Finite Bisimulations for Dynamical Systems with Overlapping Trajectories

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Abstract
Having a finite bisimulation is a good feature for a dynamical system, since it can lead to the decidability of the verification of reachability properties. We investigate a new class of o-minimal dynamical systems with very general flows, where the classical restrictions on trajectory intersections are partly lifted. We identify conditions, that we call Finite and Uniform Crossing: When Finite Crossing holds, the time-abstract bisimulation is computable and, under the stronger Uniform Crossing assumption, this bisimulation is finite and definable.

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1 Introduction

Hybrid automata. Hybrid systems [16] combine continuous dynamics, i.e. evolution of variables according to flow functions (possibly described by differential equations) in control locations, and discrete jumps between these locations, equipped with guards and variable updates. For this very expressive class of models, where the associated transition system has an uncountable state space, most verification questions are undecidable [19, 4], in particular the reachability of some error states. For the last twenty-five years, a large amount of research has been devoted to approximation methods [34, 12] and to the identification of subclasses with decidable properties obtained by restricting the continuous dynamics and/or the discrete behaviour of the systems [2]. Among these subclasses lie the well-known timed automata [1], where all variables are clocks evolving with rate 1 with respect to a global time, guards are comparisons of clocks with rational constants, and updates are resets. Decidability results

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were also obtained for larger classes (see [18, 14, 25, 2, 8, 15, 10, 9, 3]), usually (but not always) by building a finite abstraction based on some bisimulation equivalence, preserving a specific class of properties, like reachability or those expressed by temporal logic formulas.

Ingredients for decidability results. We now describe the restrictions mentioned above. The first one consists in constraining the variable updates on discrete transitions between locations by some “strong reinitialization”, to make the dynamics of locations independent from each other, hence decoupling the discrete and continuous components. Considering a single location with its dynamics is then sufficient; in the next step, the aim is to identify subclasses of the dynamical systems governing the variables on a fixed location, for which a finite bisimulation can be found.

With the decoupling conditions, powerful flows, like the linear flows considered in [17], become possible. The approach in [2] handles o-minimal hybrid systems, using o-minimal structures over the reals as time and variable domains. The first-order theory of reals is then exploited to produce a finite bisimulation. This direction was further explored in [25, 8, 22, 10, 9], where analytical or algebraic methods are proposed to extend the set of flow functions as well as the underlying o-minimal structures. In [10, 9], decidability of reachability is even obtained with the theory of reals while no finite bisimulation may exist. The work of [15] explores how to slightly lift the hypothesis on strong reinitialization.

A few cases feature hybrid automata with no decoupling between the discrete and the continuous parts, at the price of very simple dynamical systems: the first one is the class of timed automata, where clocks describe the most basic flow functions, and the second one is the (incomparable) class of Interrupt Timed Automata with polynomial constraints [3], where the variables are stopwatches (with rate 0 or 1 depending on the location) organized along hierarchical levels. In this latter case, classical polyhedron-based abstractions are not sufficient and the finite bisimulation is obtained via an adaptation of the cylindrical algebraic decomposition algorithm [13].

Contribution. We investigate a new class of o-minimal dynamical systems, where some classical restrictions on the trajectories are lifted: overlapping trajectories are possible, as depicted for instance in Figure 1. Our method involves a classification of intersection points, similar to the cylindrical decomposition, producing a time-abstract bisimulation leading to a finite abstraction under suitable hypotheses.

Outline. In Section 2, we recall the base properties of o-minimal structures used in our developments; we then define the dynamical systems we will study; we also define the technical tool of time-abstract bisimulation which is used to build a finite abstraction of the dynamical systems; we end up this section with a discussion on related work. In Section 3, we present the graph construction, which leads to abstract the original dynamical system with some partition of the state-space, on which we are able to check time-abstract bisimulation. In Section 4, we discuss definability and decidability issues, and show how our approach can be used to recover the original work [25]. We end up with some perspectives.

2 Definitions

We consider linearly ordered structures \( M = (M, <, \ldots) \). These structures can be dense or discrete (or mixed), with or without endpoints (i.e. minimum or maximum). Classical examples without endpoints are the set \( \mathbb{Z} \) of integers or the real line \( \mathbb{R} \), while the sets \( \mathbb{N} \)
of natural numbers and \( \mathbb{R}_+ \) of the non negative real numbers have 0 as left endpoint. We will consider the first-order theory associated with \( \mathcal{M} \): we say that some relation, subset or function is \textit{definable} when it is first-order definable in the structure \( \mathcal{M} \). Next we may abusively identify the structure \( \mathcal{M} \) with its first-order theory. A general reference for first-order logic is [20]. Moreover, we will assume that the theory of \( \mathcal{M} \) is o-minimal and we recall here the definition of o-minimality (references are [31, 21, 32, 33, 36]).

### 2.1 O-minimal structures

Recall that intervals of \( \mathcal{M} = \langle M, <, \ldots \rangle \) are convex sets with either a supremum in \( M \) or no upper bound, and either an infimum in \( M \) or no lower bound.

**Definition 1.** A linearly ordered structure \( \mathcal{M} = \langle M, <, \ldots \rangle \) has an \textit{o-minimal} theory if every definable subset of \( M \) is a finite union of intervals.

