 2$. Let vy be the second edge incident to one of u and v that was added to S; the first edge can be xu or vy'.

Suppose xu was the first edge. If $x \neq y$, then we would have augmented uv via $xuvy$ instead of adding vy to S – a contradiction. If $x = y$, then after vy was processed, $N_S(u) = N_S(v) = \{y\}$, and a third edge incident to one of u and v would not be added to S, because it would have formed a 3-augmenting path with either yu or vy, resulting in a contradiction that $\deg_S(u) + \deg_S(v) = 2$.

Otherwise, suppose vy' was the first edge; then $N_S(v) = \{y, y'\}$ after vy was processed. Since eventually $\deg_S(u) + \deg_S(v) \geqslant \lambda_{\mathrm{M}} + 1 \geqslant 3$ and $\deg_S(u), \deg_S(v) \leqslant \lambda_{\mathrm{M}}$, we would eventually have $\deg_S(u) \geqslant 1$, so let $xu \in S$. When xu arrived, it would have formed an 3-augmenting path with either vy or vy' (here, taking care of the fact that one of y and y' can be same as x), resulting in a contradiction that xu was not added to S.

Thus, we get the claim and complete the proof. ◀

As a consequence, we get the following.

▶ **Observation 9.** *In any call to* Improve-Matching(), *we need to set* $\lambda_{\mathrm{U}} > \lambda_{\mathrm{M}}$, *i.e.,* $\lambda_{\mathrm{U}} \geqslant 2$.

To see why, suppose $\lambda_{\mathrm{U}} \leqslant \lambda_{\mathrm{M}}$. Then by Lemma 8, potentially all 3-augmentable edges in M_0 could become bad edges.

Recall that a 3-augmentable edge is good, partially good, or bad; so by Lemmas 1 and 8,

$$\begin{aligned}\#\text{ good or partially good edges} &\geqslant \left(\frac{1}{2} - 3\alpha\right)|M^*| - \frac{\lambda_{\mathrm{M}}|M_0|}{\lambda_{\mathrm{U}}} \\ &= \left(\frac{1}{2} - 3\alpha\right)|M^*| - \frac{\lambda_{\mathrm{M}}}{\lambda_{\mathrm{U}}}\left(\frac{1}{2} + \alpha\right)|M^*| \\ &= \left(\frac{\lambda_{\mathrm{U}} - \lambda_{\mathrm{M}}}{2\lambda_{\mathrm{U}}} - \left(\frac{3\lambda_{\mathrm{U}} + \lambda_{\mathrm{M}}}{\lambda_{\mathrm{U}}}\right)\alpha\right)|M^*|\,. \end{aligned} \tag{6}$$

In the following lemma, we bound the number of partially good edges in M_0 that are charged to one good edge.

▶ **Lemma 10.** *At most $2\lambda_{\mathrm{U}} - 1$ partially good edges in M_0 are charged to one good edge in M_0.*

Proof. Suppose $uv \in M_0$ was augmented by edges xu and vy such that xu arrived before vy, then $xu \in S$. Now $|N_S(x)|, |N_S(y)| \leqslant \lambda_{\mathrm{U}}$. Since $xu \in S$, we have $|N_S(x) \setminus \{u\}| \leqslant \lambda_{\mathrm{U}} - 1$. Let $B := (N_S(x) \setminus \{u\}) \cup N_S(y)$, then $|B| \leqslant 2\lambda_{\mathrm{U}} - 1$. Now, the set of partially good edges that are charged to uv is a subset of $M_0(B)$. Observing that $|M_0(B)| \leqslant |B| \leqslant 2\lambda_{\mathrm{U}} - 1$ finishes the proof. ◀

The following lemma characterizes the improvement given by Improve-Matching().

