SUPERSET: A (Super)Natural Variant of the Card Game SET

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Abstract

We consider SUPERSET, a lesser-known yet interesting variant of the famous card game SET. Here, players look for SUPERSETs instead of SETs, that is, the symmetric difference of two SETs that intersect in exactly one card. In this paper, we pose questions that have been previously posed for SET and provide answers to them; we also show relations between SET and SUPERSET.

For the regular SET deck, which can be identified with $\mathbb{F}_3^4$, we give a proof for the fact that the maximum number of cards that can be on the table without having a SUPERSET is 9. This solves an open question posed by McMahon et al. in 2016. For the deck corresponding to $\mathbb{F}_3^d$, we show that this number is $\Omega(1.442^d)$ and $O(1.733^d)$. We also compute probabilities of the presence of a superset in a collection of cards drawn uniformly at random. Finally, we consider the computational complexity of deciding whether a multi-value version of SET or SUPERSET is contained in a given set of cards, and show an FPT-reduction from the problem for SET to that for SUPERSET, implying W[1]-hardness of the problem for SUPERSET.

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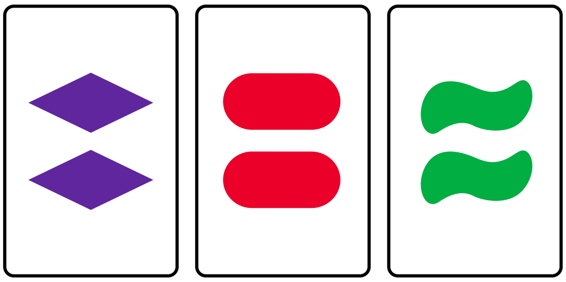


Figure 1 A SET is a collection of three cards that, in each of the properties, are either identical or distinct.

1 Introduction

The famous [1, 20] game SET[1] [5, 28] is played with cards that have four attributes each: The number, type, color, and shading of displayed shape(s). Each of these attributes can take three values, and each of the possible $3^4 = 81$ combinations of these values is contained exactly once as a card in the deck. A SET is a collection of three cards that, in each of the properties, are either identical or distinct (see Fig. 1). Among the cards that are laid out on the table, all players have to simultaneously find SETs as fast as possible. While possibly not evident from this description, we can assure you that the game is fun, even for a wider audience [31], including cats [24, Figure I.5].

However, as players get better and faster, the game becomes quite fidgety and arguably less fun. One straightforward way of making the game more difficult and thus slowing it down is adding more properties to the cards. Unfortunately, this creates decks increasing exponentially in size and possibly odor [14]. Other variants have been proposed [6, 14, 23, 24], but full-contact SET [6] seems only remotely related to mathematics, and projective SET [14] requires a completely different deck and seems incompatible with our gerontocracy [14].

Instead, we started playing a variant that is sufficiently more difficult, shows a rich mathematical structure, and can be played with the same, typically odor-free, deck of cards as SET: It is easy to see that for any pair of cards there exists exactly one *missing* card that completes the pair to a SET. Any single card, however, serves as missing card for (actually 40) different pairs. In the variant of SET considered here, the four cards from two pairs with the same missing card form the object (see Fig. 2) that the players look for instead of SETs. Until recently, when it was published in a book [24], this variant seems to have been spread mostly by word of mouth (as it was to one of the authors [7]), and appeared under different names on the Internet [6, 11, 13, 23, 32].

To continue the (admittedly short) tradition of overloading[2] mathematical terms, we choose to call the new object and the emerging variant of the game SUPERSET. We consider natural questions regarding SUPERSET: How many cards can be on the table without having a SUPERSET? More generally, what is the probability of having a SUPERSET in a collection of cards of a certain size chosen uniformly at random? What is the computational complexity of finding a SUPERSET? Are there any further connections between SUPERSET and SET?

[1] SET is a registered trademark of Cannei, LLC. The distinctive SET symbols and cards are copyrights of Cannei, LLC. All rights reserved. Used with permission from Set Enterprises, Inc.

[2] Since we typeset the two central objects of this paper as SET and SUPERSET, at least in written language we do not overload the corresponding mathematical terms. We however avoid using these mathematical terms (and thereby hopefully confusion) in this paper.

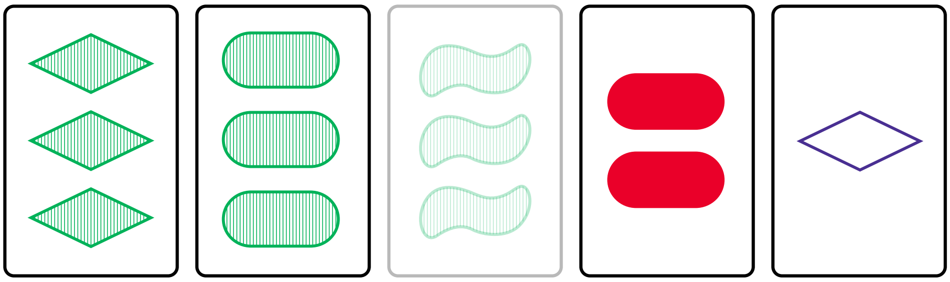


Figure 2 A SUPERSET is defined to be the symmetric difference of two SETs that intersect in exactly one card. Here, cards 1–3 and 3–5 form the two SETs; all but the third card form the SUPERSET.

These questions have, in fact, been considered for SET already, which is partly due to the applications of its study to affine spaces [14, 24], Fourier analysis [4], and error-correcting codes [19]. Clearly, the study of SUPERSET has the corresponding superapplications.

Related work. We briefly survey results related to SET. For a very accessible and lovely introduction to the mathematics of SET, we refer to McMahon et al. [24]. An also very well-written and at the time fairly comprehensive survey for mathematically more versed readers was published by Davis and Maclagan in 2003 [14].

When playing SET, one typically deals 12 cards and then looks for SETs among them. Sometimes, however, there turns out to be no SET, naturally leading to the question: How many cards have to be dealt to *guarantee* that there is a SET on the table? This question was in fact answered *before* the invention of the game SET in 1974, which is due to the following connection: One can naturally identify the deck of cards with the vector space $\mathbb{F}_3^4$ (by identifying the components with the properties) and then SETs in the deck of cards simply correspond to lines in $\mathbb{F}_3^4$. So the above question is equivalent to asking what is the maximum size of a *cap*, that is, a line-free collection of elements, in $\mathbb{F}_3^4$. This number was settled to be 20 in 1971 [26]. The most elegant proof known to date is based on counting so-called marked hyperplanes in two different ways, making use of the symmetries of the vector space [14].

It is natural to ask the same question for $\mathbb{F}_3^d$ with different d, which translates to a restricted or extended deck of cards. While this question for $d < 4$ can be easily answered using the same techniques as for $d = 4$ [14], only in 2002 did two breakthrough papers [4, 15] settle the maximum cap size for $d = 5$ to be 45 by relating the problem to the Fourier transform. For $d = 6$, the maximum cap size is 112 [27], as shown by the techniques similar to those used for $d \leq 4$ [14] along with a computer search, but the same paper claims that the Fourier-transform techniques could be used instead. Interestingly, at least up to $d = 6$, all maximum caps are from the same affine equivalence class, i.e., between any two of them there is an affine transformation that maps one to the other [15, 18, 24, 27].