In other words, the definable subsets of \( M \) are the simplest possible. This assumption implies that definable subsets of \( M^n \) (in the sense of \( \mathcal{M} \)) admit very nice structure theorems (like the \textit{cell decomposition} [21, 32]). Classical o-minimal structures are: the ordered group of rationals \( \langle \mathbb{Q}, <, +, 0, 1 \rangle \), the ordered field of reals \( \langle \mathbb{R}, <, +, \cdot, 0, 1 \rangle \), the field of reals with exponential function, the field of reals expanded by restricted pfaffian functions and the exponential function, and many more interesting structures (see [36, 37]). An example of non o-minimal structure is given by \( \langle \mathbb{R}, <, \sin, 0 \rangle \), since the definable set \( \{ x \mid \sin(x) = 0 \} \) is not a finite union of intervals. However, note that the structure\(^2 \langle \mathbb{R}, +, \cdot, 0, 1, <, \sin_{[0,2\pi]}, \cos_{[0,2\pi]} \rangle \) is o-minimal (see [35]).

We recall here a standard base result of o-minimal structures, used to build the cell decomposition, and which will be useful in the subsequent developments. While initially proved for dense structures [31, Theorem 4.2], a version for discrete structures is provided in [32, Lemmas 1.3 and 1.5], and the result holds for general mixed structures as a consequence of [33, Proposition 2.3].

**Theorem 2.** Let \( \mathcal{M} = \langle M, <, \ldots \rangle \) be a linearly ordered structure with an o-minimal theory. Let \( f : M \rightarrow M \) be a definable function. The set \( M \) can be partitioned into finitely many intervals \( I_1, \ldots, I_k \) such that, for every interval \( I_j \), (i) the restriction \( f|_{I_j} \) is either constant or one-to-one and monotonic, and (ii) the set \( f(I_j) \) is an interval of \( M \).

The other result on o-minimal structures used in the sequel is the following, restated from [33, Section 2], which provides a uniform bound on the partition size:

**Theorem 3.** Let \( \mathcal{M} = \langle M, <, \ldots \rangle \) be a linearly ordered structure with an o-minimal theory. Let \( \varphi \) be a formula with \( k \) variables. Then there exists an integer \( N_\varphi \) such that, for all \( b_2, \ldots, b_k \in M \), the set \( \{ a \in M \mid (a, b_2, \ldots, b_k) \models \varphi \} \) can be partitioned into at most \( N_\varphi \) intervals.

### 2.2 Dynamical systems

**Definition 4.** A \textit{dynamical system} is a pair \( (\mathcal{M}, \gamma) \) where:

- \( \mathcal{M} = \langle M, <, \ldots \rangle \) is a linearly ordered structure,
- \( \gamma : V_1 \times V \rightarrow V_2 \) is a function definable in \( \mathcal{M} \) (where \( V_1 \subseteq M^{k_1}, V \subseteq M \) and \( V_2 \subseteq M^{k_2} \) are definable subsets).\(^3\)

\(^2\) \( \sin_{[0,2\pi]} \) and \( \cos_{[0,2\pi]} \) correspond to the sine and cosine functions restricted to interval \([0, 2\pi]\).

\(^3\) We use these notations in the rest of the paper.
The function $\gamma$ is called the *dynamics* of the system and $(M, \gamma)$ is said to be o-minimal when the theory of $M$ is itself o-minimal.

Classically, we see $V$ as the time, $V_1$ as the input space, or set of parameters, $V_1 \times V$ as the space-time and $V_2$ as the output, or geometrical, space.

**Definition 5.** For a dynamical system $(M, \gamma)$, if we fix a point $x \in V_1$, the set $\Gamma_x = \{ \gamma(x, t) \mid t \in V \} \subseteq V_2$ is called the *trajectory* determined by $x$.

We define a transition system associated with the dynamical system. This definition is an adaptation to our context of the classical continuous transition system in the case of hybrid systems (see [25] for example).

**Definition 6.** Given $(M, \gamma)$ a dynamical system, the associated transition system $T_\gamma = (Q, \rightarrow)$ is defined by:

- its set of states $Q = V_2$;
- its transition relation $\rightarrow$, which is defined by: $y \rightarrow y'$ if $\exists x \in V_1, \exists t, t' \in V$ such that $t \leq t'$ and $\gamma(x, t) = y$, $\gamma(x, t') = y'$.

As usual, an execution is a sequence of consecutive transitions. Note that it is possible to switch between trajectories, as illustrated below.

**Example 7.** The dynamical system depicted in Figure 1 is composed of three trajectories (with $\gamma(x_1, \cdot)$ in blue, $\gamma(x_2, \cdot)$ in green and $\gamma(x_3, \cdot)$ in red), with set of parameters $V_1 = \{x_1, x_2, x_3\}$ and $V = V_2 = \mathbb{R}$. Executions take place in $\mathbb{R}$, according to the trajectories. For instance: $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_4$ with $y_1 = 3 = \gamma(x_3, -1)$ and $y_2 = 2.5 = \gamma(x_3, 0)$ on the red curve, then switching to the green curve since $y_2 = \gamma(x_2, t_2)$ for some $t_2 < 0$, $y_3 = 0.5 = \gamma(x_2, 2)$, and finally jumping to the blue curve since $y_3 = \gamma(x_1, 2)$, leading to $y_4 = \gamma(x_1, 4)$.

