Polishness of Some Topologies Related to Automata

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Abstract

We prove that the Büchi topology, the automatic topology, the alphabetic topology and the strong alphabetic topology are Polish, and provide consequences of this.

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1 Introduction

This paper is a contribution to the study of the interactions between descriptive set theory and theoretical computer science. These interactions have already been the subject of many studies, see for instance [18, 32, 21, 30, 27, 25, 29, 6, 28, 7, 11, 9, 5].

In particular, the theory of automata reading infinite words, which is closely related to infinite games, is now a rich theory which is used for the specification and verification of non-terminating systems, see [12, 25]. The space $\Sigma^{\mathbb{N}}$ of infinite words over a finite alphabet Σ being equipped with the usual Cantor topology, a natural way to study the complexity of ω-languages accepted by various kinds of automata is to study their topological complexity, and particularly to locate them with regard to the Borel and the projective hierarchies.

However, as noticed in [26] by Schwarz and Staiger and in [15] by Hoffmann and Staiger, it turned out that for several purposes some other topologies on a space $\Sigma^{\mathbb{N}}$ are useful, for instance for studying fragments of first-order logic over infinite words or for a topological characterisation of random infinite words (see also [14]). In particular, Schwarz and Staiger studied four topologies on the space $\Sigma^{\mathbb{N}}$ of infinite words over a finite alphabet Σ which are all related to automata, and refine the Cantor topology on $\Sigma^{\mathbb{N}}$: the Büchi topology, the automatic topology, the alphabetic topology, and the strong alphabetic topology.

Recall that a topological space is Polish iff it is separable, i.e. contains a countable dense subset, and its topology is induced by a complete metric. Classical descriptive set theory is about the topological complexity of definable subsets of Polish topological spaces, as well

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as the study of hierarchies of complexity (see [16, 25] for basic notions). The analytic sets, which are the projections of Borel sets, are of particular importance. Similar hierarchies of complexity are studied in effective descriptive set theory, which is based on the theory of recursive functions (see [24] for basic notions). The effective analytic subsets of the Cantor space $(2^{\mathbb{N}}, \tau_C)$ are highly related to theoretical computer science, in the sense that they coincide with the sets recognized by some special kind of Turing machines (see [31]).

In [26], Schwarz and Staiger prove that the Büchi topology, which is generated by the regular ω-languages, is metrizable. It is separable, by definition, because there are only countably many regular ω-languages. It remains to see that it is completely metrizable to see that it is Polish. This is one of the main results proved in this paper.

We now give some more details about the topologies studied by Schwarz and Staiger in [26] that we investigate in this paper. Let Σ be a finite alphabet with at least two symbols. We will consider the topology τ_δ on $\Sigma^{\mathbb{N}}$ generated by the set $\mathbb{B}_\delta$ of sets accepted by an unambiguous Büchi Turing machine. Let Σ^* be the set of finite sequences of elements of Σ. The following topologies on $\Sigma^{\mathbb{N}}$, related to automata, are considered in [26].

- The *Büchi topology* τ_B, generated by the set $\mathbb{B}_B$ of ω-regular languages,
- The *automatic topology* τ_A, generated by the set $\mathbb{B}_A$ of τ_C-closed ω-regular languages (this topology is remarkable because all τ_C-closed ω-regular languages are accepted by deterministic Büchi automata),
- The *alphabetic topology* τ_α, generated by the set $\mathbb{B}_\alpha$ of sets of the form $B_{w,A} := \{w\sigma \mid \sigma \in A^{\mathbb{N}}\}$, where $w \in \Sigma^*$ and $A \subseteq \Sigma$ (this topology is useful for investigations in restricted first-order theories for infinite words),
- The *strong alphabetic topology* τ_s, generated by the set $\mathbb{B}_s$ of sets of the form

  $$S_{w,A} := \{w\sigma \mid \sigma \in A^{\mathbb{N}} \ \wedge \ \forall a \in A \ \ \forall k \in \mathbb{N} \ \ \exists i \geq k \ \ \sigma(i) = a\},$$

  where $w \in \Sigma^*$ and $A \subseteq \Sigma$ (this topology is derived from τ_α, and considered in [4], together with τ_α).

In [26], Schwarz and Staiger prove that these topologies are metrizable. We improve this result:

▶ **Theorem 1.** *Let $z \in \{C, \delta, B, A, \alpha, s\}$. Then τ_z is Polish.*

From this result, it is already possible to infer many properties of the space $\Sigma^{\mathbb{N}}$, where Σ is a finite alphabet, equipped with the Büchi topology. In particular, we first get some results about the σ-algebra generated by the ω-regular languages. It is stratified in a hierarchy of length ω_1 (the first uncountable ordinal) and there are universal sets at each level of this hierarchy. Notice that this σ-algebra coincides with the σ-algebra of Borel sets for the Cantor topology, and that a set is Borel for the Cantor topology if and only if it is Borel for the Büchi topology, but the levels of the Borel hierarchy differ for the two topologies. For instance an ω-regular set which is non-$\mathbf{\Pi}^0_2$ for the Cantor topology is clopen (i.e., $\mathbf{\Delta}^0_1$) for the Büchi topology. Therefore the results about the existence of universal sets at each level of the σ-algebra generated by the ω-regular languages are really new and interesting. We derive many other properties from the polishness of the Büchi topology.

2 Background

We first recall the notions required to understand fully the introduction and the sequel (see for example [25, 30, 16, 24]).

2.1 Theoretical computer science

A *Büchi automaton* is a tuple $\mathcal{A} = (Q, \Sigma, \delta, Q_i, Q_f)$, where Σ is the input alphabet, Q is the finite state set, δ is the transition relation, Q_i and Q_f are the sets of initial and final states. The transition relation δ is a subset of $Q \times \Sigma \times Q$.

A *run* on some sequence $\sigma \in \Sigma^{\mathbb{N}}$ is a sequence $(q_n)_{i\in\mathbb{N}} \in Q^{\mathbb{N}}$ of states such that q_0 is initial ($q_0 \in Q_i$) and $(q_i, \sigma(i), q_{i+1})$ is a transition in δ for each $i \geq 0$. It is accepting if it visits infinitely often final states, that is, $q_i \in Q_f$ for infinitely many i. An input sequence σ is accepted if there exists an accepting run on α. The set of accepted inputs is denoted $L(\mathcal{A})$. A set of infinite words is called *ω-regular* if it is equal to $L(\mathcal{A})$ for some automaton $\mathcal{A}$.

A Büchi automaton is actually similar to a classical finite automaton. A finite word w of length n is accepted by an automaton $\mathcal{A}$ if there is sequence $(q_i)_{i\leq n}$ of $n+1$ states such that q_0 is initial ($q_0 \in Q_i$), q_n is final ($q_n \in Q_f$) and $(q_i, \sigma(i), q_{i+1})$ is a transition in δ for each $0 \leq i < n$. The set of accepted finite words is denoted by $U(\mathcal{A})$. A set of finite words is called *regular* if it is equal to $U(\mathcal{A})$ for some automaton $\mathcal{A}$.

Let us recall that the ω-power of a set U of finite words is defined by

$$U^\omega = \{\sigma \in \Sigma^{\mathbb{N}} \mid \exists\, (w_i)_{i\in\mathbb{N}} \in U^{\mathbb{N}} \text{ s.t. } \sigma = w_0 w_1 w_2 \cdots\}.$$

The ω-powers play a crucial role in the characterization of ω-regular languages (see [1]):

▶ **Theorem 2** (Büchi). *Let Σ be a finite alphabet, and $L \subseteq \Sigma^{\mathbb{N}}$. The following statements are equivalent:*
1. *L is ω-regular,*
2. *there are $2n$ regular languages $(U_i)_{i<n}$ and $(V_i)_{i<n}$ such that $L = \bigcup_{i<n} U_i V_i^\omega$.*

The closure properties of the class of ω-regular languages mentioned in the introduction are the following (see [25] and [26]). If Σ is a set and $w \in \Sigma^*$, then w defines the usual basic clopen set $N_w := \{\sigma \in \Sigma^{\mathbb{N}} \mid w \text{ is a prefix of} \sigma\}$ of the Cantor topology τ_C (so $\mathbb{B}_C := \{\emptyset\} \cup \{N_w \mid w \in \Sigma^*\}$ is a basis for τ_C).