Finding maximum supercaps for increasingly larger fixed integers d will probably keep (parts of) humanity busy for a while, but yet more forward-looking works have considered the asymptotic behavior of the maximum cap size as $d \to \infty$. While $\{0,1\}^d$ is easily seen to be a cap of size 2^d, more sophisticated product constructions [8, 16] yield caps of size $\Omega(2.217^d)$. On the upper-bound side, Fourier transforms yield $O(3^d/d)$ [25] and further far-from-trivial insights about the spectrum yield $O(3^d/d^{1+\varepsilon})$ for some $\varepsilon > 0$ [3]. However, truly improving (i.e., in terms of the base of the exponential) upon the trivial upper bound of $O(3^d)$ has been a famous open problem [29] until recently. Only in 2016, the so-called polynomial method [12] was utilized [17, 30] to show quite compactly that the maximum cap size is $O(2.756^d)$.

Table 1 Bounds on the maximum cap and supercap sizes.

dimension	1	2	3	4	5	⋯	d
maximum cap size	2	4	9	20	45	⋯	$\Omega(2.217^d) \cap O(2.756^d)$
maximum supercap size	3	4	6	9	14	⋯	$\Omega(1.442^d) \cap O(1.733^d)$

One may also wonder what the probability of the presence of a SET is in the initial layout of cards or, more generally, in a k-element collection chosen uniformly at random from $\mathbb{F}_3^d$. This question has been answered exactly for small values of k and d [24] and has been considered computationally for arbitrary values of k and d [21, 24]. An overview of the vast amount of interesting probabilistic questions, for example, the one for the expected number of SETs, can be found in the book of MacMahon et al. [24].

Let us look at the problem of deciding whether a given set of cards has a SET. As this question is boring in terms of asymptotic running time for $\mathbb{F}_3^4$, we first consider $\mathbb{F}_3^d$ where d is part of the input. Saving all cards in a dictionary and then checking the dictionary for the missing card of each pair yields an $O(n^2)$-time randomized algorithm and an $O(n^2 \log n)$-time deterministic algorithm [9]. Despite obvious resemblance to the 3SUM problem (three elements $a, b, c \in \mathbb{F}_3^d$ form a SET iff $a+b+c=0$), 3SUM-hardness has only been conjectured [9].

To still be able to write computational-complexity papers about the problem, $\mathbb{F}_v^d$ has been considered, where v is part of the input as well. Note that a SET for a larger number of values per property can be defined either through lines (line SETs, at least for v prime) or by asking for identical or distinct values in each component (combinatorial SETs), but both variants are *different* [14] (see Section 2). Indeed, defining SETs through lines only adds a factor of v in the running time (because all v elements on the line have to be checked), but the problem is NP-hard for combinatorial SETs, as shown by a reduction from a multi-dimensional matching problem [9]. This result has been subsequently improved to W[1]-hardness for parameter v [22].

Unfortunately, none of these results is super enough yet. Yet, to date SUPERSET has not been rigorously studied. There are only a few Internet sources: experiments showing that there is a collection of 9 cards for $d = 4$ that does not have a SUPERSET [32] and some providing estimates for the probabilities of the presence of a superset in random collections [13, 32].

Our contribution. Analogous to the study of caps, we initiate the (rigorous) study of *supercaps*, that is, collections of cards that do not contain SUPERSETs. The same as for caps, we are interested in the maximum size of supercaps, but our techniques are different. For $d = 2$, by simply counting the number of pairs within the supercap and then using the pigeonhole principle, one can easily show an upper bound of 4 on the size of any supercap. A bound that can be easily matched from below by hand. In three dimensions, the same upper-bound technique allows us to prove an upper bound of 7. Yet, constructed lower-bound examples imply a maximum size of 6. To prove an upper bound of 6, we develop a refined counting technique that is based on the following observation [2, 11, 24]. If two pairs of points are disjoint and their induced vectors are parallel, then they form a SUPERSET (see Lemma 1). In $\mathbb{F}_3^4$, which corresponds to the actual SET deck, we use the same counting technique and a relatively short case distinction to show an upper bound of 9, which is tight [2, 32] and thus solves an open problem [24, Question 8]. For $d = 5$, using the same techniques, we can narrow down the maximum supercap size to at most 16 and at least 14, but an exhaustive computer search shows that the maximum supercap size is indeed 14.

Regarding asymptotic results, we utilize the simple pigeonhole-principle technique to obtain a non-trivial upper bound of $O(3^{d/2}) \subset O(1.733^d)$ on the cardinality of any supercap. To obtain a non-trivial lower bound, we essentially analyze an algorithm that greedily adds elements to the supercap. The critical observation is that each card that we cannot add is excluded by a triple of cards with which it would form a SUPERSET. Counting the number of triples and the number of elements that each triple can exclude, we obtain a lower bound of $\Omega(3^{d/3}) \subset \Omega(1.442^d)$. We summarize our results on maximum supercaps along with those known for maximum caps in Table 1.

The preceding structural insights about supercaps suffice to compute the probability of the presence of the a SUPERSET in a k-element collection of elements from $\mathbb{F}_3^d$ when $k = 4$ and d is arbitrary (or, alternatively, the probability of the collection being a supercap). We also manage to compute the same probability for $d = 3$ and $k = 5, 6$, which is based on a complete characterization of the corresponding supercaps. Since this characterization is already fairly complicated, we do not compute the exact probabilities for $d = 4$ and $k > 5$ but experimentally determine estimates (10^7 samples for every k).

To consider the computational complexity of deciding whether a given collection of cards has a SUPERSET, we define a SUPERSET (more generally) to be the symmetric difference of two (line or combinatorial) SETs that intersect in exactly one element. We note that the polynomial-time algorithm known for deciding if a given collection of cards from $\mathbb{F}_v^d$ (for v possibly fixed) has a SET can be essentially generalized to the corresponding problem for SUPERSET: One iterates through the pairs. If at least $v - 3$ other elements of the emerging line are present, one then checks a dictionary for entries of the corresponding missing card(s) and, if there are none, one saves the missing card(s) there. We then establish a close relation between the two problems by providing an FPT reduction from the problem for combinatorial SET to the corresponding one for SUPERSET, so that the W[1]-hardness [9] carries over.

Overview of this paper. In Section 2, we give formal definitions and preliminary observations. Then, we provide bounds on the maximum supercap sizes for various d and the asymptotic case in Section 3. In Section 4, we use insights from the previous section and new structural properties to obtain probabilities for the presence of supersets in the random collections of cards. We obtain results on the computations complexity in Section 5 and conclude the paper in Section 6.

2 Preliminaries

Definitions. We begin with considering $\mathbb{F}_3^d$. A collection $S = \{a, b, c\} \subseteq \mathbb{F}_3^d$ is called a SET if $a + b + c = 0$. Further, a collection of elements $S = \{a, b, c, d\} \subset \mathbb{F}_3^d$ is called a SUPERSET if for some element $z \in \mathbb{F}_3^d$ and for some $x, y \in S$, both $\{x, y, z\}$ and $(S \setminus \{x, y\}) \cup \{z\}$ are SETs. We say that two pairs $\{a_1, b_1\}, \{a_2, b_2\} \subset \mathbb{F}_3^d$ are *parallel* if $a_2 - b_2 = r(a_1 - b_1)$ for some $r \in \mathbb{F}_3 \setminus \{0\}$. Note that if $S = \{a_1, b_1, a_2, b_2\}$ contains a SET, then S is (ironically) not a SUPERSET.