The definition of *dynamical system* encompasses a lot of different behaviours, examples of which can be obtained with structures enriched by additional operations like addition, multiplication (or the exponential function).
Example 8. A classical one is the continuous dynamics of timed automata [1]: In this case, \( M = \langle \mathbb{R}, <, + \rangle \) and the dynamics \( \gamma : \mathbb{R}^n_+ \times [0, +\infty] \to \mathbb{R}^n_+ \) is defined by \( \gamma(x_1, \ldots, x_n, t) = (x_1 + t, \ldots, x_n + t) \). The standard graphical view, displayed in Figure 2 left, represents the dynamical system directly on the output space: \( y \to y' \) with \( y = (2, 1) = \gamma((2, 1), 0) \) and \( y' = (3.5, 2.5) = \gamma((2, 1), 1.5) \).

Example 9. Another example, borrowed from [6] and illustrated in Figure 2 right, features a dynamical system where each point of the plane has two possible behaviours: going right or going up. The dynamics \( \gamma : \mathbb{R}^2 \times \{-1, +1\} \times \mathbb{R} \to \mathbb{R}^2 \) is defined by:

\[
\begin{align*}
\gamma(x_1, x_2, p, t) &= \begin{cases} 
(x_1 + t, x_2) & \text{if } p = +1 \\
(x_1, x_2 + t) & \text{if } p = -1
\end{cases}
\end{align*}
\]

Then \( y_1 \to y_2 \to y_3 \) for the three points \( y_1 = (0, 0), y_2 = (0, 1) \) and \( y_3 = (1, 1) \), since \( y_1 = \gamma(0, 0, -1, 0), y_2 = \gamma(0, 0, -1, 1) = \gamma(0, 1, 1, 0), \) and \( y_3 = \gamma(0, 1, 1, 1) \).

In hybrid automata, such behaviours are combined with a finite set of discrete locations, each one having its own dynamics with respect to a common structure \( M \): jumps between locations are constrained by guards and equipped with updates. As mentioned in the introduction, basic verification problems like reachability checking are undecidable in the general case, and solutions to recover decidability are often to impose strong reinitializations of trajectories at jumps (we will come back to that in subsection 2.4), which amounts to concentrating on the analysis of a single dynamical system.

### 2.3 Time-abstract bisimulation

Time-abstract bisimulation [18, 14, 2, 25] is a behavioural relation often used to obtain a quotient of the original transition system. When this quotient is finite, a large class of properties can be verified, notably reachability properties.

We associate with a dynamical system \( (M, \gamma) \) a finite set \( G \) of guards, which are definable subsets of \( V_2 \). For every \( y \in V_2 \), we define the set \( G_y \) of guards that are “satisfied” by \( y \), thus producing a finite partition of \( V_2 \) into subsets satisfying the same sets of guards.

**Definition 10.** Consider a dynamical system \( (M, \gamma) \), a finite set \( G \) of definable guards and an integer \( k \geq -1 \). A k-step time-abstract bisimulation is an equivalence relation \( R_k \subseteq V_2 \times V_2 \) such that either (i) \( k = -1 \), or (ii) \( k \geq 0 \) and there exists a \((k - 1)\)-step time-abstract bisimulation \( R_{k-1} \) such that, if \( (y_1, y_2) \in R_k \), then:
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(a) \( G_{y_1} = G_{y_2} \);
(b) if \( y_1 \rightarrow y'_1 \) then there exists \( y'_2 \) such that \( y_2 \rightarrow y'_2 \) and \( (y'_1, y'_2) \in R_{k-1} \);
(c) if \( y_2 \rightarrow y'_2 \) then there exists \( y'_1 \) such that \( y_1 \rightarrow y'_1 \) and \( (y'_1, y'_2) \in R_{k-1} \).

We further say that an equivalence relation \( R \subseteq V_2 \times V_2 \) is a \textit{time-abstract bisimulation} if \( R \) is a \( k \)-step time-abstract bisimulation for all \( k \geq -1 \). We also say that \( y_1 \) and \( y_2 \) are \textit{(k-step) time-abstract bisimilar} whenever there is a \( (k \)-step) time-abstract bisimulation \( R \subseteq V_2 \times V_2 \) such that \( (y_1, y_2) \in R \).

Note that, for every \( k \), the class of \( k \)-step time-abstract bisimulations is closed under union, and therefore there is a largest \( k \)-step time-abstract bisimulation, which can be obtained as the union of all such relations. In particular, the relation \( R_{k-1} \) used in items (b) and (c) when defining \( R_k \) can be taken as the largest \((k - 1)\)-step time-abstract bisimulation. Similarly, there is a largest time-abstract bisimulation.

2.4 Problem and existing results

We focus here on the construction of finite (time-abstract) bisimulation relations, which is a standard and powerful tool to prove decidability of classes of hybrid systems [18]. Existence of such relations is, for instance, the key property satisfied by timed automata [1], a well-established model for real-time systems. However, for hybrid systems with more complex dynamics, proving that there is a finite bisimulation might be difficult and is not possible in general. In several works willing to better understand rich continuous dynamics in a system, the idea has been to decouple the continuous and the discrete parts of the system by assuming (possibly non-deterministic) reinitializations of the trajectories when a jump between locations is performed, see e.g. [14, 24, 26, 23, 25]. This leads to only focus on bisimulation relations within a discrete location. In this work, we follow this idea, and therefore only focus on bisimulations generated by a single dynamical system.

