Patterns in Random Permutations Avoiding Some Other Patterns

Svante Janson
Department of Mathematics, Uppsala University, PO Box 480, SE-751 06 Uppsala, Sweden
svante.janson@math.uu.se
https://orcid.org/0000-0002-9680-2790

Abstract
Consider a random permutation drawn from the set of permutations of length \(n\) that avoid a given set of one or several patterns of length 3. We show that the number of occurrences of another pattern has a limit distribution, after suitable scaling. In several cases, the limit is normal, as it is in the case of unrestricted random permutations; in other cases the limit is a non-normal distribution, depending on the studied pattern. In the case when a single pattern of length 3 is forbidden, the limit distributions can be expressed in terms of a Brownian excursion.

The analysis is made case by case; unfortunately, no general method is known, and no general pattern emerges from the results.

2012 ACM Subject Classification Mathematics of computing → Permutations and combinations, Mathematics of computing → Probabilistic representations

Keywords and phrases Random permutations, patterns, forbidden patterns, limit in distribution, U-statistics

Digital Object Identifier 10.4230/LIPIcs.AofA.2018.6

Category Keynote Speakers


1 Introduction

Let \(\mathcal{S}_n\) be the set of permutations of \([n] := \{1, \ldots, n\}\), and \(\mathcal{S}_* := \bigcup_{n \geq 1} \mathcal{S}_n\). If \(\sigma = \sigma_1 \cdots \sigma_m \in \mathcal{S}_m\) and \(\pi = \pi_1 \cdots \pi_n \in \mathcal{S}_n\), then an occurrence of \(\sigma\) in \(\pi\) is a subsequence \(\pi_{i_1} \cdots \pi_{i_m}\), with \(1 \leq i_1 < \cdots < i_m \leq n\), that has the same order as \(\sigma\), i.e., \(\pi_{i_j} < \pi_{i_k} \iff \sigma_j < \sigma_k\) for all \(j, k \in [m]\). We let \(n_\sigma(\pi)\) be the number of occurrences of \(\sigma\) in \(\pi\), and note that

\[
\sum_{\sigma \in \mathcal{S}_m} n_\sigma(\pi) = \binom{n}{m} \quad (1)
\]

for every \(\pi \in \mathcal{S}_n\). For example, an inversion is an occurrence of 21, and thus \(n_{21}(\pi)\) is the number of inversions in \(\pi\).

We say that \(\pi\) avoids another permutation \(\tau\) if \(n_\tau(\pi) = 0\). Let

\[
\mathcal{S}_n(\tau) := \{\pi \in \mathcal{S}_n : n_\tau(\pi) = 0\} \quad (2)
\]

1 Partly supported by the Knut and Alice Wallenberg Foundation

© Svante Janson; licensed under Creative Commons License CC-BY

Editors: James Allen Fill and Mark Daniel Ward; Article No. 6; pp. 6:1–6:12
Leibniz International Proceedings in Informatics

Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany
the set of permutations of length $n$ that avoid $\tau$. More generally, for any set $T = \{\tau_1, \ldots, \tau_k\}$ of permutations, let

$$\mathcal{G}_n(T) = \mathcal{G}_n(\tau_1, \ldots, \tau_k) := \bigcap_{i=1}^{k} \mathcal{G}_n(\tau_i),$$

the set of permutations of length $n$ that avoid all $\tau_i \in T$. We also let $\mathcal{G}_n(T) := \bigcup_{n=1}^{\infty} \mathcal{G}_n(T)$ be the set of $T$-avoiding permutations of arbitrary length.

The classes $\mathcal{G}_n(\tau)$ and, more generally, $\mathcal{G}_n(T)$ have been studied for a long time. For examples relevant to analysis of algorithms, see e.g. [13, Exercise 2.2.1-5] ($\pi$ can be obtained by a stack if and only if $\pi \in \mathcal{G}_n(312)$; equivalently: $\pi$ is stack-sortable if and only if $\pi \in \mathcal{G}_n(312)$); [13, Exercise 2.2.1-10,11] and [17] ($\pi$ is deque-sortable if and only if $\pi \pi \in \mathcal{G}_n(2431, 4231)$; [16] ($\pi$ can be sorted by 2 parallel queues if and only if $\pi \in \mathcal{G}_n(321)$). Further examples are given in [15], Exercises 6.19 x (321), y (312), ee (321), ff (312), ii (231), oo (132), xx (321); 6.25 g (321); 6.39 k, l (1 \{2413, 3142\}), m \{1342, 1324\}; 6.47 a \{4231, 3412\}; 6.48 (1342). See also [3].

In particular, one classical problem is to enumerate the sets $\mathcal{G}_n(T)$, either exactly or asymptotically, see e.g. [3, Chapters 4–5] and [14].

The general problem that concerns us is to take a fixed set $T$ of one or several permutations and let $\pi_{T,n}$ be a uniformly random $T$-avoiding permutation, i.e., a uniformly random element of $\mathcal{G}_n(T)$, and then study the asymptotic distribution of the random variable $n_\sigma(\pi_{T,n})$ (as $n \to \infty$) for some other fixed permutation $\sigma$. (Only $\sigma$ that are themselves $T$-avoiding are interesting, since otherwise $n_\sigma(\pi_{T,n}) = 0$.)