Moreover, a collection S of elements from $\mathbb{F}_3^d$ is a *cap* if no SET is contained in it; it is a *supercap* if no SUPERSET is contained in it. We will be looking at both maxim*um* and maxim*al* (super)caps: The first type of (super)cap has largest-possible size among all (super)caps; the addition of any card to the second type of (super)cap revokes its (super)cap property. Note that any maximum (super)cap is maximal.

To consider the complexity of determining of a given collection of cards contains a SET or SUPERSET, we define two different, yet equally natural, generalizations of a SET and a SUPERSET. For an element $a \in \mathbb{F}_v^d$, we denote with $a[i]$ the value of the i-th dimension of a.

Given a collection $S = \{c_1, \ldots, c_v\} \subseteq \mathbb{F}_v^d$ of v elements, we say it is a *combinatorial* SET (or just SET when not stated differently) if for all dimensions $i \in \{1, \ldots, d\}$, either $c_1[i] = \ldots = c_v[i]$ or the values $c_1[i], \ldots, c_v[i]$ are distinct. For prime v, we say that a collection $S \subseteq \mathbb{F}_v^d$ is a *line* SET if it is a line on $\mathbb{F}_v^d$. It is not hard to see that these two generalizations are equivalent only for $v \leq 3$.

We obtain two generalizations for a SUPERSET in straightforward manner from the generalizations of a SET. A collection $S \subseteq \mathbb{F}_v^d$ of $2(v-1)$ elements is a *combinatorial (line)* SUPERSET if there is an element $z \in \mathbb{F}_v^d$ and a partition $S = A \cup B$ such that $A \cup \{z\}$ and $B \cup \{z\}$ are both *combinatorial (line)* SETs.

Preliminary observations. The following observation is used frequently in the technical part of the paper. Given two elements $a, b \in \mathbb{F}_3^d$, there is a unique third element in $\mathbb{F}_3^d$, namely $-(a+b)$, that completes $\{a, b\}$ to a SET. More generally, for $a, b \in \mathbb{F}_v^d$ there are $v-2$ unique elements $x_1, x_2, \ldots x_{v_2} \in \mathbb{F}_v^d$ such that $\{a, b, x_1, x_2, \ldots x_{v_2}\}$ is a line SET, but there are various ways of completing $\{a, b\}$ to a combinatorial SET. Regarding SUPERSETs, consider any three elements $a, b, c \in \mathbb{F}_3^d$: If they form a SET, there is no element that completes them to a SUPERSET; if they do not form a SET, they can be extended to precisely the SUPERSETS $\{a, b, c, -(a+b)\}$, $\{a, b, c, -(a+c)\}$, and $\{a, b, c, -(b+c)\}$. The situation for SUPERSETs is a bit more complicated in $\mathbb{F}_v^d$ but irrelevant for this paper.

The following lemma contains the formal statement of a fairly well-known fact for those that have concerned themselves with SUPERSETs [2, 11, 24]. For completeness, we still provide a proof.

▶ **Lemma 1.** *A collection $S \subset \mathbb{F}_3^d$ with four distinct elements is a* SUPERSET *if and only if $\{x, y\}$ and $S \setminus \{x, y\}$ are parallel pairs for some $x, y \in S$.*

Proof. Let $S = \{a, b, c, d\}$ be as in the statement. Suppose, without loss of generality, that $\{a, c\}$ and $\{b, d\}$ are parallel pairs. In this case, $x_{a,b} = -(a+b)$ and $x_{c,d} = -(c+d)$ are the (unique) elements that complete the SETs $S_{a,b} = \{a, b, x_{a,b}\}$ and $S_{c,d} = \{x, d, x_{c,d}\}$. Now, since $\{a, c\}$ and $\{b, d\}$ are parallel, we can assume that $b - d = -(a - c)$ (otherwise $b - d = a - c$, and we switch a and c). Thus we have $-(c+d) = -(a+b)$, and hence $x_{a,b} = x_{c,d}$, which implies that S is a SUPERSET.

Now, suppose that S is a SUPERSET. We may assume, without loss of generality, that there is an element $z \in \mathbb{F}_3^d$ such that $\{a, b, z\}$ and $\{c, d, z\}$ are SETs. Thus, we have $a + b + z = c + d + z = 0$, and hence $a + b = c + d$, which implies $a - c = d - b$. Therefore $\{a, c\}$ and $\{b, d\}$ are parallel. ◀

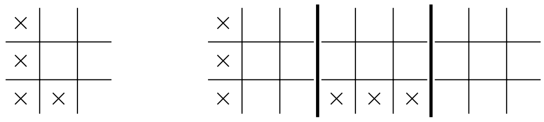
3 Bounds for supercaps

In this section we exactly determine the maximum sizes of supercaps of $\mathbb{F}_3^2$, $\mathbb{F}_3^3$, and $\mathbb{F}_3^4$. We also prove non-trivial upper and lower bounds on the asymptotic behavior of the maximum supercap size as $d \to \infty$.

3.1 Bounds for small d

In this subsection we present some auxiliary structural results along with the exact maximum sizes of supercaps of $\mathbb{F}_3^d$, for $d = 2, 3, 4$.

▶ **Proposition 2.** *A collection of four elements of $\mathbb{F}_3^2$ is a supercap if and only if it contains a* SET.



(a) **(b)**

Figure 3 Maximum supercaps in $\mathbb{F}_3^2$ and $\mathbb{F}_3^3$.

Proof. Let S be a collection with precisely four distinct elements of $\mathbb{F}_3^2$. For every pair of elements $a, b \in S$, there is a (unique) third element $x_{ab} \in \mathbb{F}_3^2$ such that $\{a, b, x_{ab}\}$ is a SET. If S does not contain a SET, then $x_{ab} \notin S$, for every pair of elements of S. Since there are precisely 6 such pairs, and $|\mathbb{F}_3^2 - S| = 3^3 - 4 = 5$, there must be two different pairs, say $\{a, b\}$ and $\{c, d\}$, such that $x_{ab} = x_{cd}$. Therefore, S contains a SUPERSET. Now, suppose that S contains a SET, say $\{a, b, c\}$, and another element d. If S is a SUPERSET, then we may suppose that there is an element w in $\mathbb{F}_3^2$ such that, without loss of generality, $\{a, b, w\}$ and $\{c, d, w\}$ are SETs. Since there is a unique w such that $\{a, b, w\}$ is a SET, we have $w = c$, which implies that $\{c, d, w\} = \{c, d\}$ is not a SET. ◀

This proposition immediately implies the lower-bound part of the following theorem.

▶ **Theorem 3.** *A maximum supercap in $\mathbb{F}_3^2$ has four elements.*

Proof. By Proposition 2, there exists a supercap of size 4 in $\mathbb{F}_3^2$. We illustrate one in Figure 3a.

We now prove that any collection S of elements of $\mathbb{F}_3^2$ of size 5 contains a SUPERSET. First, note that, if S contains two SETS S_1 and S_2, they need to intersect, because S has only size 5. Since S_1 and S_2 are non-identical, they intersect exactly in one element w, so $(S_1 \cup S_2) \setminus \{w\}$ is a SUPERSET. Thus, if S does not contain a SUPERSET, then S contains at most one SET. If S contains a SET, say P, then let x be an element of P, otherwise, let x be any element of S. Now, note that $S \setminus \{x\}$ contains four elements, and no SET. Therefore, by Proposition 2, $S \setminus \{x\}$ is a SUPERSET. ◀

Next, note that if $\varphi\colon \mathbb{F}_3^d \to \mathbb{F}_3^d$ is an invertible affine transformation and $S \subset \mathbb{F}_3^d$, then $\varphi(S)$ is a SET (resp. SUPERSET) if and only if S is a SET (resp. SUPERSET), because φ preserves addition. The following result implies a lower bound for the size of a maximum supercap of $\mathbb{F}_3^3$.