A standard methodology for proving that there is a finite time-abstract bisimulation is to compute successive approximations of the bisimulation relation (see [18, 24, 26, 23, 25, 7, 5]), and show that the procedure terminates. In (almost) all the references mentioned below, this is the way the problem is attacked. While the methodology seems rather universal, it is amazing to see the variety of arguments which are used to show termination of the procedure. They range from analytical and geometrical arguments [24, 26, 25] to model theory arguments [23], algebraic and topological arguments [14] or, more recently, arguments based on word combinatorics [7, 5].

While the precise domains of applicability of the approaches might vary, in most mentioned related works (except [8, 7, 5]), time-determinism is assumed, in the sense that there is a single trajectory going through some point of the output space. In [8, 7, 5], several trajectories may intersect or self-intersect, but rather strong assumptions need to be made. For instance, in the \textit{suffix-determinism} assumption, all trajectories starting from a given point of the output space visit the same pieces of the initial partition in a similar way; in the \textit{loop-determinism} assumption, two trajectories cannot intersect each other, but a trajectory can intersect itself in finitely many points.

In our work, lots of self-intersecting and overlapping trajectories are possible, but we bound the number of trajectories one can reach by switching between them (we will formalize this later). For example, the dynamical system of Figure 1 does not satisfy any of the above assumptions, but typically fits our framework.

The generic symbolic approach of [5] (and in particular the 2-subword refinement procedure) is a semi-procedure for finding finite bisimulations of o-minimal dynamical systems: it finds a finite bisimulation relation if there is one, but cannot tell that there is no finite
bisimulation. But only the two above-mentioned assumptions (suffix-determinism and loop-determinism) guarantee termination of the computation. For instance, even though it does not satisfy any of the sufficient conditions above, the system of Example 9 has a finite bisimulation, which can be computed by the refinement procedure with respect to the single guard $y = (0, 0)$. Since there is no bound on the intersections of trajectories, this system does not belong to our class. On the other hand, both the present work and the approach of [5] encompass the original result [25].

Also, while the theory of o-minimality has been developed in any linearly ordered structure [30, 31, 21, 33], initial settings [25, 14] assume expansion of the reals. Here, similarly to [5], our results hold in the general setting.

In this paper we provide a method which is only based on geometrical properties of o-minimal systems. It does not assume the field of real numbers, nor dense or discrete structures. Furthermore, we are able to deal not only with (self-)crossing trajectories but also with partly stationary trajectories.

### 3 The graph construction

In this section, we fix an o-minimal dynamical system $(M, \gamma)$ and a finite set $G$ of guards as defined above, and we build a graph representing the time-abstract behaviour of $\gamma$.

We define a relation $\sim$ on $V_1$, where $x \sim x'$ if and only if the trajectories $\Gamma_x$ and $\Gamma_{x'}$ cross each other, i.e. if there exist $t, t' \in V$ such that $\gamma(x, t) = \gamma(x', t')$. We also set $V_1(x) \overset{\text{def}}{=} \{x' \in V_1 : x \sim x'\}$.

To build the graph we distinguish between points of $V_2$ with (at most) finitely many predecessors by $\gamma$ on any trajectory and points of $V_2$ with infinitely many predecessors on some trajectory. We will show that those two sets are definable, and that they can be used to provide a nice finite decomposition of the state-space, fine enough to characterize the time-abstract bisimulation. After defining suitable notions of intervals, we independently provide a finite decomposition result and the construction of the graph itself.

#### 3.1 Towards a decomposition

In what follows, we need to distinguish two kinds of intervals: singletons, i.e. intervals with one unique element, and intervals with at least two elements, which we call large intervals.

> **Definition 11.** An interval $I \subseteq V$ is called $x$-static if either (i) $I$ is large and $|\gamma(x, I)| = 1$, or (ii) $I$ is a singleton and there exist a parameter $x' \in V_1$ and a large interval $J \subseteq V$ such that $\gamma(x', J) = \gamma(x, I)$. We further say that an element $t$ of $V$ is $x$-static if $t$ belongs to some $x$-static interval, and that a state $y \in V_2$ is static if there exists $x \in V_1$ and $t \in V$ such that $t$ is $x$-static and $y = \gamma(x, t)$.

On the contrary, we say that an element $t$ of $V$ is $x$-dynamic if $t$ is not $x$-static, and we say that an interval $I$ is $x$-dynamic if every element of $I$ is $x$-dynamic. We further say that $I$ is $x$-suitable if (i) $I$ is $x$-dynamic, (ii) the function $t \mapsto G_{\gamma(x, t)}$ is constant on $I$, and (iii) the function $\gamma(x, \cdot)$ is one-to-one on $I$.

This produces a classification of points in $V_2$: static, if some trajectory stops at that position, or dynamic. It also induces a classification of timepoints and intervals along trajectories: a static point $y$ of $V_2$ generates $x$-static timepoints on $\Gamma_x$, even though the trajectory $\Gamma_x$ may not be responsible for making $y$ static.
Example 12. We illustrate the various notions on the example of Figure 1. Value \( y = 0.5 \) is static, because of \( x_1 \) and \( x_2 \). In particular, interval \((1,2)\) is \( x_1 \)-static and interval \((1.5,3)\) is \( x_2 \)-static. Time \( t = 4 \) is \( x_3 \)-static because \( \gamma(x_3, 4) = 0.5 \) is static, even though \( \gamma(x_3, \cdot) \) itself crosses \( y = 0.5 \) only in one point. And thus, interval \( \{4\} \) is also \( x_3 \)-static but not large.

