On a Problem of Danzer

Nabil H. Mustafa[1]
Université Paris-Est, Laboratoire d'Informatique Gaspard-Monge, Equipe A3SI, ESIEE Paris
mustafan@esiee.fr

Saurabh Ray
Department of Computer Science, NYU Abu Dhabi, United Arab Emirates
saurabh.ray@nyu.edu

Abstract

Let C be a bounded convex object in $\mathbb{R}^d$, and P a set of n points lying outside C. Further let c_p, c_q be two integers with $1 \leq c_q \leq c_p \leq n - \left\lfloor \frac{d}{2} \right\rfloor$, such that every $c_p + \left\lfloor \frac{d}{2} \right\rfloor$ points of P contains a subset of size $c_q + \left\lfloor \frac{d}{2} \right\rfloor$ whose convex-hull is disjoint from C. Then our main theorem states the existence of a partition of P into a small number of subsets, each of whose convex-hull is disjoint from C. Our proof is constructive and implies that such a partition can be computed in polynomial time.

In particular, our general theorem implies polynomial bounds for Hadwiger-Debrunner (p, q) numbers for balls in $\mathbb{R}^d$. For example, it follows from our theorem that when $p > q \geq (1 + \beta) \cdot \frac{d}{2}$ for $\beta > 0$, then any set of balls satisfying the $\mathrm{HD}(p, q)$ property can be hit by $O\left(q^2 p^{1+\frac{1}{\beta}} \log p\right)$ points. This is the first improvement over a nearly 60-year old exponential bound of roughly $O\left(2^d\right)$.

Our results also complement the results obtained in a recent work of Keller *et al.* where, apart from improvements to the bound on $\mathrm{HD}(p, q)$ for convex sets in $\mathbb{R}^d$ for various ranges of p and q, a polynomial bound is obtained for regions with low union complexity in the plane.

2012 ACM Subject Classification Theory of computation → Computational geometry

Keywords and phrases Convex polytopes, Hadwiger-Debrunner numbers, Epsilon-nets, Balls

Digital Object Identifier 10.4230/LIPIcs.ESA.2018.64

1 Introduction

Given a finite set $\mathcal{C}$ of geometric objects in $\mathbb{R}^d$, we say that $\mathcal{C}$ satisfies the $\mathrm{HD}(p, q)$ property if for any set $\mathcal{C}' \subseteq \mathcal{C}$ of size p, there exists a point in $\mathbb{R}^d$ common to at least q objects of $\mathcal{C}'$. The goal then is to show that there exists a small set Q of points in $\mathbb{R}^d$ such that each object of $\mathcal{C}$ contains some point of Q; such a Q is called a hitting set for $\mathcal{C}$.

These bounds for a set $\mathcal{C}$ of convex sets in $\mathbb{R}^d$ have been studied since the 1950s (see the surveys [7, 8, 15]), and it was only in 1991 that Alon and Kleitman [1], in a breakthrough result, gave an upper-bound that is *independent* of $|\mathcal{C}|$. Unfortunately it depends exponentially on p, q and d. For the case where $\mathcal{C}$ consists of arbitrary convex objects, the current best bounds remain exponential in p, q and d.

[1] The work of Nabil H. Mustafa in this paper has been supported by the grant ANR SAGA (JCJC-14-CE25-0016-01).

26th Annual European Symposium on Algorithms (ESA 2018).
Editors: Yossi Azar, Hannah Bast, and Grzegorz Herman; Article No. 64; pp. 64:1–64:8
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

▶ **Theorem A** ([1, 9]). *Let $\mathcal{C}$ be a finite set of convex objects in $\mathbb{R}^d$ satisfying the* HD(p,q) *property, where p, q are two integers with $p \geq q \geq d+1$. Then there exists a hitting set for $\mathcal{C}$ of size*

$$\begin{cases} O\left(p^{d\frac{q-1}{q-d}} \cdot \log^{c' d^3 \log d} p\right), & \\ (p-q) + O\left(\left(\frac{p}{q}\right)^d \log^{c' d^3 \log d}\left(\frac{p}{q}\right)\right), & \text{for } q \geq \log p \\ p-q+2, & \text{for } q \geq p^{1-\frac{1}{d}+\epsilon}, p \geq p(d,\epsilon). \end{cases}$$

where c' is an absolute constant independent of $|\mathcal{C}|, p, q$ and d, and $p(d,\epsilon)$ is a function depending only on d and ϵ.