▶ **Lemma 11.** *Let $|M_0| = (1/2 + \alpha)|M^*|$ and $M =$ Improve-Matching$(M_0, \lambda_{\mathrm{U}}, \lambda_{\mathrm{M}})$, then*

$$|M| \geqslant \left(\frac{1}{2} + \frac{\lambda_{\mathrm{U}} - \lambda_{\mathrm{M}}}{4\lambda_{\mathrm{U}}^2} + \left(1 - \frac{3\lambda_{\mathrm{U}} + \lambda_{\mathrm{M}}}{2\lambda_{\mathrm{U}}^2}\right)\alpha\right)|M^*| \geqslant \left(\frac{1}{2} + \frac{\lambda_{\mathrm{U}} - \lambda_{\mathrm{M}}}{4\lambda_{\mathrm{U}}^2}\right)|M^*|\,.$$

Proof. By (6) and Lemma 10, the total number of augmentations during one call to Improve-Matching() is at least

$$\frac{1}{2\lambda_{\mathrm{U}}}\left(\frac{\lambda_{\mathrm{U}} - \lambda_{\mathrm{M}}}{2\lambda_{\mathrm{U}}} - \left(\frac{3\lambda_{\mathrm{U}} + \lambda_{\mathrm{M}}}{\lambda_{\mathrm{U}}}\right)\alpha\right)|M^*| = \left(\frac{\lambda_{\mathrm{U}} - \lambda_{\mathrm{M}}}{4\lambda_{\mathrm{U}}^2} - \left(\frac{3\lambda_{\mathrm{U}} + \lambda_{\mathrm{M}}}{2\lambda_{\mathrm{U}}^2}\right)\alpha\right)|M^*|\,.$$

Hence, we get the following bound on the size of the output matching M:

$$\begin{aligned}|M| &\geqslant |M_0| + \left(\frac{\lambda_{\mathrm{U}} - \lambda_{\mathrm{M}}}{4\lambda_{\mathrm{U}}^2} - \frac{3\lambda_{\mathrm{U}} + \lambda_{\mathrm{M}}}{2\lambda_{\mathrm{U}}^2}\alpha\right)|M^*| \\ &= \left(\frac{1}{2} + \frac{\lambda_{\mathrm{U}} - \lambda_{\mathrm{M}}}{4\lambda_{\mathrm{U}}^2} + \left(1 - \frac{3\lambda_{\mathrm{U}} + \lambda_{\mathrm{M}}}{2\lambda_{\mathrm{U}}^2}\right)\alpha\right)|M^*| \quad \text{because } |M_0| = (1/2 + \alpha)|M^*|\ , \\ &\geqslant \left(\frac{1}{2} + \frac{\lambda_{\mathrm{U}} - \lambda_{\mathrm{M}}}{4\lambda_{\mathrm{U}}^2}\right)|M^*| \quad \text{since } \lambda_{\mathrm{U}} \geqslant 2 \text{ by Observation 9}\,. \end{aligned}$$ ◀

Now we state and prove our main result.

▶ **Theorem 12.** *Algorithm 3 uses two passes and has an approximation ratio of $1/2 + 1/16$ for triangle-free graphs and an approximation ratio of $1/2 + 1/32$ for general graphs for maximum matching.*

Proof. After the second pass, the output size $|M| \geqslant (1/2 + (\lambda_{\mathrm{U}} - \lambda_{\mathrm{M}})/(4\lambda_{\mathrm{U}}^2))|M^*|$ due to Lemma 11; we use $\lambda_{\mathrm{U}} = 2$ and $\lambda_{\mathrm{M}} = 1$ for triangle-free graphs and $\lambda_{\mathrm{U}} = 4$ and $\lambda_{\mathrm{M}} = 2$ (see Observation 5) for general graphs to get the claimed approximation ratios. ◀

Algorithm 4 Multi-pass algorithm: input graph G

```
In the first pass, find a maximal matching M_1.
M ← M_1
if G is triangle-free then
    for i = 2 to ⌈2/(3ε)⌉ do
        M ← Improve-Matching(M, i, 1)
else
    for i = 2 to ⌈4/(3ε)⌉ do
        M ← Improve-Matching(M, i + 1, 2)
Return M.
```

6 Multi Pass Algorithm

We run the function IMPROVE-MATCHING() in Algorithm 3 with increasing values of λ_{U}, and the approximation ratio converges to $1/2 + 1/6$. We note that this multi-pass algorithm is not just a repetition of the function IMPROVE-MATCHING(). Such a repetition will give an asymptotically worse number of passes (see, for example, the multi-pass algorithm due to Feigenbaum et al. [13]). We carefully choose the parameter λ_{U} for each pass to get the required number of passes.