Note that \( y = y_\ast \) is dynamic, since no trajectory of the dynamical system is constant on a large interval on which its value is \( y_\ast \).

Assuming no guard in the system (or a single guard \( y = 0.5 \)), the intervals \((-\infty,1)\) and \((2,\infty)\) are \( x_1 \)-suitable (and maximal for that condition). Similarly, the intervals \((-\infty,1.5)\) and \((3,\infty)\) are \( x_2 \)-suitable; the intervals \((-\infty,4)\) and \((4,\infty)\) are \( x_3 \)-suitable.

Then, since we want a finite representation of important points of the dynamical system, we need to get uniform (definable) descriptions of the above classification of points.

First, we gather all portions of trajectories corresponding to dynamic parts of the system. Note that such trajectories, while they visit the same state-space (in \( V_2 \)), might follow different directions (hence the value \( \varepsilon = \pm 1 \) below).

Definition 13. Consider two parameters \( x, x' \in V_1 \), one \( x \)-suitable interval \( I \subseteq V \) and one \( x' \)-suitable interval \( I' \subseteq V \). We say that the pairs \((x,I)\) and \((x',I')\) are adapted to each other if:

(i) the sets \( \{\gamma(x,t) \mid t \in I\} \) and \( \{\gamma(x',t') \mid t' \in I'\} \) are equal to each other;

(ii) there exists \( \varepsilon = \pm 1 \) such that: for all \( t,u \in I \) with \( t < u \), there exist \( t',u' \in I' \) such that \( \gamma(x,t) = \gamma(x',t') \), \( \gamma(x,u) = \gamma(x',u') \), and \( t' < u' \Leftrightarrow \varepsilon = 1 \).

In general, we say that a family of pairs \((x_k,I_k)\) \( k \in K \) is strongly adapted if:

(iii) every two pairs \((x_k,I_k)\) and \((x_k,I_k)\) are adapted to each other;

(iv) for all \( k \in K \), \( \{(x,t) \in V_1 \times M \mid \gamma(x,I_k) \in \gamma(x_k,I_k)\} = \bigcup_{t \in I} \{x_t\} \times I_t \).

Finally, we say that an interval \( I \) is \( x \)-adaptable if the pair \((x,I)\) belongs to a strongly adapted family.

Example 14. Going back to the previous example:

- the pairs \((x_1,(-\infty,1))\) and \((x_2,(-\infty,1.5))\) are adapted to each other (with \( \varepsilon = +1 \));

- the pairs \((x_1,(2,\infty))\) and \((x_2,(-\infty,1.5))\) are also adapted to each other (with \( \varepsilon = -1 \));

- the pairs \((x_2,(3,\infty))\) and \((x_3,(4,\infty))\) are adapted to each other (with \( \varepsilon = +1 \)).

By extension, we get that:

- the pairs \((x_1,(-\infty,1)), (x_1,(2,\infty)), (x_2,(-\infty,1.5))\) and \((x_3,(-\infty,4))\) form a strongly adapted family;

- the interval \((2,\infty)\) is \( x_1 \)-adaptable, due to the strongly adapted family above;

- the interval \((3,\infty)\) is \( x_2 \)-adaptable, due to the family formed of \((x_2,(3,\infty))\) and \((x_3,(4,\infty))\);

- the singleton \( \{t_\ast\} \) is both \( x_1 \)-adaptable and \( x_3 \)-adaptable, due to the family formed of \((x_1,\{t_\ast\}), (x_2,\{t_\ast\}), (x_1,\{t_\ast\})\) and \((x_3,\{t_\ast\})\).

An interval \( I \) is said maximal \( x \)-static (resp. maximal \( x \)-adaptable), whenever it is \( x \)-static (resp. \( x \)-adaptable), and is contained in no larger \( x \)-static (resp. \( x \)-adaptable) interval.

It turns out that maximal \( x \)-static and \( x \)-adaptable intervals form a covering of the time domain \( V \).

Lemma 15. Consider a parameter \( x \in V_1 \) and a timepoint \( t \in V \). There exists an interval \( I \subseteq V \), which contains \( t \), and such that \( I \) is a maximal \( x \)-static interval (if \( t \) is \( x \)-static) or a maximal \( x \)-adaptable interval (if \( t \) is \( x \)-dynamic).
Proof. If $t$ is $x$-static, then the singleton $\{t\}$ is $x$-static. If $t$ is $x$-dynamic, then the family of pairs $(x', t')$ such that $\gamma(x', t') = \gamma(x, t)$ is strongly adapted, and therefore the singleton $\{t\}$ is $x$-adaptable. Moreover, both the class of $x$-static intervals and the class of $x$-adaptable intervals are closed under increasing union: this is clear for $x$-static intervals, and can be argued as follows for $x$-adaptable intervals.