Consider the basic case where $\mathcal{C}$ is a set of balls in $\mathbb{R}^d$ satisfying the HD(p,q) property. Theorem A implies – ignoring logarithmic factors and for general values of p and q – the existence of a hitting set of size no better than $O\left(p^d\right)$. Furthermore, it requires $q \geq d+1$ – a necessary condition for arbitrary convex objects[2] but not for balls.

Almost 60 years ago, Danzer [4, 5] considered the HD(p,q) problem for balls. The best bound that we are aware of, derived from the survey of Eckhoff [7] by combining inequalities (4.2), (4.4) and (4.5), is stated below. It is better than the one from Theorem A quantitatively, but also in that it gives a bound requiring only that $q \geq 2$. Further, for a very specific case – namely when $p = q$ and $(d-q)$ is $O(\log d)$ – it succeeds in giving polynomial bounds.

▶ **Theorem B** ([7]). *Let $\mathcal{B}$ be a finite set of balls in $\mathbb{R}^d$. If $\mathcal{B}$ satisfies the* HD(p,q) *property for some $d \geq p \geq q \geq 2$, then there exists a hitting set for $\mathcal{B}$ of size at most*

$$\sqrt{\frac{3\pi}{2}} \cdot 2^{d-q} \cdot \left((p-q) \cdot 2^q \cdot d^{\frac{3}{2}} \cdot g(d) + 4\,(d-q+2)^{\frac{3}{2}} \cdot g(d-q+2)\right)$$

where $g(x) = \log x + \log\log x + 1$. Ignoring logarithmic terms, the above bound is of the form $\Theta\left((p-q) \cdot 2^d \cdot d^{\frac{3}{2}} + 2^{d-q} \cdot (d-q)^{\frac{3}{2}}\right)$. If $p \neq q$ the first term dominates, otherwise the second term dominates.

Turning towards the lower-bound for the case where $\mathcal{C}$ is a set of unit balls in $\mathbb{R}^d$, Bourgain and Lindenstrauss [2] proved a lower-bound of 1.0645^d when $p = q = 2$ in $\mathbb{R}^d$, i.e., one needs at least 1.0645^d points to hit all pairwise intersecting unit balls in $\mathbb{R}^d$.

Our Result

We consider a more general set up for the HD(p,q) problem, as follows.

Let C be a convex object in $\mathbb{R}^d$, and P a set of n points lying outside C. For each $p \in P$, let H_p be the set of hyperplanes separating p from C. Let C_p be the set of points in $\mathbb{R}^d$ *dual* to the hyperplanes in H_p (see [12, Chapter 5.1]), and let $\mathcal{S} = \{C_p : p \in P\}$.

Our goal is to study the HD(p,q) property for $\mathcal{S}$ – namely, that out of every p objects of $\mathcal{S}$, there exists a point in $\mathbb{R}^d$ common to at least q of them. This is equivalent to the property of C and P that out of every p-sized set $P' \subseteq P$, there exists a hyperplane separating C from a q-sized subset $P'' \subset P'$ – or equivalently, conv(P'') is disjoint from C.

Our main theorem is the following. For a simpler expression, let c_q, c_p be two positive integers such that $p = c_p + \left\lfloor \frac{d}{2} \right\rfloor$ and $q = c_q + \left\lfloor \frac{d}{2} \right\rfloor$.

[2] There are easy examples, e.g. when the convex objects are hyperplanes in $\mathbb{R}^d$.

▶ **Theorem 1.** *Let C be a bounded convex object in $\mathbb{R}^d$ and P a set of n points lying outside C. Further let c_p, c_q be two integers, with $1 \leq c_q \leq c_p \leq n - \left\lfloor \frac{d}{2} \right\rfloor$, such that for every $c_p + \left\lfloor \frac{d}{2} \right\rfloor$ points of P, there exists a subset of size $c_q + \left\lfloor \frac{d}{2} \right\rfloor$ whose convex-hull is disjoint from C. Then the points of P can be partitioned into*

$$\lambda_d\left(c_p, c_q\right) = K_2 \, \frac{d}{c_q} \cdot \left(\sqrt{2}K_1\right)^{\frac{d}{c_q}} \cdot \left(\lfloor d/2 \rfloor + c_q\right)^2 \; \cdot \left(\lfloor d/2 \rfloor + c_p\right)^{1+\frac{\lfloor d/2 \rfloor - 1}{c_q}} \cdot \log\left(\lfloor d/2 \rfloor + c_p\right)$$

sets, each of whose convex-hull is disjoint from C. Here K_1, K_2 are absolute constants independent of n, d, c_p and c_q. Furthermore, such a partition can be computed in polynomial time.