Here we study the cases when $T$ is a set of permutations of length 3. The cases when $T$ contains a permutation of length $\leq 2$ are trivial, since then there is at most one permutation in $\mathcal{G}_n(T)$ for any $n$. The case of forbidding one or several permutations of length $\geq 4$ seems much more complicated, but there are recent impressive results for $\mathcal{G}_n(2413, 3142)$ (separable permutations) by Bassino, Bouvel, Féray, Gerin, and Pierrot [2], with generalizations to some other classes in [1].

There are $2^6 = 64$ sets $T$ of permutations of length 3. Of these, every $T$ that contains $\{123, 321\}$, and every $T$ with $|T| \geq 4$ is trivial, in the sense that $\mathcal{G}_n(T)$ contains at most 2 elements for any $n \geq 5$ (see [14]). Ignoring these cases, there are $1 + 6 + 14 + 16 = 37$ remaining cases (with $|T| = 0, 1, 2, 3$, respectively), and by symmetries, see Appendix A, these reduce to $1 + 2 + 4 + 4 = 11$ non-equivalent cases, which are treated in Sections 2–12. For further details, see [12], [8], [9], [10]; these papers also contain further references to related work, and to some of the many papers by various authors that study other properties of random $T$-avoiding permutations.

The cases studied here, i.e., the non-trivial cases with $T \subset \mathcal{G}_3$, all have asymptotic distributions of one of the following two types.

I. Normal limits: For every $\sigma \in \mathcal{G}_n(T)$, there exists constants $\alpha, \beta, \gamma$ such that, as $n \to \infty$,

$$\frac{n_\sigma(\pi_{T,n}) - \beta n^\alpha}{\gamma n^{1/2}} \xrightarrow{d} N(0, \gamma^2),$$

with convergence of all moments. Furthermore, assuming $|\sigma| \geq 2$, $\gamma^2 > 0$, so the limit is not deterministic, except possibly for one $\sigma \in \mathcal{G}_m(T)$ for each length $m \geq 2$.

In particular, $E n_\sigma(\pi_{T,n}) \sim \beta n^\alpha$. Note that (4) implies concentration, in the sense

$$\frac{n_\sigma(\pi_{T,n})}{E n_\sigma(\pi_{T,n})} \xrightarrow{p} 1.$$

5
The table shows whether \( n_\sigma(\pi_{T,n}) \) has limits of type I or II; furthermore, the exponent \( \alpha = \alpha(\sigma) \) is given in the column for the type. The last column shows the exceptional cases, if any, where the asymptotic variance vanishes. \( C_n := \frac{1}{n+1} \binom{2n}{n} \) is a Catalan number; \( F_{n+1} \) is a Fibonacci number \((F_0 = 0, F_1 = 1)\); \( s_{n-1} \) is a Schröder number; \( D(\sigma) \) is the number of descents and \( B(\sigma) \) is the number of blocks in \( \sigma \).

| \( T \) | \( |\mathfrak{S}_n(T)| \) | type I | type II | as. variance = 0 |
|-------|-----------------|--------|---------|-----------------|
| \( \emptyset \) | \( n! \) | \(|\sigma|\) | \(|\sigma| + D(\sigma)/2\) | \( m \cdots 1 \) |
| \{132\} | \( C_n \) | \(|\sigma| + B(\sigma)/2\) | \( 1 \cdots m \) |
| \{321\} | \( 2^{n-1} \) | \(|\sigma|\) | \(|\sigma| + B(\sigma)\) | \( 1 \cdots m \) |
| \{32, 312\} | \( n \) | \(|\sigma|\) | \(|\sigma| - 1 \) or \(|\sigma|\) | \( 1 \cdots m \) |
| \{32, 23, 312\} | \( n \) | \(|\sigma|\) | \(|\sigma|\) | \( 1 \cdots m \) |
| \{32, 23, 321\} | \( n \) | \(|\sigma|\) | \(|\sigma|\) | \( 1 \cdots m \) |
| \{2413, 3142\} | \( s_{n-1} \) | \(|\sigma|\) | \(|\sigma|\) | \(|\sigma|\) |

II. Non-normal limits without concentration: For every \( \sigma \in \mathfrak{S}_m(T) \), there exists a constant \( \alpha \) such that

\[
\frac{n_\sigma(\pi_{T,n})}{n^\alpha} \xrightarrow{d} W_\sigma ,
\]

with convergence of all moments, for some random variable \( W_\sigma > 0 \). Hence, also

\[
\frac{n_\sigma(\pi_{T,n})}{\mathbb{E} n_\sigma(\pi_{T,n})} \xrightarrow{d} W'_\sigma ,
\]

with convergence of all moments, for some random variable \( W'_\sigma > 0 \) (necessarily with \( \mathbb{E} W'_\sigma = 1 \)). Furthermore, assuming \(|\sigma| \geq 2\), \( \operatorname{Var} W_\sigma > 0 \), so \( W_\sigma \) and \( W'_\sigma \) are not deterministic, except possibly for one \( \sigma \in \mathfrak{S}_m(T) \) for each length \( m \geq 2 \).