Let $(I^\alpha)_{\alpha}$ be an increasing family of $x$-adaptable intervals. For every $\alpha$, let $\mathcal{F}^\alpha = (x^\alpha_k, I^\alpha_k)_{k \in K^\alpha}$ be a corresponding strongly adapted family. There is an obvious one-to-one correspondence between elements of $\mathcal{F}^\alpha$ and elements of $\mathcal{F}^{\alpha'}$ for any pair of indices $(\alpha, \alpha')$, hence one can rewrite the family $\mathcal{F}^\alpha$ uniformly as $(x_k, I^\alpha_k)_{k \in K}$. One can therefore take $\mathcal{F} = (x_k, \bigcup_{\alpha} I^\alpha_k)_{k \in K}$ as a strongly adapted family for $(x, \bigcup_{\alpha} I^\alpha)$. The result follows. ▶

### 3.2 Finite decomposition result

Our goal here is to prove the following decomposition, which refines Lemma 15.

► Proposition 16. Consider a parameter $x \in V_1$ such that $V_1(x)$ is finite. Then, the set $V$ is a finite, disjoint and definable union of intervals $I_1, \ldots, I_k$ such that every interval $I_j$ is either

1. a maximal $x$-static interval, or
2. a maximal $x$-adaptable interval.

We first focus on static (geometrical, i.e. in $V_2$) points and show that there can only be finitely many such points along a trajectory.

► Lemma 17. There exists an integer $K$ such that, for every parameter $x \in V_1$, there exist at most $K$ large maximal $x$-static intervals.

Proof. We first observe that, if $I_1$ and $I_2$ are maximal large $x$-static intervals, with $I_1 \neq I_2$, then $I_1 \cap I_2 = \emptyset$. Otherwise, the union $I_1 \cup I_2$ would also be $x$-static, contradicting the maximality of $I_1$ and $I_2$. Henceforth, we denote by $< \ell$ the linear order on maximal large $x$-static intervals, defined by

$I_1 < I_2$ if and only if $\forall t \in I_1, \forall t' \in I_2, t < t'$.

If $I_1 < I_2$, and if $I_1$ and $I_2$ have respective lower bounds $\ell_1$ and $\ell_2$, then $t \leq \ell_2$ for all $t \in I_1$ and therefore $\ell_1 < \ell_2$ (since $I_1$ is large). Consequently, if $\ell_2 \in I_1$, then $I_1$ must have a maximal element, and $\ell_2 = \max(I_1)$.

Now, let $\mathcal{L}(x)$ be the set of lower bounds of maximal $x$-static intervals. Observe that $\mathcal{L}(x)$ is definable, and therefore by Theorem 3, there exists an integer $K_1$ such that, for all $x \in V_1$, $\mathcal{L}(x)$ is a disjoint union of at most $K_1$ intervals. We claim that each of these intervals has (strictly) less than three elements.

Assume on the contrary that there exists a sub-interval $J$ of $\mathcal{L}(x)$ containing three elements $\ell_1 < \ell_2 < \ell_3$. For all $t \in J$, we denote by $I(t)$ the maximal large $x$-static interval with lower bound $t$. Since $I(\ell_1)$ is large, it contains some element $t$ such that $\ell_1 < t$. Up to replacing both $t$ and $\ell_2$ by $\min\{t, \ell_2\}$, we assume that $t = \ell_2$. It follows, as noted above, that $\ell_2 = \max(I(\ell_1))$. Since $I(\ell_2)$ is large too, consider some element $u$ of $I(\ell_2)$ that is not maximal in $I(\ell_2)$. Since $\ell_1 \in I(\ell_1)$ and $I(\ell_1) \prec I(\ell_2)$, we know that $\ell_2 < u$. Up to replacing both $u$ and $\ell_3$ by $\min\{u, \ell_3\}$, we also assume that $u = \ell_3$, hence that $\ell_3 \in I(\ell_2)$. However, since $\ell_2 < \ell_3$, our initial remark proves that $u = \ell_3$ must be the maximal element of $I(\ell_2)$, contradicting the definition of $u$. This proves our claim.
The set $\mathcal{L}(x)$ is therefore of cardinality at most $2K_1$. Observing that at most one maximal large $x$-static interval has no lower bound proves that there exist at most $K$ large maximal $x$-static intervals, where $K = 2K_1 + 1$.

Lemma 18. There exists an integer $L$ such that, for every parameter $x \in V_1$, the set of $x$-static elements of $V$ is a disjoint union of at most $L |V_1(x)|$ maximal $x$-static intervals.

Proof. Fix $x \in V_1$. Let $\mathcal{S}$ denote the set of static elements of $\gamma(x, V)$. With each element $y$ of $\mathcal{S}$ we can associate a pair $(x', I)$, where $x' \in V_1(x)$ and $I$ is a maximal large $x'$-static interval such that $\gamma(x', I) = \{y\}$. This association is one-to-one, and therefore $|\mathcal{S}| \leq K |V_1(x)|$.

Moreover, there exists an integer $L_1$ such that, for every $y \in V_2$, the definable set $\{t \in V \mid \gamma(x, t) = y\}$ is a finite union of at most $L_1$ intervals (Theorem 3). Assuming, without loss of generality, that these intervals are pairwise disjoint, proves Lemma 18 for $L = K L_1$.

We now turn to the case of dynamic elements. We start with the following combinatorial lemma, whose proof is immediate by induction on $k + \ell$.

Lemma 19. Let $I = (I_1, \ldots, I_k)$ and $J = (J_1, \ldots, J_\ell)$ be two partitions of $V$ into sub-intervals. There exists a partition $K = (K_1, \ldots, K_m)$ of $V$ into sub-intervals that refines both $I$ and $J$, and such that $m + 1 \leq k + \ell$.