▶ **Theorem 3** (Büchi). *The class of ω-regular languages contains the usual basic clopen sets and is closed under finite unions and intersections, taking complements, and projections.*

We now turn to the study of Turing machines (see [3, 30]).

A *Büchi Turing machine* is a tuple $\mathcal{M} = (\Sigma, Q, \Gamma, \delta, q_0, Q_f)$, where Σ and Γ are the input and tape alphabets satisfying $\Sigma \subsetneq \Gamma$, Q is the finite state set, δ is the transition relation, q_0 is the initial state and Q_f is the set of final states. The relation δ is a subset of $(Q \times \Gamma) \times (Q \times \Gamma \times \{-1, 0, 1\})$.

A *configuration* of $\mathcal{M}$ is a triple (q, γ, j) where $q \in Q$ is the current state, $\gamma \in \Gamma^{\mathbb{N}}$ is the content of the tape and the non-negative integer $j \in \mathbb{N}$ is the position of the head on the tape.

Two configurations (q, γ, j) and (q', γ', j') of $\mathcal{M}$ are *consecutive* if there exists a transition $(q, a, q', b, d) \in \delta$ such that the following conditions are met:
1. $\gamma(j) = a$, $\gamma'(j) = b$ and $\gamma(i) = \gamma'(i)$ for each $i \neq j$. This means that the symbol a is the replaced by symbol b at position j and that all other symbols on the tape remain unchanged.
2. the two positions j and j' satisfy the equality $j' = j + d$.

A *run* of the machine $\mathcal{M}$ on some input $\sigma \in \Sigma^{\mathbb{N}}$ is a sequence $(p_i, \gamma_i, j_i)_{i\in\mathbb{N}}$ of consecutive configurations such that $p_0 = q_0$, $\gamma_0 = \sigma$ and $j_0 = 0$. It is *complete* if the head visits all positions of the tape. This means that for each integer N, the exists an integer i such that

$j_i \geq N$. The run is *accepting* if it visits infinitely often final states, that is, $p_i \in Q_f$ for infinitely many integers i. The ω-language accepted by $\mathcal{M}$ is the set of inputs σ such that there exists an accepting and complete run on σ.

Notice that other accepting conditions have been considered for the acceptance of infinite words by Turing machines, like the 1' or Muller ones (the latter one was firstly called 3-acceptance), see [3, 30]. Moreover several types of required behaviour on the input tape have been considered in the literature, see [31, 10, 8].

A *Büchi automaton* $\mathcal{A}$ is in fact a Büchi Turing machine whose head only move forwards. This means that each of its transition has the form (p, a, q, b, d) where $d = 1$. Note that the written symbol b does not matter since each position of the tape is just visited once and the symbol b is never read.

2.2 Descriptive set theory

Classical descriptive set theory takes place in Polish topological spaces.
We first recall that if d is a distance on a set X, and $(x_n)_{n\in\mathbb{N}}$ is a sequence of elements of X, then the sequence $(x_n)_{n\in\mathbb{N}}$ is called a Cauchy sequence if

$$\forall k \in \mathbb{N} \;\; \exists N \in \mathbb{N} \;\; \forall p, p' \geq N \;\; d(x_p, x_{p'}) < \frac{1}{2^k}.$$

In a topological space X whose topology is induced by a distance d, the distance d and the metric space (X, d) are said to be *complete* if every Cauchy sequence in X is convergent.

▶ **Definition 4.** A topological space X is a *Polish space* if it is

1. separable (there is a countable dense sequence (x_n) in X),
2. completely metrizable (there is a complete distance d on X which is compatible with the topology of X).

Effective descriptive set theory is based on the notion of recursive function. A function from $\mathbb{N}^k$ to $\mathbb{N}^l$ is said to be *recursive* if it is total and computable. By extension, a relation is called *recursive* if its characteristic function is recursive.

▶ **Definition 5.** A *recursive presentation* of a Polish space X is a pair $\big((x_n)_{n\in\mathbb{N}}, d\big)$ such that

1. $(x_n)_{n\in\mathbb{N}}$ is dense in X,
2. d is a compatible complete distance on X such that the following relations P and Q are recursive:

$$P(i, j, m, k) \iff d(x_i, x_j) \leq \frac{m}{k+1},$$
$$Q(i, j, m, k) \iff d(x_i, x_j) < \frac{m}{k+1}.$$

A Polish space X is *recursively presented* if there is a recursive presentation of it.

Note that the formula $(p, q) \mapsto 2^p(2q + 1) - 1$ defines a recursive bijection $\mathbb{N}^2 \to \mathbb{N}$. One can check that the coordinates of the inverse map are also recursive. They will be denoted $n \mapsto (n)_0$ and $n \mapsto (n)_1$ in the sequel. These maps will help us to define some of the basic effective classes.

▶ **Definition 6.** Let $\big((x_n)_{n\in\mathbb{N}}, d\big)$ be a recursive presentation of a Polish space X.

1. We fix a countable basis of X: $B(X, n)$ is the open ball $B_d(x_{(n)_0}, \frac{((n)_1)_0}{((n)_1)_1+1})$.

2. A subset S of X is *semirecursive*, or *effectively open* (denoted $S \in \varSigma^0_1$) if

$$S = \bigcup_{n \in \mathbb{N}} B(X, f(n)),$$

for some recursive function f.

3. A subset S of X is *effectively closed* (denoted $S \in \varPi^0_1$) if its complement $\neg S$ is semirecursive.
4. One can check that a product of two recursively presented Polish spaces has a recursive presentation, and that the Baire space $\mathbb{N}^\mathbb{N}$ has a recursive presentation. A subset S of X is *effectively analytic* (denoted $S \in \varSigma^1_1$) if there is a $\varPi^0_1$ subset C of $X \times \mathbb{N}^\mathbb{N}$ such that

$$S = \pi_0[C] := \{x \in X \mid \exists \alpha \in \mathbb{N}^\mathbb{N} \ (x, \alpha) \in C\}.$$

5. A subset S of X is *effectively co-analytic* (denoted $S \in \varPi^1_1$) if its complement $\neg S$ is effectively analytic, and *effectively Borel* if it is in $\varSigma^1_1$ and $\varPi^1_1$ (denoted $S \in \varDelta^1_1$).
6. We will also use the following *relativized classes*: if X, Y are recursively presented Polish spaces and $y \in Y$, then we say that $A \subseteq X$ is in $\varSigma^1_1(y)$ if there is $S \in \varSigma^1_1(Y \times X)$ such that $A = S_y := \{x \in X \mid (y, x) \in S\}$. The class $\varPi^1_1(y)$ is defined similarly. We also set $\varDelta^1_1(y) := \varSigma^1_1(y) \cap \varPi^1_1(y)$.