 Remark. In all cases studied here, if there are any exceptional \( \sigma \in \mathfrak{S}_m(T) \) with \( \sigma \geq 2 \) such that the limit in (4) or (6) is deterministic, i.e., the asymptotic variance is 0, then the exceptional \( \sigma \) are either all identity permutations \( 1 \cdots m \), or all decreasing permutations \( m \cdots 1 \). Furthermore, these exceptional cases arise because almost all of the \( \binom{n}{|\sigma|} \) patterns in \( \pi_{T,n} \) of length \(|\sigma|\) are occurrences of \( \sigma \); more precisely, \( \mathbb{E} \left( \binom{n}{|\sigma|} - n_\sigma(\pi_{T,n}) \right) = O(n^{|\sigma|-1}) \) for the exceptional cases of type I and \( O(n^{|\sigma|-1/2}) \) for the cases of type II. (It follows that (5) holds also for the latter.)

We summarize the results for \( T \) consisting of permutations of length 3 in Table 1; for reference, we include the number \( |\mathfrak{S}_n(T)| \) of \( T \)-avoiding permutations of length \( n \), see e.g. [13, Exercises 2.2.1-4.5], [15, Exercise 6.19ee,ff], [3, Corollary 4.7], and [14]. We include also the case \( T = \{2413, 3142\} \) from [2]; see [17] for the enumeration.

We see no obvious pattern in the existence of limits of type I or II in Table 1. Moreover, the proofs, sketched below, are done case by case; we have not succeeded to prove any general results, treating all (or at least some) forbidden sets \( T \) at the same time.

 Remark. We do not know whether a general set of forbidden permutations \( T \) has limits in distribution of \( n_\sigma(\pi_{T,n}) \) (after normalization) at all, and even if limits exist, there is no known reason implying that they have to be of type I or II above; other types of limits are conceivable.
6:4 Patterns in Random Permutations Avoiding Some Other Patterns

Remark. The non-normal limits in the cases \{132\}, \{321\} and \{2413, 3142\} can all be expressed as functionals of a Brownian excursion \(e\), see [8, 9, 2]. However, the expressions in these three cases are, in general, quite different (and obtained by quite different arguments), so there is no obvious hope for a unification. (The other cases of non-normal limits in Table 1 are different, and of a more elementary kind.)

1.1 Some notation

Let \(i = i_n\) be the identity permutation of length \(n\).

If \(\sigma \in \mathcal{S}_m\) and \(\tau \in \mathcal{S}_n\), their composition \(\sigma * \tau \in \mathcal{S}_{m+n}\) is defined by letting \(\tau\) act on \([m+1, m+n]\) in the natural way; more formally, \(\sigma * \tau = \pi \in \mathcal{S}_{m+n}\) where \(\pi_i = \sigma_i\) for \(1 \leq i \leq m\), and \(\pi_{j+m} = \tau_j + m\) for \(1 \leq j \leq n\). We say that a permutation \(\pi \in \mathcal{S}_n\) is decomposable if \(\pi = \sigma * \tau\) for some \(\sigma, \tau \in \mathcal{S}_m\), and indecomposable otherwise; we also call an indecomposable permutation a block.

It is easy to see that any permutation \(\pi \in \mathcal{S}_n\) has a unique decomposition \(\pi = \pi_1 * \cdots * \pi_\ell\) into indecomposable permutations (blocks) \(\pi_1, \ldots, \pi_\ell\); we call these the blocks of \(\pi\). (These are useful to characterize the permutations in some of the classes below.)

2 No restriction, \(T = \emptyset\)

As a background, consider first the case \(T = \emptyset\), so \(\mathcal{S}_n(T) = \mathcal{S}_n\); the set of all \(n!\) permutations of length \(n\). It is well-known, see Bóna [4, 5] and [12, Theorem 4.1], that if \(\pi_n\) is a uniformly random permutation in \(\mathcal{S}_n\), then \(n_{\sigma}(\pi_n)\) has an asymptotic normal distribution as \(n \to \infty\) for every fixed permutation \(\sigma\):

\[\frac{n_{\sigma}(\pi_n) - \frac{1}{2}(\binom{n}{m})}{\sqrt{n}^{m-1/2}} \xrightarrow{d} N(0, \gamma^2).\]  

(8)

Sketch of proof. A random permutation \(\pi_n\) can be obtained by taking i.i.d. random variables \(X_1, \ldots, X_n \sim U(0, 1)\) and considering their ranks. Then

\[n_{\sigma}(\pi_n) = \sum_{i_1 < \cdots < i_m} f(X_{i_1}, \ldots, X_{i_m})\]  

(9)

for a suitable (indicator) function \(f\). This sum is an asymmetric \(U\)-statistic, and the result follows by general results on \(U\)-statistics, see [6] and [11].

Remark. The asymptotic variance \(\gamma^2\) depends on \(\sigma\). It can be calculated explicitly, and the same holds for all parameters \(\gamma^2\) (or \(\mu\)) in the limit theorems below. Moreover, the convergence (8) holds with convergence of all moments, and it holds jointly for any set of \(\sigma\); also this holds for all later limit theorems too.