▶ **Proposition 4.** *If S is a collection of elements of $\mathbb{F}_3^3$ consisting of two skew (disjoint non-parallel)* SETs*, then S is a supercap.*

Proof. Let S be as in the statement. Since these SETs are skew, their two direction vectors and an arbitrary vector connecting them are linearly independent. So we can construct an invertible linear transformation that maps these vectors into $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$, and $(0, 0, 1)$, respectively. We can further determine a translation such that the emerging invertible affine transformation φ maps the SETs into $P_1 = \{iv_1 : i \in \mathbb{F}_3\}$ and $P_2 = \{(0, 0, 1) + jv_2 : j \in \mathbb{F}_3\}$. Therefore, the element $(-i, -j, 2)$ is the unique element that forms a SET with $iv_1 \in P_1$ and $(0, 0, 1) + jv_2 \in P_2$. Since there are precisely nine pairs consisting of a vertex of P_1 and a vertex of P_2, no element in $\{(-i, -j, 2) : i, j \in \mathbb{F}_3\}$ may complete to two different such pairs to SETs. This implies that $P_1 \cup P_2$ is a supercap of $\mathbb{F}_3^3$, so $S = \{\varphi^{-1}(s) : s \in P_1 \cup P_2\}$ is as well. ◀

Let S be a collection of elements of $\mathbb{F}_3^d$, for a fixed integer d. Each pair $a, b \in S$ defines a direction $a - b$. We say that S *generates* a vector $v \in \mathbb{F}_3^d$ if there is a pair $a, b \in S$ such that $v = a - b$. In this case, we also say that v is generated by the pair $\{a, b\}$. By Lemma 1, if there are distinct $a, b, c, d \in S$ such that $a - b$ is parallel to $c - d$, then S contains a SUPERSET. So, to obtain an upper bound on the size of a supercap S, one can compare the number of parallel vectors that S generates and the number of equivalence classes of parallel vectors in the entire space. The following lemma formalizes this idea and will be used for the next upper-bound proofs.

▶ **Lemma 5.** *Let S be a supercap in $\mathbb{F}_3^d$ with s elements and r SETs. Then*

$$r \geq \left\lceil \frac{s^2 - s - 3^d + 1}{4} \right\rceil.$$

Proof. Let S, s, and r be as in the statement. The number of pairs of elements of S is $\binom{s}{2}$. Note that each SET in S generates exactly 3 parallel vectors without creating a SUPERSET, but there are r SETs in S. Only considering one vector per SET, S still generates $\binom{s}{2} - 2r$ vectors. Note that, since we only consider one vector per SET and all SETs are pairwise disjoint (otherwise there would be a SUPERSET), any two pairs that generate parallel vectors need to be disjoint. So, by Lemma 1, any two of the $\binom{s}{2} - 2r$ must not be parallel. On the other hand, we give an upper bound on the equivalence classes of parallel vectors in $\mathbb{F}_3^d$ by counting the SETs that go through the origin $0 = (0, \ldots, 0)$: Since, for any other $a \in \mathbb{F}_3^d$, there is a unique SET containing 0 and a, and each SET has size 3, there are exactly $(3^d - 1)/2$ such SETs. Thus

$$\binom{s}{2} - 2r \leq \frac{3^d - 1}{2},$$

and the result follows by solving for r. ◀

We now apply Proposition 4 and Lemma 5 to get the following theorem.

▶ **Theorem 6.** *A maximum supercap in $\mathbb{F}_3^3$ has six elements.*

Proof. By Proposition 2, there exists a supercap of size 6 in $\mathbb{F}_3^3$. We illustrate one in Figure 3b.

Now assume that $S \subset \mathbb{F}_3^3$ is a supercap of size 7. By Lemma 5, the number of SETs in S is at least 4 but there are at most two non-intersecting SETs in S, a contradiction. ◀

The proof of the next theorem goes one step further. In this case, the application of Lemma 5 does not directly imply a tight upper bound.

▶ **Theorem 7.** *A maximum supercap in $\mathbb{F}_3^4$ has nine elements.*

Proof. A supercap of $\mathbb{F}_3^4$ of size 9 was previously known [32, 2] and is illustrated in Figure 4.

For the upper bound, let S be a supercap with precisely ten different elements of $\mathbb{F}_3^4$. By Lemma 5, the number of SETs in S is at least $\lceil (100 - 10 - 81 + 1)/4 \rceil = 3$. Analogously to the proof of Proposition 4, by applying a certain invertible affine transformation, we can suppose that two of these SETs are $P_1 = \{kv_1 : k \in \mathbb{F}_3\}$ and $P_2 = \{(0, 0, 1, 0) + kv_2 : k \in \mathbb{F}_3\}$, where $v_1 = (1, 0, 0, 0)$ and $v_2 = (0, 1, 0, 0)$.

Now, let $P_3 = \{(a, b, c, d) + kv_3 : k \in \mathbb{F}_3\}$. We first show that $v_3 = (e_1, e_2, 0, 0)$, where $e_1, e_2 \in \mathbb{F}_3 \setminus \{0\}$. Let $v_3 = (e_1, e_2, e_3, e_4)$. If $e_4 \neq 0$, then P_3 has an element q of the form $(x, y, z, 0)$. Thus, the restriction S' of S to the affine subspace $F_0 = \{(x, y, z, 0) : x, y, z \in$

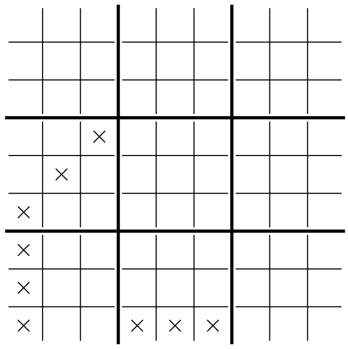


Figure 4 Maximum supercap in $\mathbb{F}_3^4$.

$\mathbb{F}_3\}$ contains the seven elements $P_1 \cup P_2 \cup \{q\}$. Since F_0 is isomorphic to $\mathbb{F}_3^3$, by Theorem 6, S' contains a SUPERSET, a contradiction. In fact, S may not contain any element of the form $(\cdot,\cdot,\cdot,0)$ different from the elements in $P_1 \cup P_2$. Now, suppose that $e_3 \neq 0$. Then P_3 contains elements q_1 and q_2 of the form $(\cdot,\cdot,0,d)$ and $(\cdot,\cdot,2,d)$, with $d \neq 0$. Let $A_1 = (0,0,0,0)$ and $A_2 = (0,0,1,0)$, and for $i = 1,2$, consider the SETs P_1' and P_2' defined by

$$\begin{aligned}
P_i' &= \{r\colon s + q_i + r = 0, s \in P_i\} \\
&= \{-(s + q_i)\colon s \in P_i\} \\
&= \{-(A_i + kv_i + q_i)\colon k \in \mathbb{F}_3\} \\
&= \{-(A_i + q_i) + 2kv_i\colon k \in \mathbb{F}_3\} \\
&= \{-(A_i + q_i) + kv_i\colon k \in \mathbb{F}_3\}.
\end{aligned}$$