▶ **Theorem 13.** *For any $\varepsilon > 0$, Algorithm 4 is a semi-streaming $(1/2+1/6-\varepsilon)$-approximation algorithm for maximum matching that uses $2/(3\varepsilon)$ passes for triangle-free graphs and $4/(3\varepsilon)$ passes for general graphs.*

Proof. We prove the theorem for triangle-free case; the general case is similar. Let M_i be the matching computed by Algorithm 4 after ith pass, and let $p := \lceil 2/(3\varepsilon)\rceil$, so $\varepsilon \leqslant 2/(3p)$. Since M_1 is maximal, it is $(1/2)$-approximate. Let $\alpha_1 := 0$, and for $i \in \{2, 3, \ldots, p\}$, let

$$\alpha_i := \frac{i-1}{4i^2} + \left(1 - \frac{3i+1}{2i^2}\right)\alpha_{i-1}$$

(see Lemma 11 with $\lambda_{\mathrm{U}} = i$ and $\lambda_{\mathrm{M}} = 1$). Then, by Lemma 11 and the logic of Algorithm 4, for $i \in [p]$, the matching M_i is $(1/2 + \alpha_i)$-approximate (by a trivial induction). Now we bound α_p by induction. We claim that for $i \in [p]$,

$$\alpha_i \geqslant \frac{1}{6} - \frac{2}{3i}\,,$$

which we prove by induction on i.

Base case: For $i = 1$, we have $1/6 - \alpha_1 = 1/6 - 0 = 1/6 \leqslant 2/(3 \cdot 1)$.

For inductive step, we want to show that

$$\frac{1}{6} - \alpha_i = \frac{1}{6} - \frac{i-1}{4i^2} - \left(1 - \frac{3i+1}{2i^2}\right)\alpha_{i-1} \leqslant \frac{2}{3i}\,,$$

which is implied by the following (using inductive hypothesis)

$$\frac{1}{6} - \frac{i-1}{4i^2} + \left(1 - \frac{3i+1}{2i^2}\right)\left(\frac{2}{3(i-1)} - \frac{1}{6}\right) \leqslant \frac{2}{3i}\,,$$

implied by

$$\frac{1}{6} - \frac{i-1}{4i^2} + \left(\frac{2i^2-3i-1}{2i^2}\right)\left(\frac{4-i+1}{6(i-1)}\right) \leqslant \frac{2}{3i}\,,$$

multiplying both sides by $12i^2(i-1)$, we then need to show that,

$$
\begin{aligned}
& 2i^2(i-1) - 3(i-1)^2 + (2i^2 - 3i - 1)(-i+5) \leqslant 8i(i-1)\,, \\
\text{implied by} \quad & 2i^3 - 2i^2 - 3(i^2 - 2i + 1) + (-2i^3 + 10i^2 + 3i^2 - 15i + i - 5) \leqslant 8i^2 - 8i\,, \\
\text{implied by} \quad & 2i^3 - 5i^2 + 6i - 3 + (-2i^3 + 13i^2 - 14i - 5) \leqslant 8i^2 - 8i\,, \\
\text{implied by} \quad & 8i^2 - 8i - 8 \leqslant 8i^2 - 8i\,,
\end{aligned}
$$

which is true, so we get the claim. Therefore $\alpha_p \geqslant 1/6 - 2/(3p) \geqslant 1/6 - \varepsilon$, and by our earlier observation, M_p is $(1/2 + \alpha_p)$-approximate, and this finishes the proof for triangle-free case. The proof for general case is very similar. We define $p := \lceil 4/(3\varepsilon) \rceil$ and $\alpha_1 := 0$, and for $i \in \{2, 3, \ldots, p\}$, we define

$$\alpha_i := \frac{i-1}{4(i+1)^2} + \left(1 - \frac{3(i+1)+2}{2(i+1)^2}\right)\alpha_{i-1}\,,$$

i.e., we use $\lambda_{\mathrm{U}} = i+1$ and $\lambda_{\mathrm{M}} = 2$. The corresponding claim then is that for $i \in [p]$,

$$\alpha_i \geqslant \frac{1}{6} - \frac{4}{3i}\,,$$

which can be verified by induction on i. ◀

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Algorithm 5 Three-pass algorithm for triangle-free graphs

In the first pass, find a maximal matching M_0.