The proof, presented in Section 2, is a combination of three ingredients: the Alon-Kleitman technique [1], bounds on independent sets in hypergraphs [9] and bounds on $(\leq k)$-sets for half-spaces [3]. It is an extension of the proof in [14] which studied Carathéodory's theorem in this setting.

▶ Remark. The restriction that $q \geq \left\lfloor \frac{d}{2} \right\rfloor + 1$ is necessary – as can be seen when P form the vertices of a cyclic polytope in $\mathbb{R}^d$ and C is a slightly shrunk copy of $\mathrm{conv}(P)$.

▶ Remark. Note that when $c_q \geq \beta \cdot \frac{d}{2}$ for any absolute constant $\beta > 0$, the above bound is *polynomial* in the dimension d – it is upper-bounded by $O\left(q^2 p^{1+\frac{1}{\beta}} \log p\right)$.

▶ Remark. It was shown in [13] that C_p is a convex object in $\mathbb{R}^d$ and thus the bounds of Theorem A apply. As before, Theorem 1 substantially improves upon this, as the bounds following from Theorem A are exponential in d and furthermore, require $q \geq d + 1$.

As an immediate corollary of Theorem 1, we obtain the first improvements to the old bound on the (p, q) problem for balls in $\mathbb{R}^d$. The bound in Theorem B is exponential in d – except in special cases where $p = q$ *and* $(d - q)$ is[3] $O(\log d)$. On the other hand, our result gives polynomial bounds as long as $q \geq \beta d$ for any constant $\beta > \frac{1}{2}$.

▶ **Corollary 2** (Hadwiger-Debrunner (p, q) bound for balls in $\mathbb{R}^d$)**.** *Let $\mathcal{B}$ be collection of balls in $\mathbb{R}^d$ such that for every subset of $c_p + \left\lfloor \frac{d+1}{2} \right\rfloor$ balls in $\mathcal{B}$, some $c_q + \left\lfloor \frac{d+1}{2} \right\rfloor$ have a common intersection, where c_p and c_q are integers such that $1 \leq c_q \leq c_p \leq n - \left\lfloor \frac{d+1}{2} \right\rfloor$. Then there exists a set X of $\lambda_{d+1}(c_p, c_q)$ points that form a hitting set for the balls in $\mathcal{B}$. Here $\lambda_{d+1}(\cdot, \cdot)$ is the function defined in the statement of Theorem 1.*

Proof. Observe that one can stereographically 'lift' balls in $\mathbb{R}^d$ to caps of a sphere S in $\mathbb{R}^{d+1}$, where a cap of a sphere is a portion of the sphere contained in a half-space that doesn't contain the center of the sphere. Thus we will prove a slightly more general result where $\mathcal{B}$ consists of caps of a d-dimensional sphere S embedded in $\mathbb{R}^{d+1}$.

For a point $x \in S$, let h_x denote the hyperplane tangent to S at x. For any point y lying outside S, define the *separating set of* y to be

$$S_y = \{z \in S \colon h_z \text{ separates } y \text{ and } S\}.$$

Geometrically, S_y is the set of points of S 'visible' from y, and form a cap of S. Furthermore, for any cap K of S, there is a unique point w such that $K = S_w$. We denote this point w by $\mathrm{apex}(K)$.

[3] Recall that Theorem B assumes $q \leq p \leq d$.

Given the set of caps $\mathcal{B}$ on S, consider the point set

$$\text{apex}\,(\mathcal{B}) = \{\text{apex}(B)\colon B \in \mathcal{B}\}\,.$$

Observe that for a point $x \in S$ and a cap $B \in \mathcal{B}$, $x \in B$ if and only if $x \in S_{\text{apex}(B)}$. As $\mathcal{B}$ satisfies the (p,q) property – namely that for every p-sized subset $\mathcal{B}'$ of $\mathcal{B}$, there exists a point $x \in S$ lying in some q elements of $\mathcal{B}'$ – we have that for every p-sized subset A' of apex$(\mathcal{B})$, there exists a point $x \in S$ lying in the separating set of some q points of A'. In other words, h_x separates these q points from S.