Note that hence P_i' is parallel to P_i, for $i = 1,2$. Moreover, since the vertices of P_1 and P_2 are of the form $(\cdot,\cdot,\cdot,0)$, the vertices of P_1' and P_2' are of the form $(\cdot,\cdot,\cdot,3-d)$. Also, since the vertices of P_1 and q_1 are of the form $(\cdot,\cdot,0,\cdot)$, the vertices of P_1' are of the form $(\cdot,\cdot,0,\cdot)$; and since the vertices of P_2 are of the form $(\cdot,\cdot,1,\cdot)$, and q_2 is of the form $(\cdot,\cdot,2,\cdot)$, the vertices of P_2' are of the form $(\cdot,\cdot,0,\cdot)$. We conclude that P_1' and P_2' belong to the 2-dimensional affine subspace $F_{0,3-d} = \{(x,y,0,3-d)\colon x,y \in \mathbb{F}_3\}$. Note that P_1' and P_2' are not disjoint because they are parallel, respectively, to P_1 and P_2. Thus, there is a vertex q^* in $P_1' \cap P_2'$ such that $s_1 + q_1 + q^* = s_2 + q_2 + q^* = 0$, for some $s_1 \in P_1$ and $s_2 \in P_2$. Therefore, S contains a SUPERSET, a contradiction. Now, if $e_1 = 0$ or $e_2 = 0$, then P_3 is parallel to either P_1 or P_2, a contradiction.

Now, let $F_{i,j} = \{(x,y,i,j)\colon x,y \in \mathbb{F}_3\}$, for $i,j \in \mathbb{F}_3$. Since $v_3 = (e_1,e_2,0,0)$, the SET P_3 must be contained in some affine subspace F_{i^*,j^*}. Further, we must have $j \neq 0$ since otherwise there are again seven elements of the form $(\cdot,\cdot,\cdot,0)$. Assume, by adapting the invertible affine transformation accordingly, that $i^* = 0$ and $j^* = 1$. Analogously to the proof of Proposition 4, each element of $F_{2,0}$ is the unique element that forms a SET with a vertex of P_1 and of P_2; each element of $F_{0,2}$ is the unique element that forms a SET with a vertex of P_1 and of P_3; and each element of $F_{2,2}$ is the unique element that forms a SET with a vertex of P_2 and of P_3; Recall that S has ten elements, i.e., there is an element q in $S \setminus (P_1 \cup P_2 \cup P_3)$. Note that the collections $F_{0,0} \cup F_{1,0} \cup F_{2,0}$, $F_{0,0} \cup F_{0,1} \cup F_{0,2}$, and $F_{1,0} \cup F_{0,1} \cup F_{2,2}$ are affine subspaces isomorphic to $\mathbb{F}_3^3$, and, by Theorem 6, q may not belong to any of these collections. Now, suppose that $q \in F_{1,1}$. For each $q_1 \in P_1 \subset F_{0,0}$, there is a vertex $q_{2,2} \in F_{2,2}$ such that $q_1 + q + q_{2,2} = 0$. As noted above, each element of $F_{2,2}$ forms a SET with a vertex of P_2 and of P_3, say q_2, q_3. Therefore $\{q_1, q_2, q_3, q\}$ is a SUPERSET. Analogously, if q belongs

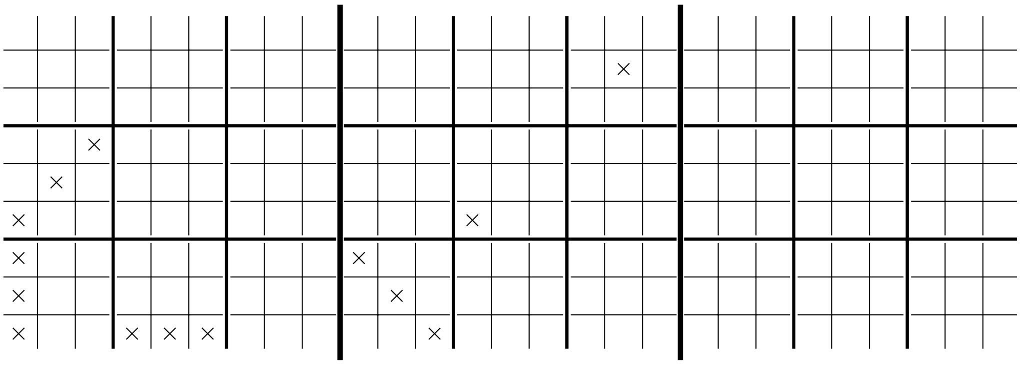


Figure 5 Maximum supercap in $\mathbb{F}_3^5$.

to $F_{2,1}$ or $F_{1,2}$, we can find elements $q_1 \in P_1$, $q_2 \in P_2$, and $q_3 \in P_3$ such that $\{q_1, q_2, q_3, q\}$ is a SUPERSET. This contradicts the assumption that S is a supercap and concludes the proof. ◀

We conclude the subsection with a discussion of open questions, preliminary answers, and fascinating phenomena. In $\mathbb{F}_3^5$, the largest supercap we can construct has size 14 (see Figure 5), but Lemma 5 only shows an upper bound of 16 on the size of supercaps. While an exhaustive computer search shows that 14 is indeed the right answer, we still believe in (super) elegant proofs. Indeed, looking at the lower bounds in this section, one may notice that, interestingly, all of them contain the maximum number of SETs possible. Also, Lemma 5 gives the loosest upper bound when the maximum number of SETs are present. So one may conjecture that, for each d, a maximum supercap is attained by a union of SETs and at most 2 additional points.

On the other hand, it has been observed [24] that the maximum supercap in four dimensions can be partitioned into ten pairs each of which is completed to a SET by the same element. So it seems that caps and supercaps are somewhat complementary in that maximum supercaps are far from being caps and vice versa. Unfortunately, however, we need to push back on this line of thought a bit. As we will see in Section 4, already in $\mathbb{F}_3^3$ there are maximum supercaps with only one (instead of two) SETs. Also, there is a maximum supercap in $\mathbb{F}_3^4$ that does not have a SET at all. On a slightly different matter, this situation is somewhat different from the one for caps in that, for any $d \in \{1, \dots, 6\}$, there is an affine transormation that takes any maximum cap to any other maximum cap [15, 18, 24, 27].

As all of the phenomena pointed at here may simply be due to the (small) dimensions we are working with, we now look at the asymptotic case.

3.2 Asymptotic supercaps

In this section we present upper and lower bounds for the size of a maximum supercap in $\mathbb{F}_3^d$. The next theorem gives the upper bound; its proof is analogous to the proof of Theorems 6 and to some of the cases of the proof of Theorem 7.