The crucial link between the effective classes and the classical corresponding classes is as follows: the class of analytic (resp., co-analytic, Borel) subsets of Y is equal to $\bigcup_{\alpha \in \mathbb{N}^\mathbb{N}} \varSigma^1_1(\alpha)$ (resp., $\bigcup_{\alpha \in \mathbb{N}^\mathbb{N}} \varPi^1_1(\alpha)$, $\bigcup_{\alpha \in \mathbb{N}^\mathbb{N}} \varDelta^1_1(\alpha)$). This allows to use effective descriptive set theory to prove results of classical type. In the sequel, when we consider an effective class in some $\Sigma^\mathbb{N}$ with Σ finite, we will always use a fixed recursive presentation associated with the Cantor topology. The following result is proved in [31], see also [8]:

▶ **Theorem 7.** *Let Σ be a finite alphabet, and $L \subseteq \Sigma^\mathbb{N}$. The following statements are equivalent:*
1. *$L = L(\mathcal{M})$ for some Büchi Turing machine $\mathcal{M}$,*
2. *$L \in \varSigma^1_1$.*

We now recall the *Choquet game* played by two players on a topological space X. Players 1 and 2 play alternatively. At each turn i, Player 1 plays by choosing an open subset U_i and a point $x_i \in U_i$ such that $U_i \subseteq V_{i-1}$, where V_{i-1} has been chosen by Player 2 at the previous turn. Player 2, plays by choosing an open subset V_i such that $x_i \in V_i$ and $V_i \subseteq U_i$. Player 2 wins the game if $\bigcap_{i \in \mathbb{N}} V_i \neq \emptyset$. We now recall some classical notions of topology.

▶ **Definition 8.** A topological space X is said to be
- T_1 if every singleton of X is closed,
- *regular* if for every point of X and every open neighborhood U of x, there is an open neighborhood V of x with $\overline{V} \subseteq U$,
- *second countable* if its topology has a countable basis,
- *zero-dimensional* if there is a basis made of clopen sets,
- *strong Choquet* if X is not empty and Player 2 has a winning strategy in the Choquet game.

Note that every zero-dimensional space is regular. The following result is 8.18 in [16].

▶ **Theorem 9** (Choquet). *A nonempty, second countable topological space is Polish if and only if it is T_1, regular, and strong Choquet.*

Let X be a nonempty recursively presented Polish space. The *Gandy-Harrington topology* on X is generated by the Σ^1_1 subsets of X, and denoted Σ_X. By Theorem 7, this topology is also related to automata and Turing machines. As there are some effectively analytic sets whose complement is not analytic, the Gandy-Harrington topology is not metrizable (in fact not regular) in general (see [3E.9] in [24]). In particular, it is not Polish.

3 Proof of the main result

The proof of Theorem 1 is organized as follows. We provide below four properties which ensure that a given topological space is strong Choquet. Then we use Theorem 9 to prove that the considered spaces are indeed Polish.

Let Σ be a countable alphabet. The set $\Sigma^{\mathbb{N}}$ is equipped with the product topology of the discrete topology on Σ, unless another topology is specified. This topology is induced by a natural metric, called the *prefix metric* which is defined as follows. For $\sigma \neq \sigma' \in \Sigma^{\mathbb{N}}$, the distance d is given by

$$d(\sigma, \sigma') = 2^{-r} \quad \text{where} \quad r = \min\{n \mid \sigma(n) \neq \sigma'(n)\}.$$

When Σ is finite this topology is the classical Cantor topology. When Σ is countably infinite the topological space is homeomorphic to the Baire space $\mathbb{N}^{\mathbb{N}}$.

Let Σ and Γ be two alphabets. The function which maps each pair $(\sigma, \gamma) \in \Sigma^{\mathbb{N}} \times \Gamma^{\mathbb{N}}$ to the element $(\sigma(0), \gamma(0)), (\sigma(1), \gamma(1)), \ldots$ of $(\Sigma \times \Gamma)^{\mathbb{N}}$ is a homeomorphism between $\Sigma^{\mathbb{N}} \times \Gamma^{\mathbb{N}}$ and $(\Sigma \times \Gamma)^{\mathbb{N}}$ allowing us to identify these two spaces.

If Σ is a set, $\sigma \in \Sigma^{\mathbb{N}}$ and $l \in \mathbb{N}$, then $\sigma|l$ is the prefix of σ of length l.

We set $2 := \{0, 1\}$ and $\mathbb{P}_\infty := \{\alpha \in 2^{\mathbb{N}} \mid \forall k \in \mathbb{N} \ \ \exists i \geq k \ \ \alpha(i) = 1\}$. This latter set is simply the set of infinite words over the alphabet $2 := \{0, 1\}$ having infinitely many symbols 1.

We will work in the spaces of the form $\Sigma^{\mathbb{N}}$, where Σ is a finite set with at least two elements. We will fix a topology τ_Σ on $\Sigma^{\mathbb{N}}$, and a basis $\mathbb{B}_\Sigma$ for τ_Σ. We consider the following properties of the family $(\tau_\Sigma, \mathbb{B}_\Sigma)_\Sigma$, using the previous identification:

(P1) $\mathbb{B}_\Sigma$ contains the usual basic clopen sets N_w,

(P2) $\mathbb{B}_\Sigma$ is closed under finite unions and intersections,

(P3) $\mathbb{B}_\Sigma$ is closed under projections, in the sense that if Γ is a finite set with at least two elements and $L \in \mathbb{B}_{\Sigma \times \Gamma}$, then $\pi_0[L] \in \mathbb{B}_\Sigma$,

(P4) for each $L \in \mathbb{B}_\Sigma$ there is a closed subset C of $\Sigma^{\mathbb{N}} \times \mathbb{P}_\infty$ (i.e. C is the intersection of a closed subset of the Cantor space $\Sigma^{\mathbb{N}} \times 2^{\mathbb{N}}$ with $\Sigma^{\mathbb{N}} \times \mathbb{P}_\infty$) which is in $\mathbb{B}_{\Sigma \times 2}$, and such that $L = \pi_0[C]$.

▶ **Theorem 10.** *Assume that the family $(\tau_\Sigma, \mathbb{B}_\Sigma)_\Sigma$ satisfies the properties (P1)-(P4). Then the topologies τ_Σ are strong Choquet.*

Proof. We first describe a strategy τ for Player 2. Player 1 first plays $\sigma_0 \in \Sigma^{\mathbb{N}}$ and a τ_Σ-open neighborhood U_0 of σ_0. Let L_0 in $\mathbb{B}_\Sigma$ with $\sigma_0 \in L_0 \subseteq U_0$. Property (P4) gives C_0 with $L_0 = \pi_0[C_0]$. This gives $\alpha_0 \in \mathbb{P}_\infty$ such that $(\sigma_0, \alpha_0) \in C_0$. We choose $l^0_0 \in \mathbb{N}$ big enough to ensure that if $s^0_0 := \alpha_0|l^0_0$, then s^0_0 has at least a coordinate equal to 1. We set $w_0 := \sigma_0|1$ and $V_0 := \pi_0[C_0 \cap (N_{w_0} \times N_{s^0_0})]$. By properties (P1)-(P3), V_0 is in $\mathbb{B}_\Sigma$ and thus τ_Σ-open. Moreover, $\sigma_0 \in V_0 \subseteq L_0 \subseteq U_0$, so that Player 2 respects the rules of the game if he plays V_0.

Now Player 1 plays $\sigma_1 \in V_0$ and a τ_Σ-open neighborhood U_1 of σ_1 contained in V_0. Let L_1 in $\mathbb{B}_\Sigma$ with $\sigma_1 \in L_1 \subseteq U_1$. Property (P4) gives C_1 with $L_1 = \pi_0[C_1]$. This gives $\alpha_1 \in \mathbb{P}_\infty$ such that $(\sigma_1, \alpha_1) \in C_1$. We choose $l^1_0 \in \mathbb{N}$ big enough to ensure that if $s^1_0 := \alpha_1|l^1_0$, then s^1_0 has at least one coordinate equal to 1. As $\sigma_1 \in V_0$, there is $\alpha'_0 \in \mathbb{P}_\infty$

such that $(\sigma_1, \alpha_0') \in C_0 \cap (N_{w_0} \times N_{s_0^0})$. We choose $l_1^0 > l_0^0$ big enough to ensure that if $s_1^0 := \alpha_0'|l_1^0$, then s_1^0 has at least two coordinates equal to 1. We set $w_1 := \sigma_1|2$ and $V_1 := \pi_0[C_0 \cap (N_{w_1} \times N_{s_1^0})] \cap \pi_0[C_1 \cap (N_{w_0} \times N_{s_0^1})]$. Here again, V_1 is τ_Σ-open. Moreover, $\sigma_1 \in V_1 \subseteq U_1$ and Player 2 can play V_1.