Applying Theorem 1 with $C = S$ and $P = \text{apex}\,(\mathcal{B})$ in dimension $d+1$, we conclude that P can be partitioned into a family Ξ of $\lambda_{d+1}(c_p, c_q)$ sets, each of whose convex hull is disjoint from S. Consider a set $P' \in \Xi$. Since the convex hull of P' is disjoint from S, we can find a hyperplane h_x tangent to S at x such that h_x separates P' from S. This implies that all the caps in $\mathcal{B}$ corresponding to the points in P' contain the point x. Thus for each set of Ξ we obtain a point which is contained in all the caps corresponding to the points in that set. These $|X| = \lambda_{d+1}(c_p, c_q)$ points form the required set X. ◀

Our results complement the recent results of Keller, Smorodinsky and Tardos [9, 10] who obtain polynomial bounds for regions of low union complexity in the plane.

2 Proof of Theorem 1

Given a set P of points in $\mathbb{R}^d$ and an integer $k \geq 1$, a set $P' \subseteq P$ is called a k-set of P if $|P'| = k$ and if there exists a half-space h in $\mathbb{R}^d$ such that $P' = P \cap h$. A set $P' \subseteq P$ is called a $(\leq k)$-set if P' is a l-set for some $l \leq k$. The next well-known theorem gives an upper-bound on the number of $(\leq k)$-sets in a point set (see [17]).

▶ **Theorem 3** (Clarkson-Shor [3]). *For any integer $k \geq \left\lfloor \frac{d}{2} \right\rfloor + 1$, the number of $(\leq k)$-sets of any set of n points in $\mathbb{R}^d$ is at most*

$$\kappa_d\,(n,k) = 2\left(\frac{K_1}{\lceil d/2 \rceil}\right)^{\lceil d/2 \rceil} \binom{n}{\lfloor d/2 \rfloor} (k + \lceil d/2 \rceil)^{\lceil d/2 \rceil} \;\; \leq \;\; \kappa'_d\,(k) \cdot n^{\lfloor d/2 \rfloor}, \tag{1}$$

where $\kappa'_d\,(k) = 2K_1^d \lfloor d/2 \rfloor^{-\lfloor d/2 \rfloor} \left(1 + \frac{k}{\lceil d/2 \rceil}\right)^{\lceil d/2 \rceil}$ and $K_1 \geq e$ is an absolute constant independent of n, d and k.

▶ **Definition 4** (Depth). Given a set P of n points in $\mathbb{R}^d$ and any set $Q \subseteq P$, define the *depth* of Q with respect to P, denoted depth$_P(Q)$, to be the minimum number of points of P contained in any half-space containing Q.

For two parameters $l \geq k \geq 2$, let $\tau_d\,(n,k,l)$ denote the maximum number of subsets of size k and depth at most l with respect to P in any set P of n points in $\mathbb{R}^d$:

$$\tau_d\,(n,k,l) = \max_{\substack{P \subseteq \mathbb{R}^d \\ |P| = n}} |\{Q \subseteq P \colon |Q| = k \;\text{ and }\; \text{depth}_P\,(Q) \leq l\}|\,.$$

The following statement is easily implied by an application of the Clarkson-Shor technique [3] (e.g., see [16]).

▶ **Theorem 5.** *For parameters $l \geq k \geq \left\lfloor \frac{d}{2} \right\rfloor + 1$,*

$$\tau_d(n,k,l) \leq e \cdot \kappa_d(n,k) \cdot l^{k - \lfloor d/2 \rfloor},$$

where the function $\kappa(\cdot,\cdot)$ is as defined in Equation (1).

Proof. Let P be any set of n points in $\mathbb{R}^d$. Let t be the number of sets of P of size k and depth at most l. Pick each element of P independently with probability $\rho = \frac{1}{l}$ to get a random sample R. The expected number of k-sets in R satisfies