In the second pass, find a maximal matching M_1 in $F_1 := \{uv : u \in V \setminus V(M_0), v \in V(M_0)\}$.

After the second pass:

- $M_1' \leftarrow$ arbitrary largest subset of M_1 such that there is no 3-augmenting path in $M_1' \cup M_0$ with respect to M_0
- $V_2 \leftarrow \{x \in V(M_0) : \exists v, w$ such that $vw \in M_1'$ and $wx \in M_0\}$
- For $x \in V_2$, denote by $P(x)$ the vertex v such that there exists w with $vw \in M_1'$ and $wx \in M_0$. See x and $P(x)$ in Figure 6.

In the third pass: $F_2 := \{xy : x \in V_2, y \in V \setminus V(M_0)\}$

$M_2 \leftarrow \emptyset$

for edge $xy \in F_2$ **do**

  if x, and y are unmarked **then**

    $M_2 \leftarrow M_2 \cup \{xy\}$; since the graph is triangle free, $y \neq P(x)$, and we can augment M_0 using xy.

    Mark $P(x)$, x, y, and $P^{-1}(y)$ (if exists).

Let M be largest of M_3 and M_3' which are computed below.

- Augment M_0 using edges in M_1 to get M_3.
- Augment M_0 using edges in M_1' and M_2 to get M_3'.

Output M.

A Three Pass Algorithm for Triangle Free Graphs

For completeness, we present our three-pass algorithm for triangle-free graphs.

▶ **Theorem 14.** *Algorithm 5 is a three-pass, semi-streaming, $(1/2 + 1/10)$-approximation algorithm for maximum matching in triangle-free graphs, and the analysis is tight.*

Proof. Let $|M_0| = (1/2 + \alpha)|M^*|$. The number of edges in M^* incident on $V(M^*) \setminus V(M_0)$ is

$$|V(M^*) \setminus V(M_0)| \geqslant |V(M^*)| - |V(M_0)| = 2|M^*| - 2|M_0| = (1 - 2\alpha)|M^*|\,; \tag{7}$$

and these edges also belong to F_1. Since M_1 is a maximal matching in F_1,

$$|M_1| \geqslant (1 - 2\alpha)|M^*|/2 = (1/2 - \alpha)|M^*|\,. \tag{8}$$

Let c be the number of 3-augmenting paths in $M_1 \cup M_0$, so $|M_1'| = |M_1| - c$ by the definition of M_1'. By Lemma 1, there are at most $4\alpha|M^*|$ non-3-augmentable edges in M_0. So at least $|M_1| - c - 4\alpha|M^*|$ edges of M_1' are incident on 3-augmentable edges of M_0. Therefore there is a matching of size at least $|M_1| - c - 4\alpha|M^*|$ in F_2; consider one such matching M_F. We claim that $|M_2| \geqslant |M_F|/4$. See Figure 6. Let $xy \in M_2$; we note that xy disallows at most four edges in M_F from being added to M_2 due to the (at most) four marks that it adds, because a marked vertex can disallow at most one edge in M_F (due to it being a matching), which shows the claim. Hence:

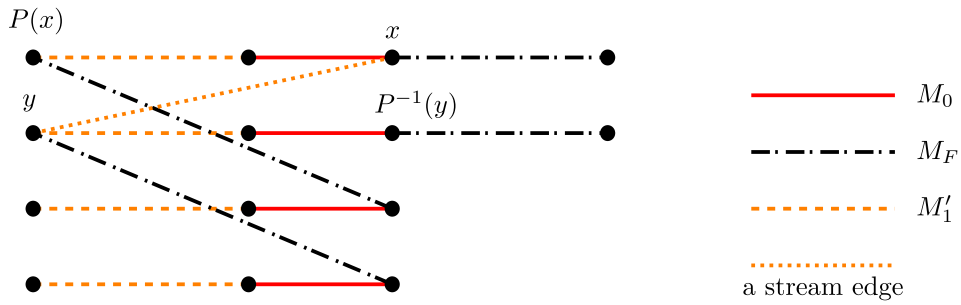

Figure 6 An edge $xy \in M_2$ disallows at most four edges in M_F from being added to M_2.