Next, Player 1 plays $\sigma_2 \in V_1$ and a τ_Σ-open neighborhood U_2 of σ_2 contained in V_1. Let L_2 in $\mathbb{B}_\Sigma$ with $\sigma_2 \in L_2 \subseteq U_2$. Property (P4) gives C_2 with $L_2 = \pi_0[C_2]$. This gives $\alpha_2 \in \mathbb{P}_\infty$ such that $(\sigma_2, \alpha_2) \in C_2$. We choose $l_0^2 \in \mathbb{N}$ big enough to ensure that if $s_0^2 := \alpha_2|l_0^2$, then s_0^2 has at least one coordinate equal to 1. As $\sigma_2 \in V_1$, there is $\alpha_1' \in \mathbb{P}_\infty$ such that $(\sigma_2, \alpha_1') \in C_1 \cap (N_{w_0} \times N_{s_0^1})$. We choose $l_1^1 > l_0^1$ big enough to ensure that if $s_1^1 := \alpha_1'|l_1^1$, then s_1^1 has at least two coordinates equal to 1. As $\sigma_2 \in V_1$, there is $\alpha_0'' \in \mathbb{P}_\infty$ such that $(\sigma_2, \alpha_0'') \in C_0 \cap (N_{w_1} \times N_{s_1^0})$. We choose $l_2^0 > l_1^0$ big enough to ensure that if $s_2^0 := \alpha_0''|l_2^0$, then s_2^0 has at least three coordinates equal to 1. We set $w_2 := \sigma_2|3$ and $V_2 := \pi_0[C_0 \cap (N_{w_2} \times N_{s_2^0})] \cap \pi_0[C_1 \cap (N_{w_1} \times N_{s_1^1})] \cap \pi_0[C_2 \cap (N_{w_0} \times N_{s_0^2})]$. Here again, V_2 is τ_Σ-open. Moreover, $\sigma_2 \in V_2 \subseteq U_2$ and Player 2 can play V_2.

If we go on like this, we build $w_l \in \Sigma^{l+1}$ and $s_l^n \in 2^*$ such that $w_0 \subseteq w_1 \subseteq \ldots$ and $s_0^n \subsetneqq s_1^n \subsetneqq \ldots$ This allows us to define $\sigma := \lim_{l\to\infty} w_l \in \Sigma^\mathbb{N}$ and, for each $n \in \mathbb{N}$, $\alpha_n := \lim_{l\to\infty} s_l^n \in 2^\mathbb{N}$. Note that $\alpha_n \in \mathbb{P}_\infty$ since s_l^n has at least $l+1$ coordinates equal to 1. As (σ, α_n) is the limit of (w_l, s_l^n) as l goes to infinity and $N_{w_l} \times N_{s_l^n}$ meets C_n (which is closed in $\Sigma^\mathbb{N} \times \mathbb{P}_\infty$), $(\sigma, \alpha_n) \in C_n$. Thus

$$\sigma \in \bigcap_{n\in\mathbb{N}} \pi_0[C_n] = \bigcap_{n\in\mathbb{N}} L_n \subseteq \bigcap_{n\in\mathbb{N}} U_n \subseteq \bigcap_{n\in\mathbb{N}} V_n,$$

so that τ is winning for Player 2. ◀

3.1 The Gandy-Harrington topology

We have already mentioned that the Gandy-Harrington topology is not Polish in general. However, it is almost Polish as it fulfills the four properties (P1)–(P4).

Let Σ be a finite alphabet with at least two elements and let X be the space $\Sigma^\mathbb{N}$ equipped with the topology $\tau_\Sigma := \Sigma_X$ generated by the family $\mathbb{B}_\Sigma$ of Σ_1^1 subsets of X. Note that the assumption of Theorem 10 are satisfied. Indeed, (P1)-(P3) come from 3E.2 in [24]. For (P4), let F be a Π_1^0 subset of $X \times \mathbb{N}^\mathbb{N}$ such that $L = \pi_0[F]$. Let φ be the function from $\mathbb{N}^\mathbb{N}$ to $2^\mathbb{N}$ defined by $\varphi(\beta) = 0^{\beta(0)}10^{\beta(1)}1\ldots$. Note that φ is a homeomorphism from $\mathbb{N}^\mathbb{N}$ onto $\mathbb{P}_\infty$, and recursive (which means that the relation $\varphi(\beta) \in N(2^\mathbb{N}, n)$ is semirecursive in β and n). This implies that $C := (\mathrm{Id} \times\varphi)[F]$ is suitable (see 3E.2 in [24]).

Note that τ_Σ is second countable since there are only countably many Σ_1^1 subsets of X (see 3F.6 in [24]), T_1 since it is finer than the usual topology by the property (P1), and strong Choquet by Theorem 10.

One can show that there is a dense basic open subset Ω_X of (X, τ_Σ) such that $S \cap \Omega_X$ is a clopen subset of (Ω_X, τ_Σ) for each Σ_1^1 subset S of X (see [19]). In particular, (Ω_X, τ_Σ) is zero-dimensional, and regular. As it is, just like (X, τ_Σ), second countable, T_1 and and strong Choquet, (Ω_X, τ_Σ) is a Polish space, by Theorem 9.

3.2 The Büchi topology

Let Σ be a finite alphabet with at least two symbols, and X be the space $\Sigma^\mathbb{N}$ equipped with the Büchi topology τ_B generated by the family $\mathbb{B}_B$ of ω-regular languages in X. Theorem 29 in [26] shows that τ_B is metrizable. We now give a distance which is compatible with τ_B. This metric was used in [14] (Theorem 2 and Lemma 21 and several corollaries following

Lemma 21). A similar argument for subword metrics is in [[15], Section 4]. If $\mathcal{A}$ is a Büchi automaton, then we denote $|\mathcal{A}|$ the number of states of $\mathcal{A}$. We say that a Büchi automaton **separates** x and y iff

$$\big(x \in L(\mathcal{A}) \wedge y \notin L(\mathcal{A})\big) \vee \big(y \in L(\mathcal{A}) \wedge x \notin L(\mathcal{A})\big).$$

The distance δ on $\Sigma^{\mathbb{N}}$ is then defined as follows: for $x, y \in \Sigma^{\mathbb{N}}$, $\delta(x,y) = \begin{cases} 0 \text{ if } x = y, \\ 2^{-n} \text{ if } x \neq y, \end{cases}$ where $n := \min\{|\mathcal{A}| \mid \mathcal{A} \text{ is a Büchi automaton which separates } x \text{ and } y\}$. We now describe some properties of the map δ. This is the occasion to illustrate the notion of complete metric.