3 Avoiding 132

Consider next the cases when \(T\) consists of a single permutation of length 3. The symmetries in Appendix A leave two non-equivalent cases. In this section we avoid \(T = \{132\}\); equivalent cases are \{213\}, \{231\}, \{312\}. Recall that the standard Brownian excursion \(e(x)\) is a random non-negative function on \([0, 1]\). Let

\[\lambda(\sigma) := |\sigma| + D(\sigma)\]  

(10)
where \( D(\sigma) \) is the number of descents in \( \sigma \), i.e., indices \( i \) such that \( \sigma_i > \sigma_{i+1} \) or (as a convenient convention) \( i = |\sigma| \). Note that \( 1 \leq D(\sigma) \leq |\sigma| \), and thus

\[
|\sigma| + 1 \leq \lambda(\sigma) \leq 2|\sigma|,
\]
with the extreme values \( \lambda(\sigma) = |\sigma| + 1 \) if and only if \( \sigma = 1 \cdots k \), and \( \lambda(\sigma) = 2|\sigma| \) if and only if \( \sigma = k \cdots 1 \), for some \( k = |\sigma| \).

\textbf{Theorem 2} ([8]). There exist strictly positive random variables \( \Lambda_\sigma \) such that as \( n \to \infty \),

\[
n_\sigma(\pi_{132; n})/n^{\lambda(\sigma)/2} \xrightarrow{d} \Lambda_\sigma.
\]

\textbf{Sketch of proof.} The analysis is based on a well-known bijection with binary trees and Dyck paths, and the, also well-known, convergence in distribution of random Dyck paths to a Brownian excursion. For (not so simple) details, see [8].

The limit variables \( \Lambda_\sigma \) in Theorem 2 can be expressed as functionals of a Brownian excursion \( e(x) \), see [8]; the description is, in general, rather complicated, but some cases are simple. Moments of the variables \( \Lambda_\sigma \) can be calculated by a recursion formula given in [8].

\textbf{Example 3.} In the special case \( \sigma = 12 \), \( \Lambda_{12} = \sqrt{2} \int_0^1 e(x) \, dx \), see [8, Example 7.6]; this is (apart from the factor \( \sqrt{2} \)) the well-known Brownian excursion area, see e.g. [7] and the references there.

By symmetries, see Appendix A, the left-hand side can also be seen as the number of inversions \( n_{21}(\pi_{231; n}) \) or \( n_{21}(\pi_{312; n}) \), normalized by \( n^{3/2} \), where we instead avoid 231 or 312.

\section{Avoiding 321}

In this section we avoid \( T = \{321\} \). The case \( T = \{123\} \) is equivalent.

\( \mathfrak{S}_n(321) \) is treated in detail in [9]. As for \( \mathfrak{S}_n(132) \) in Section 3, the analysis is based on a well-known bijection with Dyck paths, but the details are very different, and so are in general the resulting limit distributions.

\textbf{Theorem 4} ([9]). Let \( \sigma \in \mathfrak{S}_n(321) \). Let \( m := |\sigma| \), and suppose that \( \sigma \) has \( \ell \) blocks of lengths \( m_1, \ldots, m_\ell \). Then, as \( n \to \infty \),

\[
n_\sigma(\pi_{321; n})/n^{(m+\ell)/2} \xrightarrow{d} W_\sigma
\]

for a positive random variable \( W_\sigma \) that can be represented as

\[
W_\sigma = w_\sigma \int_{0< t_1< \cdots< t_\ell< 1} e(t_1)^{m_1-1} \cdots e(t_\ell)^{m_\ell-1} \, dt_1 \cdots dt_\ell,
\]

where \( w_\sigma \) is a positive constant.

\textbf{Sketch of proof.} As for Theorem 2, the analysis is based on a bijection with Dyck paths, and the convergence in distribution of random Dyck paths to a Brownian excursion. For details, see [8].
Patterns in Random Permutations Avoiding Some Other Patterns

In this case, we have an explicit general formula (15) for the limit variables. On the other hand, we do not know how to compute even the mean $E W_\sigma$ in general; see [9] for calculations in various special cases.

Example 5. Let $\sigma = 21$. Then $w_{21} = 2^{-1/2}$, see [9], and thus (14)–(15), with $\ell = 1$ and $m_1 = m = 2$, yield for the number of inversions,

$$\frac{n_{21}(\pi_{321, n})}{n^{3/2}} \xrightarrow{d} 2^{-1/2} \int_0^1 e(x) \, dx.$$

(16)

Note that the limit in (16) differs from the one in (13) by a factor 2.

5 Avoiding $\{132, 312\}$

In this section we avoid $T = \{132, 312\}$. Equivalent sets are $\{132, 231\}$, $\{213, 231\}$, $\{213, 312\}$.