$$
\begin{aligned}
|M_2| &\geqslant \frac{|M_F|}{4} \\
&\geqslant \frac{|M_1| - c - 4\alpha|M^*|}{4} \\
&\geqslant \frac{1}{4}\left(\left(\frac{1}{2} - \alpha\right)|M^*| - c - 4\alpha|M^*|\right) && \text{by (8)}, \\
&= \frac{1}{4}\left(\left(\frac{1}{2} - 5\alpha\right)|M^*| - c\right) .
\end{aligned}
$$

Now, each edge in M_2 gives one augmentation after the second pass. To see this, we observe that for any $x \in V_2$, at any point in the algorithm, x and $P(x)$ are either both marked or both unmarked. So when an edge $xy \in M_2$ arrives, x and y are unmarked, and $P(x)$ and $P^{-1}(y)$ (if it exists) are also unmarked, otherwise one of x and y would have been marked and xy would not have been added to M_2. Since both $P(x)$ and $P^{-1}(y)$ were unmarked, we can use the augmenting path $\{M_1'(\{P(x)\}), M_0(\{x\}), xy\}$. Hence we get at least

$$
\max\left\{c, \frac{1}{4}\left(\left(\frac{1}{2} - 5\alpha\right)|M^*| - c\right)\right\}
$$

augmentations after the third pass. This is minimized by setting

$$
\begin{aligned}
c &= \frac{1}{4}\left(\left(\frac{1}{2} - 5\alpha\right)|M^*| - c\right) \\
&= \frac{1}{5}\left(\left(\frac{1}{2} - 5\alpha\right)|M^*|\right) \\
&= \left(\frac{1}{10} - \alpha\right)|M^*| .
\end{aligned}
$$

So we get the following bound:

$$
|M| \geqslant |M_0| + \left(\frac{1}{10} - \alpha\right)|M^*| \geqslant \left(\frac{1}{2} + \alpha\right)|M^*| + \left(\frac{1}{10} - \alpha\right)|M^*| = \left(\frac{1}{2} + \frac{1}{10}\right)|M^*| . \blacktriangleleft
$$

The tight example is shown in Figure 7.

B Three Pass Algorithm for General Graphs

We find a maximal matching M_1 in the first pass. Then we use Improve-Matching() function from Algorithm 3, i.e.,

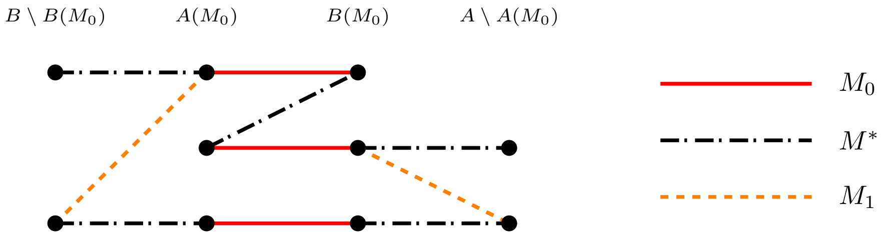

Figure 7 Tight example for Algorithm 5: M_1 has only two edges that land on bad vertices and cannot be augmented in the third pass. So $|M| = |M_0| = 3$ and $|M^*| = 5$.

- in the second pass, $M_2 \leftarrow$ IMPROVE-MATCHING$(M_1, 4, 2)$, and
- in the third pass, $M_3 \leftarrow$ IMPROVE-MATCHING$(M_2, 5, 2)$.

We observe that M_1 is $(1/2)$-approximate. Then by double application of Lemma 11, we get that M_3 is $(1/2 + 81/1600) \approx (1/2 + 1/19.753)$-approximate.

C Three Pass Algorithm for Bipartite Graphs: Suboptimal Analysis

We now give an analysis of Algorithm 1 that shows approximation ratio of only $1/2 + 1/18$ that is based on Konrad et al.'s [20] analysis for their two-pass algorithm for bipartite graphs. Afterward, we demonstrate that by not considering the distribution of lengths of augmenting paths, we may prove an approximation ratio of at most $1/2 + 1/14$. The better and tight analysis appears in Section 3.