Lemma 20. There exists an integer $M$ such that, for every parameter $x \in V_1$, every maximal $x$-dynamic interval of $V$ is a disjoint union of at most $M(1 + |V_1(x)|)$ maximal $x$-adaptable intervals.

Proof. First, recall that there exists an integer $L_1$ such that, for all $x \in V_1$ and all $y \in V_2$, the definable set $\{t \in V \mid \gamma(x', t) = y\}$ is a disjoint union of at most $L_1$ intervals. If $y$ is not static, then these intervals must be singletons, and therefore $|\{t \in V \mid \gamma(x', t) = y\}| \leq L_1$.

Now, for all $t \in V$ and $x, x' \in V_1$, we denote by $f_1(x, x', t) < \ldots < f_{L_1}(x, x', t)$ the elements of the set $\{t' \in V \mid \gamma(x, t') = \gamma(x', t)\}$, where $f_i(x, x', t)$ is undefined if $\gamma(x, t') = \gamma(x', t')$. Observe that every function $f_i$ is definable. Consequently, there exists an integer $M_1$ such that, for all $x \in V_1$ and $x' \in V_1(x)$, there exists a partition $P_i(x, x')$ of $V$ into at most $M_1$ intervals on which the function $t \mapsto f_i(x, x', t)$ is either undefined, constant, or continuous and strictly monotonic (Theorems 2 and 3).

Similarly, since the function $(x, t) \mapsto G_{\gamma(x, t)}$ is definable, there exists an integer $M_2$ such that, for all $x \in V_1$, there exists a partition $P'(x)$ of $V$ in at most $M_2$ intervals on which $t \mapsto G_{\gamma(x, t)}$ is constant.

Now, consider some $x \in V_1$. By Lemma 19, there exists a partition $P$ of $V$, which refines $P'(x)$ and every partition $P_i(x, x')$, for $i \leq L_1$ and $x' \in V_1(x)$, and which contains at most $M_1 L_1 |V_1(x)| + M_2$ intervals. By construction, every interval of the partition $P$ is either $x$-adaptable or $x$-static, and by choosing $P$ to contain as few intervals as possible, these intervals are guaranteed to be maximal $x$-adaptable intervals. Lemma 20 follows, by choosing $M = \max\{M_1 L_1, M_2\}$.

Since maximal $x$-adaptable intervals and maximal $x$-static intervals are definable, we derive from Lemmas 18 and 20 the targeted Proposition 16.
3.3 Construction of the graph

> **Definition 21.** We call bisimulation graph for the o-minimal dynamical system \((\mathcal{M}, \gamma)\) and the set of definable guards \(G\) the (possibly infinite) labeled graph with \(\varepsilon\)-transitions \(\mathcal{G} = (N, E, E_\varepsilon, L)\) defined as follows:
- the set of nodes is
  \[ N = \{(x, I) : x \in V_1, I \text{ is a maximal } x\text{-static or } x\text{-adaptable interval}\}; \]
- the set of edges is
  \[ E = \{(x, I), (x, J) \in N \times N : \exists t \in I, \exists t' \in J, t \leq t'\}; \]
- the set of \(\varepsilon\)-transitions is
  \[ E_\varepsilon = \{(x, I), (x', I') \in N \times N : \exists t \in I, \exists t' \in I', \gamma(x, t) = \gamma(x', t')\}; \]
- the labeling function is \(L : (x, I) \mapsto \{g : \exists t \in I, g \in G_{\gamma(x, t)}\}.\)

Next, we write \(\rightarrow\) (resp. \(\rightarrow_\varepsilon\)) the transition relation defined by \(E\) (resp. \(E_\varepsilon\)), and we denote by \(\leadsto\) the relation defined by: \(n_1 \leadsto n_4\) if there exist nodes \(n_2\) and \(n_3\) such that \(n_1 \rightarrow n_2 \rightarrow n_3 \rightarrow_\varepsilon n_4\).

> **Definition 22.** Consider an integer \(k \geq -1\). A \(k\)-step \(\varepsilon\)-bisimulation is an equivalence relation \(\mathcal{R}_k \subseteq N \times N\) such that either (i) \(k = -1\), or (ii) \(k \geq 0\) and there exists a \((k - 1)\)-step \(\varepsilon\)-bisimulation \(\mathcal{R}_{k-1}\) such that, if \(n_1 \mathcal{R}_k n_2\), then:
  (a) \(L(n_1) = L(n_2)\);
  (b) if \(n_1 \leadsto n'_1\) then there exists \(n'_2\) such that \(n_2 \leadsto n'_2\) and \(n'_1 \mathcal{R}_{k-1} n'_2\);
  (c) if \(n_2 \leadsto n'_2\) then there exists \(n'_1\) such that \(n_1 \leadsto n'_1\) and \(n'_1 \mathcal{R}_{k-1} n'_2\).

We further say that an equivalence relation \(\mathcal{R} \subseteq N \times N\) is a \(\varepsilon\)-bisimulation if \(\mathcal{R}\) is a \(k\)-step \(\varepsilon\)-bisimulation for all \(k \geq -1\). We also say that two nodes \(n_1\) and \(n_2\) are \((k\text{-step}) \varepsilon\)-bisimilar whenever there is a \((k\text{-step})\) \(\varepsilon\)-bisimulation \(\mathcal{R} \subseteq N \times N\) such that \(n_1 \mathcal{R} n_2\).