▶ **Theorem 8.** *A maximum supercap in $\mathbb{F}_3^d$ has less than $2 \cdot 3^{\frac{d}{2}}$ elements.*

Proof. It is sufficient to prove for $d \geq 2$ that, if a collection $S \subseteq \mathbb{F}_3^d$ has size $s = 2 \cdot 3^{\frac{d}{2}}$, then it contains a SUPERSET. Let S be such a collection, and suppose that S is a supercap. By

Lemma 5, the number of non-intersecting SETs in S is at least

$$\left\lceil \frac{4 \cdot 3^d - 2 \cdot 3^{d/2} - 3^d + 1}{4} \right\rceil.$$

On the other hand, there are at most $\lfloor s/3 \rfloor$ non-intersecting SETs in S. Thus, we have

$$\left\lceil \frac{4 \cdot 3^d - 2 \cdot 3^{d/2} - 3^d + 1}{4} \right\rceil \leq \left\lfloor \frac{2 \cdot 3^{d/2}}{3} \right\rfloor.$$

Note that for any $d \geq 2$ we have

$$\frac{4 \cdot 3^d - 2 \cdot 3^{d/2} - 3^d + 1}{4} > \frac{3 \cdot 3^d - 2 \cdot 3^{d/2}}{4} > \frac{3^d}{4} > \frac{2 \cdot 3^{d/2}}{3}.$$

Therefore, we have

$$\left\lceil \frac{4 \cdot 3^d - 2 \cdot 3^{d/2} - 3^d + 1}{4} \right\rceil \geq \frac{4 \cdot 3^d - 2 \cdot 3^{d/2} - 3^d + 1}{4} > \frac{2 \cdot 3^{d/2}}{3} \geq \left\lfloor \frac{2 \cdot 3^{d/2}}{3} \right\rfloor,$$

a contradiction. Therefore, for any SUPERSET S in $\mathbb{F}_3^d$ we have $|S| < 2 \cdot 3^{d/2}$. ◀

The next theorem gives a lower bound for the size of a maximum supercap in $\mathbb{F}_3^d$.

▶ **Theorem 9.** *A maximum supercap in $\mathbb{F}_3^d$ has more than $3^{\frac{d}{3}}$ elements.*

Proof. We prove that every maximal supercap has size at least $3^{\frac{d}{3}}$. Given a supercap S in $\mathbb{F}_3^d$, let $\bar{S}$ be the collection of elements v of $\mathbb{F}_3^d - S$ for which there is at least one triple T in S such that $T \cup \{v\}$ is a SUPERSET. Note that if $x \in \mathbb{F}_3^d \setminus (S \cup \bar{S})$, then $S \cup \{x\}$ is a supercap. Thus, if S is a maximal supercap, then $S \cup \bar{S} = \mathbb{F}_3^d$. Given $a, b \in \mathbb{F}_3^d$, let x_{ab} be the (unique) element of $\mathbb{F}_3^d$ such that $\{a, b, x_{ab}\}$ is a SET; and given a triple $\{a, b, c\} \subset \mathbb{F}_3^d$ that is not a SET, let y_c be the (unique) element of $\mathbb{F}_3^d$ such that $\{c, x_{ab}, y_c\}$ is a SET. Note that for every such triple $\{a, b, c\}$ in a supercap S, we have $y_a, y_b, y_c \in \bar{S}$. Moreover, if $\{a, b, c, y\}$ is a SUPERSET, then $y = y_z$ for some $z \in \{a, b, c\}$. Therefore, $|\bar{S}| \leq 3\binom{|S|}{3}$ for every supercap S in $\mathbb{F}_3^d$.

Now, suppose that S is a maximal supercap and that $|S| = s \leq 3^{\frac{d}{3}}$. Since S is maximal, we have $3^d = |\mathbb{F}_3^d| \leq |S| + |\bar{S}|$. Thus, we have

$$s^3 \leq 3^d \leq s + 3\binom{s}{3}.$$

Yet, $s^3 > s + 3\binom{s}{3}$ for all $s > 1$, contradicting our assumption, since a maximal supercap has at least three elements. We conclude that if S is maximal, then $|S| > 3^{\frac{d}{3}}$. ◀

4 Probabilities of the presence of a superset in random collections

In the section, we compute probabilities of k-element collections in $\mathbb{F}_3^d$ being supercaps. Using structural insights from Section 3, we get the following result, settling the question for $d = 2$.

▶ **Theorem 10.** *A collection of four elements drawn uniformly at random without replacement from $\mathbb{F}_3^d$ is a supercap with probability $\frac{3^d-5}{3^d-2}$.*

Proof. Let $S = \{a, b, c, d\}$ be a collection of four elements drawn uniformly at random without replacement from $\mathbb{F}_3^d$. Consider the four elements of S in (alphabetical) order. As noted earlier in Proposition 2, if S contains a SET, then it is a supercap. Without loss of generality, fix the first two elements $\{a, b\}$. The third element, c, completes a SET with probability $\frac{1}{3^d-2}$, since exactly one of the remaining $3^d - 2$ elements from $\mathbb{F}_3^d$ forms a SET with $\{a, b\}$. If $\{a, b, c\}$ does not form a SET, then there are three pairs, $\{a, b\}$, $\{b, c\}$, and $\{a, c\}$ that define different elements with which they form a SET. Thus, there are exactly three elements that can complement $\{a, b, c\}$ into a SUPERSET. Therefore

$$\Pr(S \text{ is a supercap}) = \frac{1}{3^d - 2} + \frac{3^d - 3}{3^d - 2} \cdot \frac{3^d - 6}{3^d - 3} = \frac{3^d - 5}{3^d - 2}. \qquad \blacktriangleleft$$

For $d = 3$, we require new structural insights.

▶ **Proposition 11.** *Let S be a collection with five elements in $\mathbb{F}_3^3$. Then S is a supercap if and only if either*

- *S contains a SET P and the elements not in P form a pair skew with P*
- *or S does not contain a SET and there is no hyperplane in $\mathbb{F}_3^3$ containing at least four elements of S.*

Proof. Let S be a collection of five elements with a SET P. It is clear that if $S \setminus P$ forms a pair not skew with P, then S contains a SUPERSET either by the intersection of P and the SET containing $S \setminus P$, or by P being parallel to $S \setminus P$. Now, suppose that $S \setminus P = \{a, b\}$ is skew with P. It is not hard to check that a SUPERSET admits three partitions into two pairs, and one of these partitions consists of two pairs that miss the same third element to complete a SET; and the other two of these partitions consist of two parallel pairs (see Lemma 1). Since $\{a, b\}$ is skew with P, for any $c, d \in P$, the pair $(\{a, b\}, \{c, d\})$ forms a partition of $\{a, b, c, d\}$ that does not consist of two pairs with a common missing third element, and does not consist of two parallel pairs. Thus, $\{a, b, c, d\}$ is not a SUPERSET. Suppose now that S does not contain a SET. Since a hyperplane in $\mathbb{F}_3^3$ is isomorphic to $\mathbb{F}_3^2$ and every SUPERSET is in a hyperplane, by proposition 2, S contains a SUPERSET if and only if there is a set of four vertices contained in a hyperplane. ◀

▶ **Proposition 12.** *Let S be a collection of six elements in $\mathbb{F}_3^3$. Then S is a supercap if and only if either*

- *S contains two SETs that are skew, or*
- *there are three parallel planes H_1, H_2, H_3 that partition $\mathbb{F}_3^3$ such that $S \cap H_1 = \{a, b, c, d\}$, $S \cap H_2 = \{e\}$, and $S \cap H_3 = \{f\}$ where $\{a, b, c\} = P$ is a SET and $f \notin \{-(x + e) : x \in S \cap H_1\} \cup \{x + d - e : x \in P\}$.*

Proof. Let S be as in the statement and first suppose that S is a supercap. First note that Lemma 5 implies that S must contain at least one SET. If S contains two SETs, they must be skew, because otherwise the two SETs (and thus at least five elements) are within a two-dimensional affine subspace, contradicting Theorem 3. If S contains precisely one SET $\{a, b, c\} = P$, we can find a plane H_1 that contains P and any fourth element $d \in S$. Note that H_1 may not contain any other element of S, because this would be a contradiction to Theorem 3 again. Next, consider the case that there is a plane H' parallel to H_2 such that $|S \cap H'| = 2$. But this is not possible: Since H_1 and H' generate 5 vectors parallel to H_1 and, among the vectors parallel to H_1, there are only 4 equivalence classes of parallel vectors, we get a contradiction to Lemma 1. Hence, there are planes H_2 and H_3 parallel to H_1 with $S \cap H_2 = \{e\}$ and $S \cap H_3 = \{f\}$ for some $e, f \in \mathbb{F}_3^3$. Now, since S contains only the set P, $x + e + f \neq 0$ for all $x \in H_1$, so $f \notin \{-(x + e) : x \in S \cap H_1\}$. Similarly, since S is a supercap $x + d \neq e + f$ for all $x \in P$, so $f \notin \{x + d - e : x \in P\}$. Thus we are in the second situation.