Theorem 6. For any $m \geq 2$ and $\sigma \in S_m(132, 312)$, as $n \to \infty$,

$$\frac{n_\sigma(\pi_{132, 312, n}) - 2^{1-m} n^m / m!}{n^{m-1/2}} \xrightarrow{d} N(0, \gamma^2).$$

(17)

Sketch of proof. It was shown by [14, Proposition 12] (in an equivalent formulation) that a permutation $\pi$ belongs to the class $S_\sigma(132, 312)$ if and only if every entry $\pi_i$ is either a maximum or a minimum. We encode a permutation $\pi \in S_n(132, 312)$ by a sequence $\xi_2, \ldots, \xi_n \in \{\pm 1\}^{n-1}$, where $\xi_j = 1$ if $\pi_j$ is a maximum in $\pi$, and $\xi_j = -1$ if $\pi_j$ is a minimum. This is a bijection, and hence the code for a uniformly random $\pi_{132, 312, n}$ has $\xi_2, \ldots, \xi_n$ i.i.d. with the symmetric Bernoulli distribution $P(\xi_j = 1) = P(\xi_j = -1) = 1/2$.

Let $\sigma \in S_m(132, 312)$ have the code $\eta_2, \ldots, \eta_m$. Then $\pi_{i_1} \cdots \pi_{i_m}$ is an occurrence of $\sigma$ in $\pi$ if and only if $\xi_{i_j} = \eta_j$ for $2 \leq j \leq m$. Consequently, $n_\sigma(\pi_{132, 312, n})$ is a $U$-statistic

$$n_\sigma(\pi_{132, 312, n}) = \sum_{i_1 < \cdots < i_m} f(\xi_{i_1}, \ldots, \xi_{i_m}),$$

(18)

where

$$f(\xi_1, \ldots, \xi_m) := \prod_{j=2}^m 1(\xi_j = \eta_j).$$

(19)

Note that $f$ does not depend on the first argument.

The result now follows from the theory of $U$-statistics [6], [11].

Example 7. For the number of inversions, we have $\sigma = 21$ and $m = 2$, $\eta_2 = -1$. A calculation yields $\mu = 1/2$ and $\gamma^2 = 1/12$, and thus Theorem 6 yields

$$\frac{n_{21}(\pi_{132, 312, n}) - n^{2/4}}{n^{3/4}} \xrightarrow{d} N(0, \frac{1}{12}),$$

(20)

6 Avoiding $\{231, 312\}$

In this section we avoid $T = \{231, 312\}$. The only equivalent set is $\{132, 213\}$.

Theorem 8. Let $\sigma \in S_m(231, 312)$ have block lengths $\ell_1, \ldots, \ell_b$. Then, as $n \to \infty$,

$$\frac{n_\sigma(\pi_{231, 312, n}) - n^{b/4}}{n^{b-1/2}} \xrightarrow{d} N(0, \gamma^2).$$

(21)
Sketch of proof. It was shown by [14, Proposition 12] (in an equivalent form) that a permutation \( \pi \) belongs to the class \( \mathcal{G}_s(231, 312) \) if and only if every block in \( \pi \) is decreasing, i.e., of the type \( \ell(\ell - 1) \cdots 21 \) for some \( \ell \). Hence there exists exactly one block of each length \( \ell \geq 1 \), and a permutation \( \pi \in \mathcal{G}_s(231, 312) \) can be encoded by its sequence of block lengths. In this section, let \( \pi_{\ell_1, \ldots, \ell_b} \) denote the permutation in \( \mathcal{G}_s(231, 312) \) with block lengths \( \ell_1, \ldots, \ell_b \).

A uniformly random permutation \( \pi_{231,312,n} \) can be generated as \( \pi_{L_1, \ldots, L_B} \), where the block lengths \( L_1, \ldots, L_B \) are obtained from an infinite i.i.d. sequence \( L_1, L_2, \ldots \sim \text{Ge}(\frac{1}{3}) \), stopped at \( B \) such that \( L_1 + \cdots + L_B \geq n \), and then adjusting \( L_B \) such that \( L_1 + \cdots + L_B = n \).

Let \( \sigma \in \mathcal{G}_s(231, 312) \) have block lengths \( \ell_1, \ldots, \ell_b \), so that \( \sigma = \pi_{\ell_1, \ldots, \ell_b} \). Then,

\[
n_\sigma(\pi_{L_1, \ldots, L_B}) = \sum_{1 \leq i_1 < \cdots < b \leq B} \prod_{j=1}^{b} \binom{L_{i_j}}{\ell_{i_j}}.
\]  

(22)

This is again a kind of U-statistic, but it is based on the sequence \( L_1, \ldots, L_B \) of random length \( B \), obtained by stopping the infinite sequence \( L_i \). Nevertheless, general results for U-statistics cover this modification and yield the result, see [11].

Example 9. For the number of inversions, we have \( \sigma = 21 \) and \( b_1 = 1, \ell_1 = 2 \). A calculation yields \( \gamma^2 = 6 \), and Theorem 8 yields

\[
\frac{n_{21}(\pi_{231,312,n}) - n}{n^{1/2}} \xrightarrow{d} N(0, 6).
\]

(23)

7 Avoiding \( \{231, 321\} \)

In this section we avoid \( T = \{231, 321\} \). Equivalent sets are \( \{123, 132\}, \{123, 213\}, \{312, 321\} \).