$$\begin{aligned}
\rho^k \cdot (1-\rho)^{l-k} \cdot t &\leq \mathbb{E}[\text{ number of } k\text{-sets in } R\ ] \\
&\leq 2\left(\frac{K_1}{\lceil d/2 \rceil}\right)^{\lceil \frac{d}{2} \rceil} \mathbb{E}\left[\binom{|R|}{\lfloor \frac{d}{2} \rfloor}\right]\left(k + \left\lceil \frac{d}{2} \right\rceil\right)^{\lceil \frac{d}{2} \rceil} \\
&= 2\left(\frac{K_1}{\lceil d/2 \rceil}\right)^{\lceil \frac{d}{2} \rceil} \binom{n}{\lfloor \frac{d}{2} \rfloor}\rho^{\lfloor \frac{d}{2} \rfloor}\left(k + \left\lceil \frac{d}{2} \right\rceil\right)^{\lceil \frac{d}{2} \rceil} \\
&= \kappa_d(n,k) \cdot \rho^{\lfloor \frac{d}{2} \rfloor} \\
\implies t &\leq \frac{\kappa_d(n,k) \cdot \rho^{\lfloor \frac{d}{2} \rfloor}}{\rho^k \cdot (1-\rho)^{l-k}} \leq e \cdot \kappa_d(n,k) \cdot l^{k - \lfloor d/2 \rfloor},
\end{aligned}$$

as $\left(1 - \frac{1}{l}\right)^{-(l-k)} \leq e$ for any $l \geq k \geq 2$. ◀

▶ **Lemma 6.** *Let C be a bounded convex object in $\mathbb{R}^d$, and P a set of n points lying outside C. Let $p \geq q \geq \lfloor \frac{d}{2} \rfloor + 1$ be parameters such that for every subset $Q \subseteq P$ of size p, there exists a set $Q' \subset Q$ of size q such that Q' can be separated from C by a hyperplane. Then there exists a hyperplane separating at least*

$$\left(2\, q\, p^{q-1} \cdot e\, \kappa'_d(q)\right)^{\frac{1}{\lfloor d/2 \rfloor - q}}$$

fraction of the points of P from C.

Proof. From [6, 9], it follows that the number of distinct q-tuples of P that can be separated from C by a hyperplane is, assuming that $n \geq 2p$, at least

$$\frac{n-p+1}{n-q+1} \frac{\binom{n}{q}}{\binom{p-1}{q-1}} \geq \frac{n^q}{2q\, p^{q-1}}.$$

Let l be the maximum depth (Definition 4) of any of these separable q-tuples. The number of such tuples is therefore at most $\tau_d(n,q,l)$. Thus by Theorem 5 we must have

$$\frac{n^q}{2q\, p^{q-1}} \leq \tau_d\,(n,q,l) \leq e\, \kappa_d\,(n,q)\ l^{q - \lfloor d/2 \rfloor}.$$

Re-arranging the terms and from inequality (1), we get

$$\begin{aligned}
l \geq \left(\frac{n^q}{2\, q\, p^{q-1} \cdot e\, \kappa_d\,(n,q)}\right)^{\frac{1}{q - \lfloor d/2 \rfloor}} &\geq \left(\frac{n^q}{2\, q\, p^{q-1} \cdot e\, \kappa'_d\,(q)\ n^{\lfloor \frac{d}{2} \rfloor}}\right)^{\frac{1}{q - \lfloor d/2 \rfloor}} \\
&= n \cdot \left(2\, q\, p^{q-1} \cdot e\, \kappa'_d\,(q)\right)^{\frac{1}{\lfloor d/2 \rfloor - q}}.
\end{aligned}$$

Thus one of the separable q-tuples, say $P' \subseteq P$, must have depth at least l; in other words, the hyperplane separating P' from C must contain at least l points of P. This is the required hyperplane. ◀

We now prove a weighted version of the above statement.

▶ **Corollary 7.** *Let C be a bounded convex object in $\mathbb{R}^d$, and P a weighted set of n points lying outside C, where the weight of each point $p \in P$ is a non-negative rational number. Let $p \geq q \geq \lfloor \frac{d}{2} \rfloor + 1$ be parameters such that for every subset $Q \subseteq P$ of size p, there exists a set $Q' \subset Q$ of size q such that Q' can be separated from C by a hyperplane. Then there exists a hyperplane separating a set of points whose weight is at least*

$$\alpha_d(p,q) = \left(2e\,\kappa'_d\,(q)\; q^q\, p^{q-1}\right)^{\frac{1}{\lfloor d/2 \rfloor - q}}$$

fraction of the total weight of the points in P.

Proof. By appropriately scaling all the rational weights, we may assume that each weight is a non-negative integer and we replace a point with weight m by m unweighted copies of the point. Let P' be the new set of points. Observe that any set S of pq points in P' either contains q copies of some point in P or it contains p distinct points from P. In either case, there is hyperplane separating q points of S from C. Thus, we can apply Lemma 6 to the point set P' with the parameter p in the lemma replaced by pq. The result follows. ◀

Finally we return to the proof of the main theorem.