If S contains two SETs that are skew, then Proposition 4 shows that S is a supercap. If the second condition is fulfilled, then let $a, \ldots, f$ and H_1, H_2, H_3 be as in the statement. Note that H_1 does not contain a SUPERSET by Proposition 2. If for a SUPERSET $Q \subset S$, we have $|Q \cap H_1| = 3$, then, for any pair $x_1, x_2 \in H_1$, $x_1 + x_2 \in H_1$, but $x_3 + x_4 \notin H_1$ where $\{x_3, x_4\} = Q \setminus \{x_1, x_2\}$, a contradiction. So, if S contains a SUPERSET Q, then $Q = \{e, f, y_1, y_2\}$ for some $y_1, y_2 \in S \cap H_1$. But $y_i + e \in H_3$ while $y_i + f \in H_2$ for any $i \in \{1, 2\}$. If $d \notin Q$, then $-(y_1 + y_2) \in P$, but $f \neq -(x + e)$ for all $x \in P$ by the choice of f; a contradiction. So $Q = \{e, f, d, x\}$ for some $x \in P$. But then we must have $x + d = e + f$; a contradiction to the choice of f. ◀

Using these insights and counting the numbers of the corresponding objects yields the following theorem, which settles the central question of this section for $d = 3$.

▶ **Theorem 13.** *A collection of five (six) elements drawn uniformly at random without replacement from $\mathbb{F}_3^3$ is a supercap with probability $\frac{54}{115} \approx 46.96\%$ ($\frac{18}{253} \approx 7.11\%$).*

Proof. We count the number of supercaps of five elements using Proposition 11. The ones that contain a SET and a pair skew with it can be constructed as follows. Choose a SET, then any of the remaining $(3^d - 3)$ cards and finally any of the $(3^d - 9)$ cards that do not complete an intersecting SET or creates a parallel vector with the SET. Since the last pair is counted twice this way, the total number is

$$N^5_{\text{SET,skew}} = \frac{3^3(3^3-1)}{6}(3^3-3)(3^3-9) \cdot \frac{1}{2} = 25272$$

For the ones that do not contain a SET and in which no four elements are in a hyperplane, we count first the number of collections with a SET: pick first one of the $\frac{1}{3}\binom{3^3}{2}$ possible SETs, then pick any pair on the remaining cards. With this procedure we double count the collections composed by two intersecting SETs, so the total number of collections of five elements with a SET is

$$N^5_{\text{SET}} = \frac{1}{3}\binom{3^3}{2} \cdot \binom{3^3-3}{2} - \frac{1}{3}\binom{3^3}{2} \cdot \frac{1}{4}\left(3^3-3\right) \cdot 3 = 30186$$

We now compute the number of collections without a SET but with a hyperplane. It is clear that only one hyperplane contains four points of such a collection. Pick then the first four elements to be the ones in the same hyper plane. There are 3^3 options for the first, $(3^3 - 1)$ for the second, $(3^3 - 3)$ for the third without forming a SET, and only 3 for the fourth so it lies in the same hyperplane and does not form a SET. We divide by the number of permutations of 4 elements to avoid multiple counting. For the fifth element the only condition is that it is outside the hyperplane, so there are $3^{3-1} \cdot 2$ options. The number of such collections is then

$$N^5_{\text{!SET,HP4}} = \frac{1}{4!}3^3(3^3-1)(3^3-3) \cdot 3 \cdot (3^{3-1}2) = 37908$$

Now, the number of collections of five elements without a SET is $N^5_{\text{!SET}} = \binom{3^3}{5} - N^5_{\text{SET}} = 50544$, and the amount of collections without a SET and with no four elements in a hyperplane is $N^5_{\text{!SET,!HP4}} = N^5_{\text{!SET}} - N^5_{\text{!SET,HP4}} = 12636$. Finally, the number of supercaps of size five is $N^5_{\text{SET,skew}} + N^5_{\text{!SET,!HP4}} = 37908$, which divided by $\binom{3^3}{5}$ gives the probability that a random collection S of five elements in $\mathbb{F}_3^3$ is a supercap, so

$$\Pr(S \text{ is a supercap}) = \frac{37908}{80730} = \frac{54}{115} \approx 46.96\%.$$

Table 2 Probabilities (and estimates thereof) of a k-element collection from $\mathbb{F}_3^d$ being a supercap, expressed as percentages rounded to two decimals. We used a computer to estimate the probability where indicated by an asterisk (10^7 samples for each corresponding cell); the other probabilities are exact.

	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
$d=2$	57.14%	0	0	0	0	0
$d=3$	88.00%	46.96%	7.11%	0	0	0
$d=4$	96.20%	81.68%*	52.08%*	19.25%*	2.34%*	0.01%*

We count the number of supercaps S of six elements using Proposition 12. Note that the number of two skew SETs is exactly a sixth the number of five cards formed by a SET and a pair that is skew with it, so

$$N^6_{2\text{SETs},\text{skew}} = \frac{\binom{3^3}{2}}{3} \cdot (3^3-3)\cdot(3^3-9)\cdot\frac{1}{12} = 4212.$$

Now consider the second situation in Proposition 12, and let $a,\dots,f$ and H_1,H_2,H_3 be as in the statement. First note that $f \notin \{-(x+e) : x \in S\cap H_1\}$ ensures that there is exactly one SET in S, so the two situations cannot happen simultaneously. We count the number of collections S that fall into this situation the following way: First fix any set $\{a,b,c\}=P$ and any fourth point d. There are 9 choices for e. Since $\{-(x+e) : x \in S\cap H_1\}$ and $\{x+d-e : x \in P\}$ are both contained in H_3 and are disjoint, there are 2 choices left for f. As we count each collection three times (any of the points outside the SET can be the fourth point), the total number of collections S that fall into the second situation is

$$N^6_{\text{Case }2} = \frac{\binom{3^3}{2}}{3}\cdot(3^3-3)\cdot 9\cdot 2\cdot\frac{1}{3} = 16848.$$

In total, we get

$$\Pr(S \text{ is a supercap}) = \frac{N^6_{2\text{SETs},\text{skew}} + N^6_{\text{Case }2}}{\binom{3^3}{6}} = \frac{4212+16848}{296010} = \frac{18}{253} \approx 7.11\%.$$

This concludes the proof. ◀

Considering the name of this conference, we leave proving similar statements for $d=4$ to future work. To not disappoint the reader, we however provide probabilities that were determined experimentally with the computer. We summarize the results in Table 2.

5 Algorithms and complexity

Chaudhuri et al. [9] as well as Lampis and Mitsou [22] consider decision problem versions for SET and show complexity results for them. In these decision problems, we are given a collection of n elements from $\mathbb{F}_v^d$ and ask if the collection contains a SET[3]. In this section, we obtain similar results for decision problem versions of SUPERSET.