▶ **Proposition 11.**
1. *the map δ defines a distance on $\Sigma^{\mathbb{N}}$,*
2. *the distance δ is compatible with τ_B,*
3. *the distance δ is not complete.*

Proof. 1. If $x, y \in \Sigma^{\mathbb{N}}$, then $\delta(x,y) = \delta(y,x)$, by definition of δ. Let $x, y, z \in \Sigma^{\mathbb{N}}$, and assume that $\delta(x,y) + \delta(y,z) < \delta(x,z) = 2^{-n}$. Then $\delta(x,y) < 2^{-n}$ and $\delta(y,z) < 2^{-n}$ hold. In particular, if $\mathcal{A}$ is a Büchi automaton with n states then it does not separate x and y and similarly it does not separate y and z. Thus either $x, y, z \in L(\mathcal{A})$ or $x, y, z \notin L(\mathcal{A})$. This implies that the Büchi automaton $\mathcal{A}$ does not separate x and z. But this holds for every Büchi automaton with n states and then $\delta(x,z) < 2^{-n}$. This leads to a contradiction and thus $\delta(x,z) \leq \delta(x,y) + \delta(y,z)$ for all $x, y, z \in \Sigma^{\mathbb{N}}$. This shows that δ is a distance on $\Sigma^{\mathbb{N}}$.

2. Recall that an open set for this topology is a union of ω-languages accepted by Büchi automata. Let then $L(\mathcal{A})$ be an ω-language accepted by a Büchi automaton $\mathcal{A}$ having n states, and $x \in L(\mathcal{A})$. We now show that the open ball $B(x, 2^{-(n+1)})$ with center x and δ-radius $2^{-(n+1)}$ is a subset of $L(\mathcal{A})$. Indeed, if $\delta(x,y) < 2^{-(n+1)} < 2^{-n}$, then x and y cannot be separated by any Büchi automaton with n states, and thus $y \in L(\mathcal{A})$. This shows that $L(\mathcal{A})$ (and therefore any open set for τ_B) is open for the topology induced by the distance δ. Conversely, let $B(x,r)$ be an open ball for the distance δ, where $r > 0$ is a positive real. It is clear from the definition of the distance δ that we may only consider the case $r = 2^{-n}$ for some natural number n. Then $y \in B(x, 2^{-n})$ if and only if x and y cannot be separated by any Büchi automaton with $p \leq n$ states. Therefore the open ball $B(x, 2^{-n})$ is the intersection of the regular ω-languages $L(\mathcal{A}_i)$ for Büchi automata $\mathcal{A}_i$ having $p \leq n$ states and such that $x \in L(\mathcal{A}_i)$ and of the regular ω-languages $\Sigma^{\mathbb{N}} \setminus L(\mathcal{B}_i)$ for Büchi automata $\mathcal{B}_i$ having $p \leq n$ states and such that $x \notin L(\mathcal{B}_i)$. The class of regular ω-languages being closed under complementation and finite intersection, the open ball $B(x, 2^{-n})$ is actually a regular ω-language and thus an open set for τ_B.

3. Without loss of generality, we set $\Sigma = \{0,1\}$ and we consider, for a natural number $n \geq 1$, the ω-word $X_n = 0^{n!} \cdot 1 \cdot 0^{\omega}$ over the alphabet $\{0,1\}$ having only one symbol 1 after $n!$ symbols 0, where $n! := n \times (n-1) \times \cdots \times 2 \times 1$. Let now $m > n > k$ and $\mathcal{A}$ be a Büchi automaton with k states. Using a classical pumping argument, we can see that the automaton $\mathcal{A}$ cannot separate X_n and X_m. Indeed, assume first that $X_n \in L(\mathcal{A})$. Then, when reading the first k symbols 0 of X_n, the automaton enters at least twice in a same state q. This implies that there is a sequence of symbols 0 of length $p \leq k$ which can be added several times to the word X_n so that the resulting word will still be accepted by $\mathcal{A}$. Formally, any word $0^{n!+lp} \cdot 1 \cdot 0^{\omega}$, for a natural number $l \geq 1$, will be accepted by $\mathcal{A}$. In particular $m! = n! \times (n+1) \times \cdots \times m = n! + n! \times \Big(\big((n+1) \times \cdots \times m\big) - 1\Big)$ is of this form and thus $X_m \in L(\mathcal{A})$. A very similar pumping argument shows that if $X_m \in L(\mathcal{A})$, then $X_n \in L(\mathcal{A})$. This shows that $\delta(X_n, X_m) < 2^{-k}$ and finally that the sequence (X_n) is a Cauchy sequence

for the distance δ. On the other hand if this sequence was converging to an ω-word x then x should be the word 0^ω because τ_B is finer than τ_C. But 0^ω is an ultimately periodic word and thus it is an isolated point for τ_B. This would lead to a contradiction, and thus the distance δ is not complete because the sequence (X_n) is a Cauchy sequence which is not convergent. ◀

Proposition 11 gives a motivation for deriving Theorem 1 from Theorem 10. Note that the assumption of Theorem 10 are satisfied. Indeed, (P1)-(P3) come from Theorem 3. We now check (P4).

▶ **Lemma 12.** *Let Σ be a finite set with at least two elements, and $L \subseteq \Sigma^\mathbb{N}$ be an ω-regular language. Then there is a closed subset C of $\Sigma^\mathbb{N} \times \mathbb{P}_\infty$, which is ω-regular as a subset of $(\Sigma \times 2)^\mathbb{N}$ identified with $\Sigma^\mathbb{N} \times 2^\mathbb{N}$, and such that $L = \pi_0[C]$.*

Proof. Let $\mathcal{A} = (\Sigma, Q, \delta, Q_i, Q_f)$ be a Büchi automaton and let $L = L(\mathcal{A})$ be its set of accepted words. Let χ_f be the characteristic function of the final states. It maps each state q to 1 if $q \in Q_f$ and to 0 otherwise. The function χ_f is extended to $Q^\mathbb{N}$ by setting $\alpha = \chi_f((q_n)_{n\in\mathbb{N}})$ where $\alpha(n) = \chi_f(q_n)$. Note that a run ρ of $\mathcal{A}$ is accepting if and only if $\chi_f(\rho) \in \mathbb{P}_\infty$.

Let C be the subset of $\Sigma^\mathbb{N} \times \mathbb{P}_\infty$ defined by

$$C := \{(\sigma, \alpha) \in \Sigma^\mathbb{N} \times \mathbb{P}_\infty \mid \exists\rho \text{ run of } \mathcal{A} \text{ on } \sigma \text{ s. t. } \alpha = \chi_f(\rho)\}.$$

By definition of C, $L = \pi_0[C]$. Let K be the subset of $\Sigma^\mathbb{N} \times 2^\mathbb{N} \times Q^\mathbb{N}$ be defined by

$$K := \{(\sigma, \alpha, \rho) \in \Sigma^\mathbb{N} \times 2^\mathbb{N} \times Q^\mathbb{N} \mid \rho \text{ is a run of } \mathcal{A} \text{ on } \sigma \text{ s. t. } \alpha = \chi_f(\rho)\}.$$

Since K is compact as a closed subset of a compact space and $C = \pi_{\Sigma^\mathbb{N}\times 2^\mathbb{N}}[K] \cap (\Sigma^\mathbb{N} \times \mathbb{P}_\infty)$, the subset C is a closed subset of $\Sigma^\mathbb{N} \times \mathbb{P}_\infty$. It remains to show that C is indeed ω-regular. Let Δ be defined by

$$\Delta := \{(p, (a, \varepsilon), q) \in Q \times (\Sigma \times 2) \times Q \mid (p, a, q) \in \delta \wedge (\varepsilon = 1 \iff p \in Q_f)\}.$$