▶ **Theorem 1.** *Let C be a bounded convex object in $\mathbb{R}^d$ and P a set of n points lying outside C. Further let c_p, c_q be two integers, with $1 \leq c_q \leq c_p \leq n - \lfloor \frac{d}{2} \rfloor$, such that for every $c_p + \lfloor \frac{d}{2} \rfloor$ points of P, there exists a subset of size $c_q + \lfloor \frac{d}{2} \rfloor$ whose convex-hull is disjoint from C. Then the points of P can be partitioned into*

$$\lambda_d\left(c_p, c_q\right) = K_2\, \frac{d}{c_q} \cdot \left(\sqrt{2}K_1\right)^{\frac{d}{c_q}} \cdot \left(\lfloor d/2 \rfloor + c_q\right)^2 \;\cdot \left(\lfloor d/2 \rfloor + c_p\right)^{1+\frac{\lfloor d/2 \rfloor - 1}{c_q}} \cdot \log\left(\lfloor d/2 \rfloor + c_p\right)$$

sets, each of whose convex-hull is disjoint from C. Here K_1, K_2 are absolute constants independent of n, d, c_p and c_q. Furthermore, such a partition can be computed in polynomial time.

Proof. Let $p = c_p + \lfloor d/2 \rfloor$ and $q = c_q + \lfloor d/2 \rfloor$. Let $\mathcal{H}$ be the set of all hyperplanes separating a distinct subset of points of P from C. As the number of subsets of P is finite, one can assume that $\mathcal{H}$ is also finite. Consider the following linear program on $|\mathcal{H}|$ variables $\{u_h \geq 0\colon h \in \mathcal{H}\}$:

$$\min \sum_{h \in \mathcal{H}} u_h, \quad \text{such that} \quad \forall r \in P\colon \sum_{\substack{h \in \mathcal{H} \\ h \text{ separates } r \text{ from } C}} u_h \geq 1. \tag{2}$$

The LP-dual to the above program, on $|P|$ variables $\{w_r \geq 0\colon r \in P\}$, is:

$$\max \sum_{p \in P} w_p, \quad \text{such that} \quad \forall h \in \mathcal{H}\colon \sum_{\substack{r \in P \\ h \text{ separates } r \text{ from } C}} w_r \leq 1. \tag{3}$$

Consider an optimal solution w^* of the dual linear program and interpret w^*_r as the weight of each $r \in P$. Since the weights are rational, by Corollary 7, there exists a hyperplane $h \in \mathcal{H}$ separating a subset of P of combined weight at least $\epsilon = \alpha_d(p,q)$ fraction of the total weight of all the points. Since the total weight of the points in any half-space is constrained to be at most 1 by the linear program, the total weight of all the points of P must be at most $\frac{1}{\epsilon}$. In other words, the optimal value of linear program (3) is at most $\frac{1}{\epsilon}$. Since the optimal values of both linear programs are equal due to strong duality, the optimal value of linear program (2) is also at most $\frac{1}{\epsilon}$.

Let u^* be the optimal solution of linear program (2). If we interpret u_h as the weight of the hyperplane h, the constraints of the program imply that each point is separated by a set of hyperplanes in $\mathcal{H}$ whose combined weight is at least 1 out of a total weight of at most $\frac{1}{\epsilon}$ – in other words, at least ϵ-th fraction of the total weight of $\mathcal{H}$. By associating with each hyperplane the half-space bounded by it and not containing C, and using the ϵ-net theorem for half-spaces in $\mathbb{R}^d$ (see [11]), there exists a set of $O\left(\frac{d}{\epsilon}\log\frac{1}{\epsilon}\right)$ hyperplanes which together separate all points of P from C. Recalling that

$$\frac{1}{\epsilon} = \frac{1}{\alpha_d(p,q)} = \left(2e\,\kappa'_d(q)\,q^q\,p^{q-1}\right)^{\frac{1}{q-\lfloor d/2\rfloor}} = \left(2e\,\kappa'_d(q)\,q^q\,p^{q-1}\right)^{\frac{1}{c_q}}.$$

and that $\kappa'_d(q) = 2K_1^d\lfloor d/2\rfloor^{-\lfloor d/2\rfloor}\left(1+\frac{q}{\lceil d/2\rceil}\right)^{\lceil d/2\rceil}$, we get