▶ **Theorem 15.** *Algorithm 1 is a three-pass, semi-streaming, $(1/2 + 1/18)$-approximation algorithm for maximum matching in bipartite graphs.*

Proof. As usual, let $|M_0| = (1/2 + \alpha)|M^*|$. Since M_0 is a maximal matching, there are $|B(M^*) \setminus B(M_0)|$ edges of M^* that are also in F_2. We have

$$|B(M^*) \setminus B(M_0)| \geqslant |B(M^*)| - |B(M_0)| = |M^*| - |M_0|\,,$$

and since M_A is maximal, we then get the following:

$$|M_A| \geqslant \frac{1}{2}|B(M^*) \setminus B(M_0)| \geqslant \frac{1}{2}(|M^*| - |M_0|) = \frac{1}{2}\left(1 - \left(\frac{1}{2} + \alpha\right)\right)|M^*| = \frac{1}{2}\left(\frac{1}{2} - \alpha\right)|M^*|. \quad (9)$$

By Lemma 1, there are at most $4\alpha|M^*|$ non-3-augmentable edges in M_0. Which means that at least $|M_A| - 4\alpha|M^*|$ edges of M_A are incident on 3-augmentable edges of M_0; therefore there is a matching of size at least $|M_A| - 4\alpha|M^*|$ in F_3. Since we output a maximal matching in F_3, we get at least $(1/2)(|M_A| - 4\alpha|M^*|)$ augmentations after the third pass. So we get the following bound:

$$
\begin{aligned}
|M| &\geqslant |M_0| + \frac{1}{2}(|M_A| - 4\alpha|M^*|) \\
&\geqslant |M_0| + \frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2} - \alpha\right) - 4\alpha\right)|M^*| && \text{by (9)}\,, \\
&= |M_0| + \left(\frac{1}{8} - \frac{9}{4}\alpha\right)|M^*| \\
&= \left(\frac{1}{2} + \alpha\right)|M^*| + \left(\frac{1}{8} - \frac{9}{4}\alpha\right)|M^*| && \text{because } |M_0| = (1/2+\alpha)|M^*|\,, \\
&= \left(\frac{1}{2} + \frac{1}{8} - \frac{5}{4}\alpha\right)|M^*|\,.
\end{aligned}
$$

We also have $|M| \geqslant |M_0| = (1/2+\alpha)|M^*|$. As α increases, the former bound deteriorates and the latter improves, so the worst case α is when these two bounds are equal, which happens at $\alpha = 1/18$, and the approximation ratio we get is $1/2 + 1/18$. ◀

C.1 Improved Analysis Without Considering Longer Augmenting Paths

We can analyze Algorithm 1 better if we bound $|M_A|$ more carefully. The claim is that at least $(1/2 - 7\alpha)|M^*|/2$ edges of M_A are incident on 3-augmentable edges of M_0. Let $A_G \subseteq A(M_0)$ be the set of vertices in A that are endpoints of 3-augmentable edges of M_0; also, let $A_N = A(M_0) \setminus A_G$. So there is a matching of size at least $|A_G|$ in F_2 that covers A_G. Any maximal matching in F_2 has at least $(|A_G| - |A_N|)/2$ edges that are incident on A_G. To see the claim, we use the facts $|A_G| \geqslant (1/2 - 3\alpha)|M^*|$ and $|A_N| \leqslant 4\alpha|M^*|$. So there is a matching of size at least $(1/2 - 7\alpha)|M^*|/2$ in F_3. We output a maximal matching in F_3; hence we get at least $(1/2 - 7\alpha)|M^*|/4$ augmentations after the third pass. So we get the following bound:

$$
\begin{aligned}
|M| &\geqslant |M_0| + \frac{1}{4}\left(\frac{1}{2} - 7\alpha\right)|M^*| \\
&= \left(\frac{1}{2} + \alpha\right)|M^*| + \frac{1}{4}\left(\frac{1}{2} - 7\alpha\right)|M^*| \\
&= \left(\frac{1}{2} + \frac{1}{8} - \frac{3}{4}\alpha\right)|M^*|\,.
\end{aligned}
$$

where the second inequality is by (9). We also have $|M| \geqslant |M_0| = (1/2+\alpha)|M^*|$, so the worst case α is when these two bounds are equal, which happens at $\alpha = 1/14$ and the approximation ratio we get is $1/2 + 1/14$, and we get the following theorem.