Theorem 10. Let \( \sigma \in \mathcal{G}_m(231, 321) \) have block lengths \( \ell_1, \ldots, \ell_b \), and let \( b_1 \) be the number of blocks of length \( \ell_1 = 1 \). Then, as \( n \to \infty \),

\[
\frac{n_\sigma(\pi_{231,321,n}) - 2^{b_1 - b_1 n^b/b!} n^{b_1 - 1/2}}{n^{b_1 - 1/2}} \xrightarrow{d} N(0, \gamma^2).
\]

(24)

Sketch of proof. It was shown by [14, Proposition 12] (in an equivalent form) that a permutation \( \pi \) belongs to the class \( \mathcal{G}_s(231, 321) \) if and only if every block in \( \pi \) is of the type \( \ell(\ell - 1) \cdots (\ell - 1) \) for some \( \ell \). Thus, as in Section 6, a permutation in \( \mathcal{G}_s(231, 321) \) is determined by its block lengths, and these can be arbitrary. Hence, a uniformly random \( \pi_{231,321,n} \) has block lengths \( L_1, \ldots, L_B \) with the same distribution as in Section 6. Letting now \( \sigma \) be the permutation in \( \mathcal{G}_s(231, 321) \) with block lengths \( \ell_1, \ldots, \ell_b \), \( n_\sigma(\pi_{231,321,n}) \) is a function of the block lengths \( L_1, \ldots, L_B \) that is similar (but not identical) to (22). This time some lower order terms appear, but they may be neglected, and the remainder is a U-statistic similar to the one in the proof of Theorem 8, and the result follows in the same way.

Example 11. For the number of inversions, we have \( \sigma = 21 \) and \( b_1 = 1, b_1 = 2, b_1 = 0 \). A calculation yields \( \gamma^2 = 1/4 \), and Theorem 10 yields

\[
\frac{n_{21}(\pi_{231,321,n}) - n/2}{n^{1/2}} \xrightarrow{d} N(0, \frac{1}{4}).
\]

(25)

In fact, in this special case it can be seen that we have the exact distribution

\[
n_{21}(\pi_{231,321,n}) \sim B(n - 1, \frac{1}{2}).
\]

(26)
Avoiding \{132, 321\}

In this section we avoid \(T = \{132, 321\}\). Equivalent sets are \{123, 321\}, \{123, 312\}, \{213, 321\}.

It was shown in [14, Proposition 13] that a permutation \(\pi\) belongs to \(\mathcal{S}_n(132, 321)\) if and only if either \(\pi = \iota_n\) for some \(n\), or \(\pi = \pi_{k,\ell,m}\) for some \(k,\ell \geq 1\) and \(m \geq 0\), where, in this section,

\[
\pi_{k,\ell,m} := (\ell + 1, \ldots, \ell + k, \ldots, \ell, k + \ell + 1, \ldots, k + \ell + m) \in \mathcal{S}_{k + \ell + m}.
\]

Recall that the Dirichlet distribution \(\text{Dir}(1, 1, 1)\) is the uniform distribution on the simplex \(\{(x, y, z) \in \mathbb{R}_+^3 : x + y + z = 1\}\).

\begin{theorem}
Let \(\sigma \in \mathcal{S}_n(132, 321)\). Then the following hold as \(n \to \infty\).
\begin{enumerate}[(i)]
\item If \(\sigma = \pi_{i,j,p}\) for some \(i, j, p\), then
\[
n^{-i-j+p}n_\sigma(\pi_{132,321,n}) \xrightarrow{d} W_{i,j,p} := \frac{1}{i!j!p!}X^iY^jZ^p,
\]
where \((X, Y, Z) \sim \text{Dir}(1,1,1)\).
\item If \(\sigma = \iota_i\), then
\[
n^{-}n_\sigma(\pi_{132,321,n}) \xrightarrow{d} W_i := \frac{1}{i!}((X + Z)^i + (Y + Z)^i - Z^i),
\]
with \((X, Y, Z) \sim \text{Dir}(1,1,1)\) as in i.
\end{enumerate}
\end{theorem}

\begin{proof}
Sketch of proof. For asymptotic results, we may ignore the case when \(\pi_{132,321,n} = \iota_n\). Conditioning on \(\pi_{132,321,n} \neq \iota_n\), we have \(\pi_{132,321,n} = \pi_{K,L,n-K-L}\), where \(K\) and \(L\) are random with \((K, L)\) uniformly distributed over the set \(\{K, L \geq 1 : K + L \leq n\}\). As \(n \to \infty\), we thus have

\[
\left(\frac{K}{n}, \frac{L}{n}, \frac{n - K - L}{n}\right) \xrightarrow{d} (X, Y, Z) \sim \text{Dir}(1,1,1).
\]

If \(\sigma = \pi_{i,j,p}\) for some \(i, j, p\), then it is easily seen that

\[
n_\sigma(\pi_{k,\ell,m}) = \left(\begin{array}{c} k \\ i \\ j \\ p \end{array} \right).
\]

Similarly, if \(\sigma = \iota_i\), then, by inclusion-exclusion,

\[
n_\sigma(\pi_{k,\ell,m}) = \left(\begin{array}{c} k + m \\ i \\ j \\ p \end{array} \right).
\]