$$\begin{aligned}
\frac{1}{\epsilon} &= \left(4K_1^d e\lfloor d/2\rfloor^{-\lfloor d/2\rfloor}\left(1+\frac{q}{\lceil d/2\rceil}\right)^{\lceil d/2\rceil} q^q\,p^{q-1}\right)^{\frac{1}{c_q}}\\
&\le \left(4K_1^{d+1}\lfloor d/2\rfloor^{-d}\,(c_q+d)^{\lceil d/2\rceil}\,q^q\,p^{q-1}\right)^{\frac{1}{c_q}} \quad \text{(using } e\le K_1 \text{ and } q=c_q+\lfloor d/2\rfloor\text{)}\\
&\le \left(4K_1^{d+1}\lfloor d/2\rfloor^{-d}(c_q+d)^{\lceil d/2\rceil} q^{c_q+\lfloor d/2\rfloor} p^{c_q+\lfloor d/2\rfloor-1}\right)^{\frac{1}{c_q}}\\
&= O\left(K_1^{\frac{d}{c_q}}\,\lfloor d/2\rfloor^{-\frac{d}{c_q}}\,(c_q+d)^{\frac{\lceil d/2\rceil}{c_q}}\,(c_q+\lfloor d/2\rfloor)^{1+\frac{\lfloor d/2\rfloor}{c_q}}\,(c_p+\lfloor d/2\rfloor)^{1+\frac{\lfloor d/2\rfloor-1}{c_q}}\right)\\
&= O\left(K_1^{\frac{d}{c_q}}\,d^{2+\frac{\lfloor d/2\rfloor-1}{c_q}}\,(1+\frac{c_q}{d})^{\frac{\lceil d/2\rceil}{c_q}}\left(1+\frac{c_q}{\lfloor d/2\rfloor}\right)^{1+\frac{\lfloor d/2\rfloor}{c_q}}\left(1+\frac{c_p}{\lfloor d/2\rfloor}\right)^{1+\frac{\lfloor d/2\rfloor-1}{c_q}}\right)\\
&= O\left(K_1^{\frac{d}{c_q}}\,d^{2+\frac{\lfloor d/2\rfloor-1}{c_q}}\,e^{\frac{c_q}{d}\cdot\frac{\lceil d/2\rceil}{c_q}}\left(1+\frac{c_q}{\lfloor d/2\rfloor}\right)e^{\frac{c_q}{\lfloor d/2\rfloor}\cdot\frac{\lfloor d/2\rfloor}{c_q}}\left(1+\frac{c_p}{\lfloor d/2\rfloor}\right)^{1+\frac{\lfloor d/2\rfloor-1}{c_q}}\right)\\
&= O\left(K_1^{\frac{d}{c_q}}\,d^{2+\frac{\lfloor d/2\rfloor-1}{c_q}}\left(1+\frac{c_q}{\lfloor d/2\rfloor}\right)\left(1+\frac{c_p}{\lfloor d/2\rfloor}\right)^{1+\frac{\lfloor d/2\rfloor-1}{c_q}}\right)\\
&= O\left(K_1^{\frac{d}{c_q}}\,2^{\frac{d}{2c_q}}\,(\lfloor d/2\rfloor+c_q)\,(\lfloor d/2\rfloor+c_p)^{1+\frac{\lfloor d/2\rfloor-1}{c_q}}\right)\\
&= O\left(\left(\sqrt{2}K_1\right)^{\frac{d}{c_q}}\,(\lfloor d/2\rfloor+c_q)\,(\lfloor d/2\rfloor+c_p)^{1+\frac{\lfloor d/2\rfloor-1}{c_q}}\right).
\end{aligned}$$

The Big-Oh notation here does not hide dependencies on d – namely we do not treat d as a constant. From the above it follows that

$$\log\frac{1}{\epsilon} = O\left(c_q^{-1}\left(\lfloor d/2\rfloor+c_q\right)\log\left(\lfloor d/2\rfloor+c_p\right)\right).$$