This allows us to define a Büchi automaton by $\mathcal{A}' := (\Sigma \times 2, Q, \Delta, Q_i, Q_f)$. Note that

$$\begin{aligned}
(\sigma, \alpha) \in L(\mathcal{A}') &\Leftrightarrow \exists (s_i)_{i\in\mathbb{N}} \in Q^\mathbb{N} \;\; \big(s_0 \in Q_i \;\wedge\; \forall i \in \mathbb{N} \;\; (s_i, (\sigma(i), \alpha(i)), s_{i+1}) \in \Delta\big) \;\wedge \\
&\hspace{18em} \forall k \in \mathbb{N} \;\; \exists i \geq k \;\; s_i \in Q_f \\
&\Leftrightarrow \alpha \in \mathbb{P}_\infty \;\wedge\; \exists (s_i)_{i\in\mathbb{N}} \in Q^\mathbb{N} \;\; \big(s_0 \in Q_i \;\wedge\; \forall i \in \mathbb{N} \;\; (s_i, \sigma(i), s_{i+1}) \in \delta \;\wedge \\
&\hspace{20em} (\alpha(i) = 1 \Leftrightarrow s_i \in Q_f)\big) \\
&\Leftrightarrow (\sigma, \alpha) \in C.
\end{aligned}$$

Thus $C = L(\mathcal{A}')$ is ω-regular. ◀

▶ **Corollary 13.** *Let Σ be a finite set with at least two elements. Then the Büchi topology τ_B is zero-dimensional and Polish.*

Proof. As there are only countably many possible automata (up to identifications), $\mathbb{B}_B$ is countable. This shows that τ_B is second countable. It is T_1 since it is finer than the usual topology by the property (P1), and strong Choquet by Theorem 10. Moreover, it is zero-dimensional since the class of ω-regular languages is closed under taking complements (see Theorem 3). It remains to apply Theorem 9. ◀

3.3 The other topologies

Proof of the main result. It is well known that $(\Sigma^{\mathbb{N}}, \tau_C)$ is metrizable and compact, and thus Polish.

- By Theorem 3.4 in [23], the implication (iii) $\Rightarrow$ (i), $\Delta^1_1(\Sigma^{\mathbb{N}})$ is a basis for a zero-dimensional Polish topology on $\Sigma^{\mathbb{N}}$. Recall that a Büchi Turing machine is **unambiguous** if every ω-word $\sigma \in \Sigma^{\mathbb{N}}$ has at most one accepting run. By Theorem 3.6 in [8], a subset of $\Sigma^{\mathbb{N}}$ is Δ^1_1 if and only if it is accepted by an unambiguous Büchi Turing machine. Therefore $\mathbb{B}_\delta = \Delta^1_1(\Sigma^{\mathbb{N}})$ is a basis for the zero-dimensional Polish topology τ_δ.
- Corollary 13 gives the result for the Büchi topology.
- Let (X, τ) be a Polish space, and $(C_n)_{n\in\mathbb{N}}$ be a sequence of closed subsets of (X, τ). By 13.2 in [16], the topology τ_n generated by $\tau \cup \{C_n\}$ is Polish. By 13.3 in [16], the topology τ_∞ generated by $\bigcup_{n\in\mathbb{N}} \tau_n$ is Polish. Thus the topology generated by $\tau \cup \{C_n \mid n \in \mathbb{N}\}$, which is τ_∞, is Polish. This shows that the automatic topology and the alphabetic topology are Polish since they refine the usual product topology on $\Sigma^{\mathbb{N}}$.
- Note that $S_{w,A}$ is a G_δ subset of $\Sigma^{\mathbb{N}}$, and thus a Polish subspace of $\Sigma^{\mathbb{N}}$, by 3.11 in [16]. Note also that $\Sigma^{\mathbb{N}} = \bigcup_{\emptyset\neq A\subseteq\Sigma} S_{\emptyset,A} \cup \bigcup_{\emptyset\neq A\subsetneqq\Sigma, w\in\Sigma^*, b\in\Sigma\setminus A} S_{wb,A}$, and that this union is disjoint. As all the sets in this union are open for the strong alphabetic topology, they are also clopen for this topology. This shows that this topology is the countable sum of its restrictions to the sets in this union. As the strong alphabetic topology coincides with the usual topology on each of these sets, it is Polish, by 3.3 in [16]. ◀

4 Consequences for our topologies

4.1 Consequences not directly related to the polishness, concerning isolated points

Notation. If $z \in \{C, \delta, B, A, \alpha, s\}$, then the space $(\Sigma^{\mathbb{N}}, \tau_z)$ is denoted $\mathcal{S}_z$. The set of ultimately periodic ω-words on Σ is denoted $\text{Ult} := \{u \cdot v^\omega \mid u, v \in \Sigma^* \setminus \{\emptyset\}\}$, and $P := \Sigma^{\mathbb{N}} \setminus \text{Ult}$.

1. As noted in [26], Ult is the set of *isolated points* of $\mathcal{S}_B$ and $\mathcal{S}_A$ (recall that a point $\sigma \in \Sigma^{\mathbb{N}}$ is isolated if $\{\sigma\}$ is an open set). Indeed, each singleton $\{u \cdot v^\omega\}$ formed by an ultimately periodic ω-word is an ω-regular language, and thus each ultimately periodic ω-word is an isolated point of $\mathcal{S}_A$. Conversely, if $\{\sigma\}$ is τ_B-open, then it is ω-regular and then the ω-word σ is ultimately periodic (because any countable ω-regular language contains only ultimately periodic ω-words, see [1, 25, 30]).
2. Every nonempty ω-regular set contains an ultimately periodic ω-word, [1, 25, 30]. In particular, the set Ult of isolated points of $\mathcal{S}_B$ and $\mathcal{S}_A$ is *dense* , and a subset of $\mathcal{S}_B$ or $\mathcal{S}_A$ is dense if and only if it contains Ult.
3. Let X be a topological space in which the set $\mathbb{I}$ of isolated points is dense, for example $\mathcal{S}_B$ by (2).
   (a) A subset of X is nowhere dense (i.e., its closure has empty interior) if and only if it is *meager* (i.e., it is a countable union of nowhere dense sets). indeed, a meager set does not meet $\mathbb{I}$.
   (b) Recall that a subset L of a topological space Y has the *Baire property* if there is an open subset O of Y such that the symmetric difference $L\Delta O := (L\setminus O) \cup (O\setminus L)$ is meager. This is equivalent to say that $L = G \cup M$, where G is G_δ (i.e. a countable intersection of open sets) and M is meager (see 8.23 in [16]). Every subset of X has the Baire property (which is very uncommon, see for exemple 8.24 in [16]). Indeed,

we can even say that every subset of X can be written $L = O \cup N$, where O is open and N is nowhere dense (O is $L \cap \mathbb{I}$, and N is $L \setminus \mathbb{I}$).

(c) Recall that a topological space is a *Baire space* if the intersection of countably many dense open sets is dense. By the Baire category theorem, every completely metrizable space is Baire. The space X is Baire in a very strong way: the intersection of any family of dense sets is dense, since a subset of X is dense if and only if it contains $\mathbb{I}$. This is very unusual, since for example we can find two disjoint countable dense sets in $\mathbb{R}$.

(d) A consequence of (b), (c) and 8.38 in [16] is that any map from X into a second countable Polish space is *continuous on a dense* G_δ subset of X. Note that if X is $\mathcal{S}_B$ or $\mathcal{S}_A$, then any map from X into a topological space is continuous on the dense open set Ult of ultimately periodic ω-words.