We define the problems COMBINATORIAL SUPERSET and LINE SUPERSET as follows: Given a collection of elements $C \subseteq \mathbb{F}_v^d$ for any $v,d>0$, is there a combinatorial (line)

[3] Previously only these decision versions of Combinatorial SETs were considered [9, 22]. More restricted versions with given number of values (k-VALUE SET) or given number of dimensions (k-DIMENSIONAL SET) have also been considered by the same authors.

SUPERSET $S \subseteq C$? We define k-VALUE COMBINATORIAL (LINE) SUPERSET as the restricted versions where $v = k$ is fixed, and k-DIMENSIONAL COMBINATORIAL (LINE) SUPERSET as the restricted versions where $d = k$ instead.

Note that, in order to get interesting complexity questions, the number of possible values v of each attribute needs to be variable, as we see from the following result.

▶ **Theorem 14.** *The problem k-VALUE COMBINATORIAL SUPERSET and k-VALUE LINE SUPERSET can be solved in $\tilde{O}(dkn^{k-1})$ time, for any given $k > 0$.*

Proof. Consider the algorithm that iteratively checks all $\binom{n}{k-1}$ subsets of size $k-1$ and keeps an AVL tree [10] of missing k-th elements to complete a SET, if such an element exists. Then checking if the ordered list contains any duplicates decides if there is a SUPERSET in the given collection of elements. This algorithm works independent of the considered type of SUPERSET (combinatorial or line) and runs in $\tilde{O}(dkn^{k-1})$ time. ◀

▶ **Theorem 15.** *The problem LINE SUPERSET can be solved in $\tilde{O}(dvn^2)$ time.*

Proof. Each pair of elements defines exactly one line, so it suffices to check for each pair if the collection contains $v-1$ elements of the line. If so, the missing element is stored in an ordered list. ◀

Note that, by similar reasoning, LINE SET can also be solved in $\tilde{O}(dvn^2)$ time.

▶ **Theorem 16.** *There is a $O(k^2 n)$-time reduction from k-DIMENSIONAL COMBINATORIAL SET to $(k+1)$-DIMENSIONAL COMBINATORIAL SUPERSET. Furthermore, it sends an instance on v values to an instance on $v+1$ values.*

Proof. Let $S \in \mathbb{F}_v^k$ be an instance of k-DIMENSIONAL COMBINATORIAL SET. That is, S is a collection of n elements in k dimensions, each with v possible values. Through the following procedure, we construct an instance of $k+1$-DIMENSIONAL COMBINATORIAL SUPERSET consisting of collection of elements $S' \in \mathbb{F}_{v+1}^{k+1}$ of size at most $(v+1)\cdot n$, in $k+1$ dimensions, each with $v+1$ possible values.

Create a copy S_0 of S on $k+1$ dimensions, filling the $(k+1)$-th dimension of every element with the value $v+1$. Then, create $(k+1)$-dimensional copies $S_1, \dots, S_k$ of S, where the $(k+1)$-th dimension of elements in S_i have value equal to the i-th dimension of that element elements. That is, for an element $c' \in S_i$ there is an element $c \in S$, such that $c' = (c[1], \dots, c[k], c[i])$.

Now, we show that S contains a SET in $\mathbb{F}_v^k$ if and only if $S' = \bigcup_{i=0}^k S_i$ contains a SUPERSET in $\mathbb{F}_{v+1}^{k+1}$ (note that the union might be non disjoint). Suppose S contains a SET in $\mathbb{F}_v^k$, say A, and let $A_i \subseteq S_i$ be the corresponding copy of A for all $i \in \{0, \dots, v\}$. Let $z \in \mathbb{F}_{v+1}^{k+1}$ be the element that for all $j \in \{1, \dots, k\}$ has $z[j] = a_1[j]$, if $a_1[j] = a_2[j]$, and $z[j] = v+1$, otherwise, and $z[k+1] = v+1$. Then $A_0 \cup \{z\}$ is a SET in $\mathbb{F}_{v+1}^{k+1}$. Since the elements $a_1, \dots, a_v \in A$ are distinct and A is a SET in $\mathbb{F}_v^k$, there must be at least one dimension $1 \le j \le k$ such that the values $a_1[j], \dots, a_v[j]$ are all distinct. Then $A_j \cup \{z\}$ forms a SET in $k+1$ dimensions and $v+1$ values, because the first k dimensions are the same as $A_0 \cup \{z\}$, and dimension $k+1$ has the same values as dimension j, so the missing value in A_j is $v+1$. Since by construction we have $A_0 \cap A_j = \emptyset$, we conclude that $A_0 \cup A_j$ is a SUPERSET in $\mathbb{F}_{v+1}^{k+1}$.

Next, let $A \cup B \subseteq S'$ be a SUPERSET in $\mathbb{F}_{v+1}^{k+1}$ such that $A \cap B = \emptyset$ and there is an element z such that $A \cup \{z\}$ and $B \cup \{z\}$ are SETs in $\mathbb{F}_{v+1}^{k+1}$. This implies that $|A| = |B| = v$. Let $A = \{a_1, \dots, a_v\}$ and let $a'_j \in S$ denote the projection of a_j onto its first k dimensions, which is the original of a_j in S. If all elements $a'_1, \dots, a'_v$ are different, then they form a SET in $\mathbb{F}_v^k$. Now, assume that there are two elements a_k and a_ℓ in A, such that $a'_k = a'_\ell$. Since $A \cup \{z\}$

is a SET, this implies that $a'_1 = a'_2 = \ldots = a'_v$. Moreover, the projection of z onto the first k dimensions is equal to a'_1. Therefore, if the same holds for B, then the projections of each of the elements of B onto the first k dimensions are all equal and these projections are also equal to a'_1. However, this is a contradiction, as S' contains at most $v + 1$ copies of the same element in S (and $|A \cup B| = 2v$). Thus, without loss of generality, we can assume that the elements in A are copies of different elements in S and the projection of A onto the first k dimensions is a SET in S. ◀

Chaudhuri et al. [9] prove that k-Dimensional Combinatorial Set is NP-complete for $k \geq 3$, and Lampis and Mitsou [22] prove that Combinatorial Set parametrized by the number of values is W[1]-hard. These two results, together with Theorem 16, yield the following hardness results for SUPERSET.

▶ **Corollary 17.** *The problem k-Dimensional Combinatorial Superset for $k \geq 4$ and Combinatorial Superset are NP-complete.*

▶ **Corollary 18.** *The problem Combinatorial Superset parametrized by the number of values v is W[1]-hard.*

6 Conclusion

While it is plausible that we have exhausted (hopefully not gone beyond) the reader's tolerance of jokes including "super" in this paper, we believe that we have not done so to their curiosity regarding SUPERSET. In fact, while we have made progress on many natural questions in this paper, a few remain open: As for caps, the gaps for the maximum supercap size for larger fixed dimensions and its asymptotic behavior would be interesting to investigate. Also figuring out whether a subquadratic algorithm for deciding the presence of a SET or SUPERSET exists in $\mathbb{F}_3^d$ seems to be an interesting open problem.

Just like SET became too easy one day, we will eventually demand a variant of SET more difficult than SUPERSET. In fact, note that the term *powerset* is yet to be overloaded. For instance, a POWERSET could be the union of three (or more) pairs that are all completed to a SET by a same element [24] or, alternatively, the symmetric difference between two SUPERSETs that intersect in exactly one element.

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