These exact formulas and (30) yield the results.
\end{proof}

\begin{corollary}
The number of inversions has the asymptotic distribution
\[
n^{-2}n_{21}(\pi_{132,321,n}) \xrightarrow{d} W := XY,
\]
with \((X, Y)\) as above; the limit variable \(W\) has density function

\[
2\log(1 + \sqrt{1 - 4x}) - 2\log(1 - \sqrt{1 - 4x}), \quad 0 < x < 1/4,
\]
and moments

\[
\mathbb{E} W^r = 2\frac{r!^2}{(2r + 2)!}, \quad r > 0.
\]
\end{corollary}
9 Avoiding \{231,312,321\}

We proceed to sets of three forbidden patterns. In this section we avoid $T = \{231, 312, 321\}$. An equivalent set is $\{123, 132, 213\}$.

**Theorem 14.** Let $\sigma \in \mathcal{S}_m(231, 312, 321)$ have block lengths $\ell_1, \ldots, \ell_b$. Then, as $n \to \infty$,

$$\frac{n_{\sigma}(\pi_{231,312,321:n}) - \mu b / \sqrt{b}}{\mu b^{1/2}} \xrightarrow{d} N(0, \gamma^2),$$

for some constants $\mu$ and $\gamma^2$.

**Sketch of proof.** It was shown in [14, Proposition 15*] (in an equivalent form) that a permutation $\pi$ belongs to the class $\mathcal{G}_n(231, 312, 321)$ if and only if every block in $\pi$ is decreasing and has length $\leq 2$, i.e., every block is 1 or 21. Hence, a permutation $\pi \in \mathcal{S}_m(231, 312, 321)$ is uniquely determined by its sequence of block lengths $L_1, \ldots, L_B$, where each $L_i \in \{1, 2\}$ and $L_1 + \cdots + L_B = n$.

Let $p := (\sqrt{5} - 1)/2$, the golden ratio, so that $p + p^2 = 1$. Let $X$ be a random variable with the distribution

$$\mathbb{P}(X = 1) = p, \quad \mathbb{P}(X = 2) = p^2.$$ (37)

Consider an i.i.d. sequence $X_1, X_2, \ldots$ of copies of $X$, and let $S_k := \sum_{i=1}^k X_i$. Let further $B(n) := \min\{k : S_k \geq n\}$. Then, conditioned on $S_{B(n)} = n$, the sequence $X_1, \ldots, X_{B(n)}$ has the same distribution as the sequence $L_1, \ldots, L_B$ of block lengths of a uniformly random permutation $\pi_{231,312,321:n}$.

Consequently, $n_{\sigma}(\pi_{231,312,321:n})$ can be expressed as a $U$-statistic based on $X_1, \ldots, X_B$, conditioned as above. This conditioning does not affect the asymptotic distribution, see [11], and the result follows again by general results for $U$-statistics.

**Example 15.** For the number of inversions, $\sigma = 21$ we have $b = 1$. A calculation yields $\mu = 1 - p = (3 - \sqrt{5})/2$ and $\gamma^2 = 5 - 3/2$. Consequently,

$$\frac{n_{21}(\pi_{231,312,321:n}) - \frac{3 - \sqrt{5}}{2} n}{n^{1/2}} \xrightarrow{d} N(0, 5 - 3/2).$$ (38)

10 Avoiding \{132,231,312\}

In this section we avoid $\{132, 231, 312\}$. Equivalent sets are $\{132, 213, 231\}, \{132, 213, 312\}, \{213, 231, 312\}$.

It was shown in [14, Proposition 16*] (in an equivalent form) that $\mathcal{G}_n(132, 231, 312) = \{\pi_{k,n-k} : 1 \leq k \leq n\}$, where, in this section,

$$\pi_{k,\ell} := (k, \ldots, 1, k+1, \ldots, k+\ell) \in \mathcal{S}_{k+n}, \quad k \geq 1, \ell \geq 0.$$ (39)

**Theorem 16.** Let $\sigma \in \mathcal{S}_n(132, 231, 312)$. Then the following hold as $n \to \infty$, with $U \sim \mathcal{U}(0, 1)$.

(i) If $\sigma = \pi_{k,m-k}$ with $2 \leq k \leq m$, then

$$n^{-m} n_{\sigma}(\pi_{132,231,312:n}) \xrightarrow{d} W_{k,m-k} := \frac{1}{k!(m-k)!} \left(1 - U\right)^{m-k}.$$ (40)
Corollary 17. The number of inversions has the asymptotic distribution
\[ n^{-2} n_{21}(\pi_{132,231,312,n}) \xrightarrow{d} \mathcal{W} := U^2 / 2 \]
with \( U \sim \mathcal{U}(0,1) \). Thus, \( 2W \sim \mathcal{B}(\frac{1}{2},1) \), and \( W \) has moments
\[ \mathbb{E} W^r = \frac{1}{2^r (2r + 1)}, \quad r > 0. \]

11 Avoiding \( \{132,231,321\} \)

In this section we avoid \( \{132,231,321\} \). Equivalent sets are \( \{123,132,231\}, \{123,213,312\}, \{213,312,231\}, \{123,132,231\}, \{132,312,231\}, \{213,231,312\} \).