(e) There is a strong version of the *Kuratowski-Ulam theorem* (see 8.41 in [16]): assume that X and Y are topological spaces in which the set of isolated points is dense. Then this property also holds in the product $X \times Y$, so that a subset of $X \times Y$ is meager if and only if it contains no isolated point, which is also equivalent to the fact that for each isolated point $x \in X$ (resp., $y \in Y$), the vertical (resp., horizontal) section at x (resp., y) contains no isolated point. An interesting example is the case of an infinitary rational relation $R(\mathcal{A}) \subseteq \Sigma^{\mathbb{N}} \times \Gamma^{\mathbb{N}}$ accepted by a 2-tape Büchi automaton $\mathcal{A}$ which may be synchronous or asynchronous. Indeed if $\Sigma^{\mathbb{N}} \times \Gamma^{\mathbb{N}}$ is equipped with the product topology of the Büchi topologies on $\Sigma^{\mathbb{N}}$ and $\Gamma^{\mathbb{N}}$ then a non-empty rational relation is always non-meager. This follows easily from the fact that $\mathrm{Dom}\big(R(\mathcal{A})\big) = \{x \in \Sigma^{\mathbb{N}} \mid \exists y \in \Gamma^{\mathbb{N}} \ \ (x,y) \in R(\mathcal{A})\}$ is a regular ω-language. Thus $\mathrm{Dom}\big(R(\mathcal{A})\big)$ contains an ultimately periodic word x and then $\{y \in \Gamma^{\mathbb{N}} \mid (x,y) \in R(\mathcal{A})\}$ is also a non-empty regular ω-language and so it contains also an ultimately periodic word y.

4.2 Consequences of the polishness

4.2.1 Consequences related to Cantor-Bendixson Theorem

We consider our topologies on $\Sigma^{\mathbb{N}}$, where Σ is a finite set with at least two elements. We give a (non exhaustive) list of results. We often refer to [16] when classical descriptive set theory is involved. The reader may read this book to see many other results.

4. (a) The union $P \cup \mathrm{Ult}$ is the *Cantor-Bendixson decomposition* of $\mathcal{S}_B$ and $\mathcal{S}_A$ (see 6.4 in [16]). This means that P is perfect (i.e., closed without isolated points) and Ult is countable open. Let us check that P is perfect. We argue by contradiction, so that we can find $\sigma \in P$ and an ω-regular language L such that $\{\sigma\} = L \setminus \mathrm{Ult}$. Note that $L \subseteq \{\sigma\} \cup \mathrm{Ult}$ is countable. But a countable regular ω-language contains only ultimately periodic words (see [25]), and thus $L \subseteq \mathrm{Ult}$, which is absurd.

(b) The closed subspace (P, τ_B) of $\mathcal{S}_B$ is *homeomorphic to the Baire space* $\mathbb{N}^{\mathbb{N}}$. Indeed, it is not empty since Ult is countable and $\Sigma^{\mathbb{N}}$ is not, zero-dimensional and Polish as a closed subspace of the zero-dimensional Polish space $\mathcal{S}_B$. By 7.7 in [16], it is enough to prove that every compact subset of (P, τ_B) has empty interior. We argue by contradiction, which gives a compact set K. Note that there is an ω-regular language L such that $P \cap L$ is a nonempty compact subset of K, so that we may assume that $K = P \cap L$. Theorem 2 gives $(U_i)_{i<n}$ and $(V_i)_{i<n}$ with $L = \bigcup_{i<n} U_i \cdot V_i^\omega$. On the other hand, L is not countable since every countable regular ω-language contains only ultimately periodic words and $K = P \cap L$ is non-empty. Thus $n > 0$ and, for

example, $U_0V_0^\omega$ is not countable.
This implies that we can find $v_0, v_1 \in V_0$ which are not powers of the same word. Indeed, we argue by contradiction. Let $v_0 \in V_0 \setminus \{\emptyset\}$, and $v \in \Sigma^*$ of minimal length such that v_0 is a power of v. We can find $w_0, w_1, ... \in V_0 \setminus \{\emptyset\}$ such that $\sigma := w_0w_1... \neq v^\omega$. Fix a natural number i. Then w_i and v_0 are powers of the same word w. By Corollary 6.2.5 in [22], v and w are powers of the same word u, and v_0 too. By minimality, $u = v$, and w_i is a power of v. Thus $\sigma = v^\omega$, which is absurd.
Let $u_0 \in U_0$, and $L' := \{u_0\}\cdot\{v_0, v_1\}^\omega$. Note that $P \cap L'$ is a τ_B-closed subset of K, so that it is τ_B-compact. As the identity map from $(P \cap L', \tau_B)$ onto $(P \cap L', \tau_C)$, (where τ_C is the Cantor topology), is continuous, $P \cap L'$ is also τ_C-compact. But the map $\alpha \mapsto u_0v_{\alpha(0)}v_{\alpha(1)}\ldots$ is a homeomorphism from the Cantor space onto L', by Corollaries 6.2.5 and 6.2.6 in [22]. Thus $P \cap L' = L' \setminus \mathrm{Ult}$ is a dense closed subset of L'. Thus $P \cap L' = L'$, which is absurd since $u_0 \cdot v_0^\omega \in L' \setminus P$.

(c) By (b) and 7.9 in [16], for each Polish space Y, we can find a τ_B-closed set $F \subseteq P$ (an intersection of ω-regular sets) and a continuous bijection from F onto Y. In particular, if Y is not empty, then there is a *continuous surjection* from P onto Y extending the previous bijection. By 7.15 in [16], if moreover Y is perfect, then there is a continuous bijection from P onto Y.

(d) A consequence of (b), Corollary 13, and 7.10 in [16], is that the space $\mathcal{S}_B$ **is not** K_σ (i.e., countable union of compact sets).

4.2.2 Consequences related to Borel sets

5. (a) The *σ-algebra* $\mathcal{A}_z$ generated by $\mathbb{B}_z$ is exactly the σ-algebra of Borel subsets of $\mathcal{S}_C$ (generated by the open subsets of $\mathcal{S}_C$). Indeed, the identity map from $\mathcal{S}_z$ onto $\mathcal{S}_C$ is a continuous bijection. By 15.2 in [16], the inverse map is Borel, so that the two Borel structures coincide. It remains to note that the σ-algebra generated by $\mathbb{B}_z$ is exactly the σ-algebra of Borel subsets of $\mathcal{S}_z$.

(b) We can define a natural *hierarchy* in $\mathcal{A}_z$: let $\Sigma_1^z := (\mathbb{B}_z)_\sigma$ be the set of countable unions of elements of $\mathbb{B}_z$, and, inductively on $1 \leq \xi < \omega_1$, $\Pi_\xi^z := \{\neg L \mid L \in \Sigma_\xi^z\}$ and $\Sigma_\xi^z := (\bigcup_{\eta<\xi} \Pi_\eta^z)_\sigma$. This hierarchy is actually the Borel hierarchy in the family of Borel subsets of $\mathcal{S}_z$, in the sense that $\Sigma_\xi^z = \mathbf{\Sigma}_\xi^0(\mathcal{S}_z)$ and $\Pi_\xi^z = \mathbf{\Pi}_\xi^0(\mathcal{S}_z)$ (see 11.B in [16]; we will also consider $\Delta_\xi^z := \Sigma_\xi^z \cap \Pi_\xi^z = \mathbf{\Delta}_\xi^0(\mathcal{S}_z)$ and $\mathbf{\Pi}_0^0(\mathcal{S}_z) := \mathbb{B}_z$). By the main result and 22.4 in [16], this hierarchy is strict and of length ω_1 (the first uncountable ordinal). This comes from the existence of universal sets for these classes (see 22.3 in [16]).
Recall that a subset $\mathcal{U}$ of $2^\mathbb{N} \times \Sigma^\mathbb{N}$ is **universal** for a class Γ of subsets of $\Sigma^\mathbb{N}$ if it is in Γ and the Γ subsets of $\Sigma^\mathbb{N}$ coincide with the vertical sections $\mathcal{U}_x = \{y \in \Sigma^\mathbb{N} \mid (x, y) \in \mathcal{U}\}$ of $\mathcal{U}$.
Note that a set is Borel for τ_C if and only if it is Borel for any topology τ_z, but the levels of the Borel hierarchy may differ for the two topologies. For instance any singleton associated with an ultimately constant word is not open for the Cantor topology and is actually open for the δ, Büchi, automatic, alphabetic and strong alphabetic topologies. Therefore the existence of universal sets for each level of the Borel hierarchy of $\mathcal{S}_z$ is different from the existence of universal sets for each level of the Borel hierarchy of $\mathcal{S}_C$ if $z \neq C$, and could lead to other consequences or new applications to other domains of theoretical computer science.