Thus, $\frac{d}{\epsilon}\log\frac{1}{\epsilon}$ is

$$O\left(d\cdot\left(\left(\sqrt{2}K_1\right)^{\frac{d}{c_q}}\,(\lfloor d/2\rfloor+c_q)\,(\lfloor d/2\rfloor+c_p)^{1+\frac{\lfloor d/2\rfloor-1}{c_q}}\right)\cdot\left(c_q^{-1}\left(\lfloor d/2\rfloor+c_q\right)\log\left(\lfloor d/2\rfloor+c_p\right)\right)\right)$$

which simplifies to

$$O\left(\frac{d}{c_q}\left(\sqrt{2}K_1\right)^{\frac{d}{c_q}}\left(\lfloor d/2\rfloor+c_q\right)^2\,\left(\lfloor d/2\rfloor+c_p\right)^{1+\frac{\lfloor d/2\rfloor-1}{c_q}}\log(\lfloor d/2\rfloor+c_p)\right).$$

Since linear programs can be solved in polynomial time and epsilon nets can be computed in polynomial time, the partition of P into the above number of sets can be achieved in polynomial time. The theorem follows. ◀

References

1 N. Alon and D. Kleitman. Piercing convex sets and the Hadwiger–Debrunner (p, q)-problem. *Adv. Math.*, 96(1):103–112, 1992.

2 J. Bourgain and J. Lindenstrauss. On covering a set in R^n by balls of the same diameter. In *Geometric Aspects of Functional Analysis*, pages 138–144. Springer Berlin Heidelberg, 1991.

3 K. Clarkson and P. Shor. Applications of random sampling in computational geometry, II. *Discrete & Computational Geometry*, 4(5):387–421, 1989.

4 L. Danzer. *Uber zwei Lagerungsprobleme; Abwandlungen einer Vermutung von T. Gallai.* PhD thesis, Techn. Hochschule Munchen, 1960.

5 L. Danzer. Uber durchschnittseigenschaften n-dimensionaler kugelfamilien. *J. Reine Angew. Math.*, 208:181–203, 1961.

6 D. de Caen. Extension of a theorem of Moon and Moser on complete subgraphs. *Ars Combin.*, 16:5–10, 1983.

7 J. Eckhoff. A survey of the Hadwiger-Debrunner (p, q)-problem. In *Discrete and Computational Geometry: The Goodman-Pollack Festschrift*, pages 347–377. Springer, 2003.

8 A. Holmsen and R. Wenger. Helly-type theorems and geometric transversals. In J. E. Goodman, J. O'Rourke, and C. D. Tóth, editors, *Handbook of Discrete and Computational Geometry*, pages 91–123. CRC Press LLC, 2017.

9 C. Keller, S. Smorodinsky, and G. Tardos. Improved bounds on the Hadwiger-Debrunner numbers. *Israel J. of Math., to appear*, 2017.

10 C. Keller, S. Smorodinsky, and G. Tardos. On max-clique for intersection graphs of sets and the hadwiger-debrunner numbers. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 2254–2263, 2017.

11 A. Kupavskii, N. H. Mustafa, and J. Pach. Near-optimal lower bounds for ϵ-nets for half-spaces and low complexity set systems. In Martin Loebl, Jaroslav Nešetřil, and Robin Thomas, editors, *A Journey Through Discrete Mathematics: A Tribute to Jiří Matoušek*, pages 527–541. Springer International Publishing, 2017.

12 J. Matoušek. *Lectures in Discrete Geometry.* Springer-Verlag, New York, NY, 2002.

13 N. H. Mustafa and S. Ray. Weak ϵ-nets have a basis of size $O(1/\epsilon \log 1/\epsilon)$. *Comp. Geom: Theory and Appl.*, 40(1):84–91, 2008.

14 N. H. Mustafa and S. Ray. An optimal generalization of the colorful Carathéodory theorem. *Discrete Mathematics*, 339(4):1300–1305, 2016.

15 N. H. Mustafa and K. Varadarajan. Epsilon-approximations and epsilon-nets. In J. E. Goodman, J. O'Rourke, and C. D. Tóth, editors, *Handbook of Discrete and Computational Geometry*, pages 1241–1268. CRC Press LLC, 2017.

16 S. Smorodinsky, M. Sulovský, and U. Wagner. On center regions and balls containing many points. In *Proceedings of the 14th annual International Conference on Computing and Combinatorics*, COCOON '08, pages 363–373, 2008.

17 U. Wagner. k-sets and k-facets. In J.E. Goodman, J. Pach, and R. Pollack, editors, *Surveys on Discrete and Computational Geometry: Twenty Years Later*, Contemporary Mathematics, pages 231–255. American Mathematical Society, 2008.