It was shown in [14, Proposition 16] (in an equivalent form) that \( \mathcal{E}_n(132,231,312) = \{\pi_{k,n-k} : 1 \leq k \leq n\} \), where, in this section,
\[ \pi_{k,\ell} := (k,1,\ldots,k-1,k+1,\ldots,k+\ell) \in \mathcal{S}_{k+\ell}, \quad k \geq 1, \ell \geq 0. \]

Theorem 18. Let \( \sigma \in \mathcal{S}_n(132,231,321) \). Then the following hold as \( n \to \infty \), with \( U \sim \mathcal{U}(0,1) \).

(i) If \( \sigma = \pi_{k,m-k} \) with \( 2 \leq k \leq m \), then
\[ n^{-(m-1)} n_{\sigma}(\pi_{132,231,321,n}) \xrightarrow{d} W_{k,m-k} := \frac{1}{(k-1)! (m-k)!} U^{k-1} (1 - U)^{m-k}. \]

(ii) If \( \sigma = \pi_{1,m-1} = t_m \), then
\[ n^{-m} n_{\sigma}(\pi_{132,231,321,n}) = \frac{1}{m!} + O(n^{-1}) \xrightarrow{d} \frac{1}{m!}. \]

Sketch of proof. The random permutation \( \pi_{132,231,321,n} := \pi_{K,n-K} \), where \( K \in [n] \) is uniformly random. The results follow similarly to the proof of Theorem 16.

Corollary 19. The number of inversions \( n_{21}(\pi_{132,231,321,n}) \) has a uniform distribution on \( \{0,\ldots,n-1\} \), and thus the asymptotic distribution
\[ n^{-1} n_{21}(\pi_{132,231,321,n}) \xrightarrow{d} U \sim \mathcal{U}(0,1). \]
\section{Avoiding \{132, 213, 321\}}

In this section we avoid \{132, 213, 321\}. An equivalent set is \{123, 231, 312\}.

It was shown in [14, Proposition 16\*] (in an equivalent form) that \(S_n(132, 213, 321) = \{\pi_{k,n-k} : 1 \leq k \leq n\}\), where, in this section,
\[
\pi_{k,\ell} := (\ell + 1, \ldots, \ell + k, 1, \ldots, \ell) \in S_{k+\ell}, \quad k \geq 1, \ell \geq 0.
\] (50)

\begin{Theorem}
Let \(\sigma \in S_n(132, 213, 321)\). Then the following hold as \(n \to \infty\), with \(U \sim \mathcal{U}(0, 1)\).
\begin{enumerate}[(i)]
\item If \(\sigma = \pi_{k,m-k}\) with \(1 \leq k \leq m - 1\), then
\[
n^{-m}n_\sigma(\pi_{132,213,321,n}) \xrightarrow{d} W_{k,m-k} := \frac{1}{k!(m-k)!}U^k(1-U)^{m-k}.
\] (51)
\item If \(\sigma = \pi_{m,0} = \iota_m\), then
\[
n^{-m}n_\sigma(\pi_{132,213,321,n}) \xrightarrow{d} W_{m,0} := \frac{1}{m!}(U^m + (1-U)^m).
\] (52)
\end{enumerate}
\end{Theorem}

\textbf{Sketch of proof.} Similarly to the proof of Theorem 16. \qed

\begin{Corollary}
The number of inversions has the asymptotic distribution
\[
n^{-2}n_{21}(\pi_{132,213,321,n}) \xrightarrow{d} W \equiv U(1-U),
\] (53)
with \(U \sim \mathcal{U}(0, 1)\). Thus, \(4W \sim B(1, \frac{1}{2})\), and \(W\) has moments
\[
\mathbb{E}W^r = \frac{\Gamma(r+1)^2}{\Gamma(2r+2)}, \quad r > 0.
\] (54)
\end{Corollary}

\textbf{References}
\begin{enumerate}
\end{enumerate}
For any permutation \( \pi = \pi_1 \cdots \pi_n \), define its \textit{inverse} \( \pi^{-1} \) in the usual way, and its \textit{reversal} and \textit{complement} by

\[
\pi' := \pi_n \cdots \pi_1, \\
\pi^c := (n + 1 - \pi_1) \cdots (n + 1 - \pi_n).
\]

(55)  
(56)

These three operations generate a group \( G \) of 8 symmetries (isomorphic to the dihedral group \( D_4 \)). It is easy to see that for any symmetry \( s \in G \),

\[
n_{s^*}(\pi^c) = n_s(\pi).
\]

(57)

Thus, if we define \( T^s := \{ \tau^s : \tau \in T \} \), then

\[
G_n(T^s) = \{ \pi^c : \pi \in G_n(T) \},
\]

(58)

and, for any permutation \( \sigma \),

\[
n_{s^*}(\pi_{T^s;n}) \overset{d}{=} n_s(\pi_{T;n}).
\]

(59)

We say that the sets of forbidden permutations \( T \) and \( T^s \) are \textit{equivalent}, and note that (59) implies that it suffices to consider one set \( T \) in each equivalence class \( \{ T^s : s \in G \} \).