(c) The *Wadge theorem* holds (see 22.10 in [16]): let $z \in \{\delta, B\}$, $\xi \geq 1$ be a countable ordinal. A subset L of $\Sigma^\mathbb{N}$ is in $\Sigma_\xi^z \setminus \Pi_\xi^z$ if and only if it is in Σ_ξ^z, and for each zero-dimensional Polish space X and any $\mathbf{\Sigma}^0_\xi$ subset B of X there is $f : X \to \mathcal{S}_z$ continuous with $B = f^{-1}(L)$ (we can exchange Σ^z_ξ and Π^z_ξ).

(d) The *Saint Raymond theorem* holds (see 35.45 in [16]): let Σ, Γ be finite sets with at least two elements, $L \subseteq \Sigma^{\mathbb{N}} \times \Gamma^{\mathbb{N}}$ be in $\mathcal{A}_z$ with Σ^z_2 vertical sections. Then $L = \bigcup_{n\in\mathbb{N}} L_n$, where L_n is in $\mathcal{A}_z$ and has Π^z_1 vertical sections.

(e) By 4.(b) and 13.7 in [16], for each L in $\mathcal{A}_B$, we can find a τ_B-closed set $F \subseteq P$ (an intersection of ω-regular sets) and a continuous bijection from F onto L. In particular, if L is not empty, then there is a *continuous surjection* from P onto L extending the previous bijection.

4.2.3 Consequences related to analytic sets

6. Recall that the analytic sets are the projections of the elements of $\mathcal{A}_z$. We saw in (5).(a) that τ_z and τ_C have the same Borel sets. This implies that τ_z and τ_C have the same *analytic* sets. Then the following items (a)-(d) are not new, but we cite them as interesting results about the topology τ_z.

(a) The class of analytic sets contains strictly A_z because of the existence of universal sets (see 14.2 in [16]).

(b) The *Lusin separation theorem* holds (see 14.7 in [16]): assume that $L, M \subseteq \Sigma^{\mathbb{N}}$ are disjoint analytic sets. Then there is S in $\mathcal{A}_z$ such that $L \subseteq S \subseteq \neg M$ (we say that S *separates* L from M).

(c) The *Souslin theorem* holds (see 14.11 in [16]): the elements of $\mathcal{A}_z$ are exactly the analytic sets whose complement is also analytic.

(d) By 14.13 in [16], an analytic set L has the *perfect set property* : either L is countable, or L contains a copy of $\mathcal{S}_C$ (this copy has size continuum and is compact).

(e) The *parametrization theorem for Borel sets* holds (see 35.5 in [16]). We say that a set is *co-analytic* if its complement is analytic. We can find a co-analytic subset $\mathcal{D}$ of $2^{\mathbb{N}}$, an analytic subset $\mathcal{S}$ of $2^{\mathbb{N}} \times \Sigma^{\mathbb{N}}$, and a co-analytic subset $\mathcal{P}$ of $2^{\mathbb{N}} \times \Sigma^{\mathbb{N}}$ such that the vertical sections $\mathcal{S}_\alpha$ and $\mathcal{P}_\alpha$ of $\mathcal{S}$ and $\mathcal{P}$ at any point α of $\mathcal{D}$ are equal, and the elements of $\mathcal{A}_z$ are exactly the sets $\mathcal{S}_\alpha$ for some α in $\mathcal{D}$.

4.2.4 Consequences related to uniformization problems

7. Let X, Y be sets, $L \subseteq X \times Y$, and $f : X \to Y$ be a partial map. We say that f *uniformizes* L if the domain of f is $\pi_0[L]$ and the graph of f is contained in L. In the sequel, Σ and Γ will be finite sets with at least two elements.

(a) Recall that if $L \subseteq \Sigma^{\mathbb{N}} \times \Gamma^{\mathbb{N}}$ is an infinitary rational relation, then L can be uniformized by a function whose graph is ω-rational (see Theorem 5 in [2]). Moreover, the inverse image of an ω-regular language by such a function is itself ω-regular, and thus the function is continuous for τ_B.

(b) The *Arsenin-Kunugui theorem* holds (see 18.18 in [16]): if $L \subseteq \Sigma^{\mathbb{N}} \times \Gamma^{\mathbb{N}}$ is in $\mathcal{A}_z$ and has K_σ vertical sections, then L can be uniformized by a function whose graph is in $\mathcal{A}_z$ (a K_σ set is a countable union of compact sets).

(c) Let $\xi \geq 1$ be a countable ordinal. The class $\mathbf{\Sigma}^0_\xi$ has the *number uniformization property* : if X is a zero-dimensional Polish space and $L \subseteq X \times \mathbb{N}$ is $\mathbf{\Sigma}^0_\xi$, then L can be uniformized by a function whose graph is $\mathbf{\Sigma}^0_\xi$. Consequently, for $z \in \{\delta, B\}$,

- Σ^z_ξ has the *generalized reduction property* : if $(L_n)_{n\in\mathbb{N}}$ is a sequence of Σ^z_ξ subsets of $\Sigma^{\mathbb{N}}$, then there is a sequence $(B_n)_{n\in\mathbb{N}}$ of pairwise disjoint Σ^z_ξ subsets of $\Sigma^{\mathbb{N}}$ with $\bigcup_{n\in\mathbb{N}} B_n = \bigcup_{n\in\mathbb{N}} L_n$ and $B_n \subseteq L_n$,

- Π^z_ξ has the *generalized separation property* : if $(L_n)_{n\in\mathbb{N}}$ is a sequence of Π^z_ξ subsets of $\Sigma^\mathbb{N}$ with empty intersection, then there is a sequence $(B_n)_{n\in\mathbb{N}}$ of Δ^z_ξ subsets of $\Sigma^\mathbb{N}$ with empty intersection and $B_n \supseteq L_n$ (see 22.16 in [16]).

Moreover,

- the class of co-analytic sets has the number uniformization property and the generalized reduction property,
- the class of analytic sets has the generalized separation property (see 35.1 in [16]).

(d) The *Novikov-Kondo theorem* holds (see 36.14 in [16]): if $L \subseteq \Sigma^\mathbb{N} \times \Gamma^\mathbb{N}$ is co-analytic, then L can be uniformized by a function whose graph is co-analytic.

5 Concluding Remarks

We have obtained in this paper new links and interactions between descriptive set theory and theoretical computer science, showing that four topologies considered in [26] are Polish, and providing many consequences of these results.

Notice that this paper is also motivated by the fact that the Gandy-Harrington topology, generated by the effective analytic subsets of a recursively presented Polish space, is an extremely powerful tool in descriptive set theory. In particular, this topology is used to prove some results of classical type (without reference to effective descriptive set theory in their statement). Among these results, let us mention the dichotomy theorems in [13, 17, 19, 20]. Sometimes, no other proof is known. Part of the power of this technique comes from the nice closure properties of the class Σ^1_1 of effective analytic sets (in particular the closure under projections).

The class of ω-regular languages has even stronger closure properties. So our hope is that the study of the Büchi topology, generated by the ω-regular languages, will help to prove some automatic versions of known descriptive results in the context of theoretical computer science.

From the main result, we know that there is a complete **distance** which is compatible with τ_z. It would be interesting to have a natural complete distance compatible with τ_B. We leave this as an open question for further study.

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