Current Trends and New Perspectives for First-Order Model-Checking*

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Abstract

The model-checking problem for a logic $\mathcal{L}$ is the problem of decidig for a given formula $\varphi \in \mathcal{L}$ and structure $\mathfrak{A}$ whether the formula is true in the structure, i.e. whether $\mathfrak{A} \models \varphi$.

Model-checking for logics such as *First-Order Logic* (FO) or *Monadic Second-Order Logic* (MSO) has been studied intensively in the literature, especially in the context of algorithmic meta-theorems within the framework of parameterized complexity. However, in the past the focus of this line of research was model-checking on classes of *sparse* graphs, e.g. planar graphs, graph classes excluding a minor or classes which are nowhere dense. By now, the complexity of first-order model-checking on sparse classes of graphs is completely understood. Hence, current research now focusses mainly on classes of dense graphs.

In this talk we will briefly review the known results on sparse classes of graphs and explain the complete classification of classes of sparse graphs on which first-order model-checking is tractable. In the second part we will then focus on recent and ongoing research analysing the complexity of first-order model-checking on classes of dense graphs.

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1 Introduction

The *model checking problem* MC($\mathcal{L}$) for a logic $\mathcal{L}$ is the problem to decide, given as input a formula $\varphi \in \mathcal{L}$ and a structure $\mathfrak{A}$, whether φ is true in $\mathfrak{A}$. Often the model checking problem is relativised to a particular class $\mathcal{C}$ of structures. We define MC($\mathcal{L}, \mathcal{C}$) as the restriction of MC($\mathcal{L}$) to structures in $\mathcal{C}$.

Understanding the model checking complexity for important logical systems such as first-order (FO) and monadic second-order logic (MSO) but also various temporal logics has been one of the prominent topics in finite and computational model theory for decades. Prominent applications of model checking include database systems or computer aided verification, where logical methods are prevalent.

For classical logics such as first-order logic (FO) or monadic second-order logic (MSO) the model checking problem is known to be computationally intractable: it is PSPACE-complete for FO and MSO. These hardness results even hold in restriction to structures with only two

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elements and no relations. As in most "model centric" contexts two element structures are not particularly interesting, it has become standard to measure the complexity of model checking problems in a more refined complexity framework, especially in the framework of parameterized complexity.

In the parameterized setting, the goal is to develop algorithms deciding $\mathrm{MC}(\mathcal{L}, \mathcal{C})$ on input $\varphi \in \mathcal{L}$ and $\mathfrak{A} \in \mathcal{C}$ in time $f(|\varphi|) \cdot |\mathfrak{A}|^c$, where f is a computable function and c is a constant. Problems that can be solved in this time are called *fixed-parameter tractable* (fpt). If, furthermore, $c = 1$ they are called *fixed-parameter linear*.

If $\mathrm{MC}(\mathcal{L}, \mathcal{C})$ is fpt, then a fixed formula φ can be evaluated efficiently even in very large structures, as the running time in the size of $\mathfrak{A}$ is only linear or polynomial. This is not only a theoretical observation but seems to be reflected also in practical applications. For instance, model checking for one of the widely used temporal logics, the *linear time logic* (LTL), is PSPACE-complete. But it is fixed-parameter tractable and it can be solved very efficiently in real world applications.

For logics such as MSO or FO, it is not hard to see that their model checking problem is not fixed-parameter tractable in general.[1] On the other hand, Courcelle [2] showed that if $\mathcal{C}$ is a class of structures of bounded *tree width*, then $\mathrm{MC}(\mathrm{MSO}, \mathcal{C})$ is fixed-parameter linear, even if quantification over sets of edges as well as sets of vertices is allowed (this logic is called MSO_2 whereas MSO_1 only has quantification over sets of vertices in a graph).

MSO_2 is a very powerful language in which many natural graph theoretical problems can be formulated easily and naturally. It has attracted significant interest in the parameterized graph algorithms community, especially in a field known as algorithmic graph structure theory where researchers develop algorithms for NP-hard problems that become efficient for special classes of graphs, such as classes with forbidden minors. Initially, Courcelle's theorem served as a quick and easy way of proving that a problem is linear time solvable for classes of bounded tree width. Subsequently it has become a valuable tool in parameterized algorithms research for proving general meta-tractability results such as meta-kernels and the core concepts of the model-theoretic proof of Courcelle's theorem have found its way into the standard tool set of parameterized graph algorithmics, e.g. in the form of *protrusions*. Furthermore, Langer et al. [16] showed that an MSO_2-solver can be implemented and used for solving real-world problems. Hence, MSO_2 can now even be used as a high-level programming language for graph problems.