▶ **Theorem 16.** *Algorithm 1 is a three-pass, semi-streaming,* $(1/2 + 1/14)$*-approximation algorithm for maximum matching in bipartite graphs.*

D A Note on the Analysis by Esfandiari et al.

We demonstrate with an example that the analysis of the algorithm by Esfandiari et al. [11] given for bipartite graphs cannot be extended for triangle-free graphs to get the same approximation ratio. See Figure 8. Lemma 6 in their paper, as they correctly claim, holds only for bipartite graphs and not for triangle-free graphs. Our algorithm in Section 4 is essentially the same algorithm except for the post-processing step; we augment the maximal matching computed in the first pass greedily, whereas they use an offline maximum matching algorithm. We have highlighted some other comparison points in Section 1.

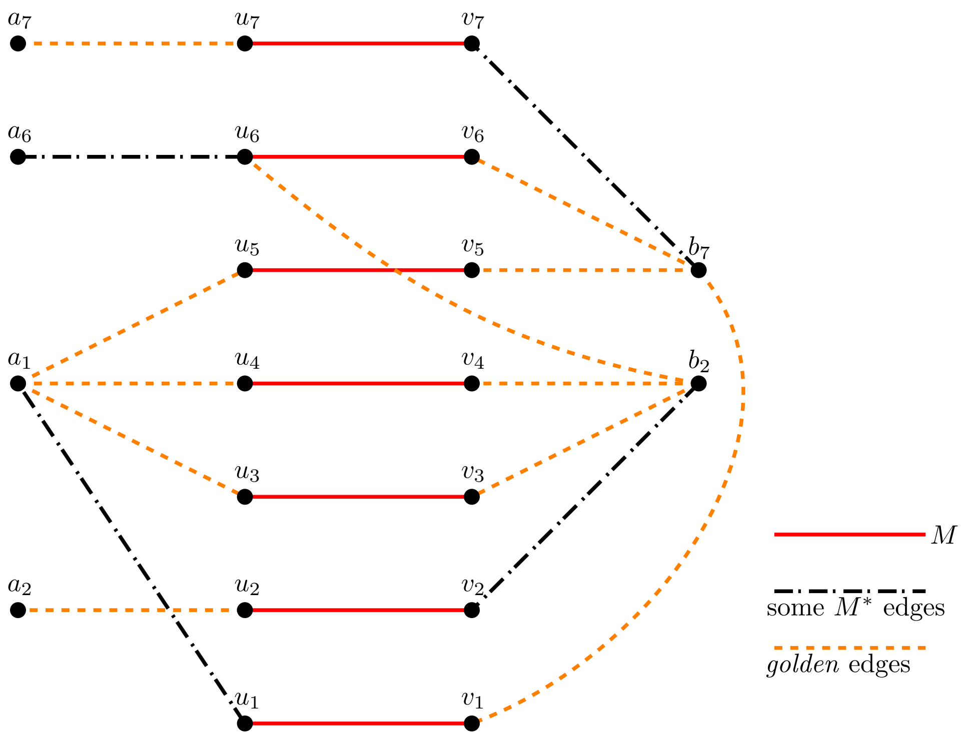

Figure 8 Example demonstrating that Lemma 6 in Esfandiari et al. [11] does not hold when the input graph is not bipartite but is triangle-free. We use $k = 3$. For an M edge $u_i v_i$, there are two M^* edges incident on it, which are $a_i u_i$ and $v_i b_i$, and some of the M^* edges are not shown, but all golden edges are shown, which we call support edges or denote by S in our terminology. It can be seen from this example that their algorithm is not a $(1/2 + 1/12)$-approximation algorithm for triangle free graphs, because out of the seven 3-augmentable edges in M, only one will get augmented, thereby giving a worse approximation ratio.