As mentioned above, for logics such as FO or MSO, $\mathrm{MC}(\mathcal{L}, \mathcal{C})$ is (presumably) not fixed-parameter tractable on the class of all finite structures or graphs but it is tractable if $\mathcal{C}$ has bounded tree width. This raises the natural question for a logic $\mathcal{L}$ such as $\mathrm{FO}, \mathrm{MSO}_1, \mathrm{MSO}_2$:

> *What are the largest classes $\mathcal{C}$ of graphs or structures for which* $\mathrm{MC}(\mathcal{L}, \mathcal{C})$ *is still fixed-parameter tractable.*

Or:

> *Can we identify a structural property P such that* $\mathrm{MC}(\mathcal{L}, \mathcal{C})$ *is fixed-parameter tractable for a class $\mathcal{C}$ if, and only if, $\mathcal{C}$ has property P?*

Such a characterisation (of course subject to complexity theoretic assumptions) would be extremely interesting as it

[1] Here and elsewhere in this abstract we assume the standard hypotheses in complexity theory such as $\mathrm{P} \neq \mathrm{NP}$ and a similar assumption in parameterized complexity theory and our hardness results are subject to these assumptions.

(a) completely explains what makes model checking for this logic hard,
(b) identifies a set of algorithmic techniques that can be employed in the tractable cases to evaluate formulas quickly, and
(c) provided that the property P is a natural and efficiently decidable parameter, yields a quick way of checking whether model checking for the logic $\mathcal{L}$ is a particular application determined by the class $\mathcal{C}$ of structures is feasible.

This leads to a systematic research programme aiming at identifying such characterisations of tractable cases in terms of parameters P for the major logics such as FO, MSO_1 and MSO_2.

Following Courcelle's theorem, this research programme has received significant attention in the area. In [15, 14] it was proved in that fixed-parameter tractability of MSO_2 cannot be extended much beyond the case of bounded tree width. This does not establish a tight characterisation but shows that tree width seems to be the characterising parameter for MSO_2 tractability.

For MSO_1, i.e. monadic second-order logic with quantification over sets of vertices only, it was proved by Courcelle et al. [3] that $\text{MC}(\text{MSO}_1, \mathcal{C})$ is fixed-parameter tractable on any class $\mathcal{C}$ of graphs of bounded clique width. Here, no matching lower bound is known and to date the right graph theoretical tools to establish such a bound are still missing.

Most research, however, has gone into understanding the tractable cases for first-order model checking. For a long time much of this research has focussed on sparse classes of graphs[2]. Seese [20] showed that $\text{MC}(\text{FO}, \mathcal{C})$ is fpt for classes $\mathcal{C}$ of graphs of bounded maximum degree. The result was generalised by Flum and Grohe to classes with excluded minors [7] and to classes of bounded local tree width by Flum, Frick and Grohe [6]. This was followed by a series of related results, e.g. in [5, 4, 21]. See also [11] for a survey of earlier work in this area.

Finally, the case for sparse classes of graphs was settled completely in [12], where it was proved that if $\mathcal{C}$ is a class of graphs that is closed under taking subgraphs then $\text{MC}(\text{FO}, \mathcal{C})$ is fixed-parameter tractable if, and only if, $\mathcal{C}$ is *nowhere dense*. Nowhere dense classes of graphs have been introduced by Nešetřil and Ossonda de Mendez [18, 19, 17] as a model for *sparseness*. A huge number of results that have been obtained in recent years support their claim that nowhere denseness captures structural sparsity.

The result in [12] completely characterises the tractable cases of first-order model checking for classes of structures closed under substructures by a simple and natural parameter. Consequently, current research activities focus on model checking on dense classes of graphs. For instance, Ganian et al. [10] showed that first-order model checking is fpt on certain classes of interval graphs. And Bova et al. [1] proved that model checking for existential FO is fpt on classes of partial orders. This was later generalised by Gajarsky et al. [8] to full FO.

These examples demonstrate tractability of first-order model checking on classes of graphs that were already well established and studied in different contexts.

A different approach is to study transformations of graph classes that preserve efficient model checking. The most notably of these are first-order interpretations. A currently very active topic in this field is to study the closure of sparse classes of graphs under first-order

[2] We call a class of graphs *sparse* if the number of edges is essentially linear in the number of vertices. There are several mathematical concepts trying to define sparseness formally, including *bounded average degree*, *degeneracy* or *nowhere dense classes* of graphs. Usually, sparse classes of graphs can be closed under taking subgraphs, i.e. if $G \in \mathcal{C}$ and $H \subseteq G$ then $H \in \mathcal{C}$. The obvious exception is bounded average degree.

interpretations designed in a way that the tractability of first-order model checking is carried over from the sparse to the dense class. This route was for instance taken by Gajarsky et al. who studied the interpretation closure of classes of graphs of bounded degree and obtained a very elegant solution to this problem. See e.g. [9].

For the sparse setting, model checking for FO was studied on classes of graphs closed under taking subgraphs. This is a perfectly natural operation in the sparse setting. For dense graphs, it seems equally natural to consider classes of graphs closed under induced subgraphs. A third approach therefore is to study graph parameters which for sparse classes of graphs are equivalent to nowhere denseness but are also well defined for classes closed under induced subgraphs. One such parameter is *stability.* Stable classes of graphs hold the promise to be a very interesting class for model checking and other algorithmic purposes. See e.g. [13]. But there is still a lot of work required to establish their basic algorithmic properties.

While currently there is significant progress and activity in the study of first-order model checking on dense classes of graphs, we are still very far from a precise characterisation of the dense classes of graphs with tractable model checking.

In this talk we will briefly review the results obtained for model-checking problems in the sparse setting and then present the new ideas and results and future directions for first-order model checking on dense classes of graphs.

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