Like time-abstract bisimulation, the class of \((k\text{-step})\) \(\varepsilon\)-bisimulations is closed under union, hence there is a largest \((k\text{-step})\) \(\varepsilon\)-bisimulation, which can be obtained as the union of all such relations. In particular, the relation \(\mathcal{R}_{k-1}\) used in items (b) and (c) when defining \(\mathcal{R}_k\) can be taken as the largest \((k - 1)\)-step \(\varepsilon\)-bisimulation.

> **Lemma 23.** Let \(n = (x, I)\) and \(n' = (x', I')\) be nodes of the bisimulation graph \(\mathcal{G}\). The following statements are equivalent: (i) \(n \rightarrow_\varepsilon n'\), (ii) \(\gamma(x, I) \cap \gamma(x', I') \neq \emptyset\), and (iii) \(\gamma(x, I) = \gamma(x', I')\).

**Proof.** The equivalence between (i) and (ii) follows directly from the definition of the set \(E_\varepsilon\) of \(\varepsilon\)-transitions, and the implication (iii) \(\Rightarrow\) (ii) is obvious.

It remains to prove (iii), under the assumption that (ii) holds. If \(I\) is \(x\)-static, then \(\gamma(x, I)\) is a singleton, hence \(I'\) contains an \(x'\)-static element, and therefore \(I'\) is not \(x'\)-suitable. This proves that \(I'\) is \(x'\)-static, hence that \(\gamma(x', I')\) is a singleton too, and (iii) follows.

If \(I\) is maximal \(x\)-adaptable, then \(I'\) cannot be \(x'\)-static, hence \(I'\) is maximal \(x'\)-adaptable too. Let \(I''\) be an interval such that \((x, I)\) and \((x', I')\) are adapted, with \(I' \cap I'' \neq \emptyset\). Since maximal \(x'\)-adaptable intervals are disjoint, it follows that \(I'' \subseteq I'\), whence \(\gamma(x, I) \subseteq \gamma(x', I')\). Similarly, we have \(\gamma(x', I') \subseteq \gamma(x, I)\), which completes the proof. ▶
Lemma 24. Let \( n = (x, I) \) and \( n' = (x', I') \) be nodes of the bisimulation graph \( \mathcal{G} \). The following statements are equivalent: (i) \( n \sim n' \), (ii) \( \exists y \in \gamma(x, I), \exists y' \in \gamma(x', I') \) s.t. \( y \rightarrow y' \), and (iii) \( \forall y \in \gamma(x, I), \exists y' \in \gamma(x', I') \) s.t. \( y \rightarrow y' \).

Proof. We first prove that (i) \( \Rightarrow \) (iii). Assume that \( n \sim n' \), and let \( n_1 = (x_1, I_1), n_2 = (x_2, I_2) \) be nodes such that \( n \rightarrow_e n_1 \rightarrow n_2 \rightarrow_e n' \). Let also \( y \in \gamma(x, I) \). By Lemma 23, there exists \( t \in I_1 \) such that \( y = \gamma(x_1, t) \). Let us prove that there exists \( t' \in I_2 \) such that \( t \leq t' \). Indeed, if \( I_1 = I_2 \), we may choose \( t' = t \). Otherwise, recall that \( I_1 \) and \( I_2 \), as maximal \( x_1 \)-static or \( x_1 \)-adaptable intervals, must be disjoint, and that there exist \( t_1 \in I_1 \) and \( t_2 \in I_2 \) such that \( t_1 \leq t_2 \); this proves in fact that \( t_1 < t_2 \) for all \( t_1 \in I_1 \) and \( t_1 \in I_2 \), and therefore that every \( t' \in I_2 \) is greater than \( t \). Finally, let \( y' = \gamma(x_2, t') \). Since \( x_1 = x_2 \) and \( t \leq t' \), we know that \( y \rightarrow y' \), and since \( n_2 \rightarrow_e n' \), Lemma 23 proves that \( y' \in \gamma(x', I') \), which proves (iii).

Second, observe that the implication (iii) \( \Rightarrow \) (ii) is immediate. It remains to prove that (ii) \( \Rightarrow \) (i). Assume that (ii) holds. Let \( x_1 \in V_1 \) be a parameter, and \( t_1 \leq t_2 \) be elements of \( V \) such that \( y = \gamma(x_1, t_1) \) and \( y' = \gamma(x_1, t_2) \). Let \( I_1 \) and \( I_2 \) be the maximal \( x_1 \)-static or \( x_1 \)-adaptable intervals to which belong \( t_1 \) and \( t_2 \), and let \( n_1 = (x_1, t_1) \) and \( n_2 = (x_1, t_2) \). By construction, and using Lemma 23, we have \( n \rightarrow_e n_1 \rightarrow n_2 \rightarrow_e n' \), which proves (i).

Theorem 25. For all integers \( k \geq 1 \), two elements \( y_1 \) and \( y_2 \) in \( V_2 \) are \((k\text{-step})\) time-abstract bisimilar if and only if there exist \((k\text{-step})\) \( \epsilon \)-bisimilar nodes \( n_1 = (x_1, I_1) \) and \( n_2 = (x_2, I_2) \) of the bisimulation graph \( \mathcal{G} \) such that \( y_i \in \gamma(x_i, I_i) \).

Proof. In the following, we conveniently write \( \gamma(n) \) instead of \( \gamma(x, I) \) when \( n \) is the node \((x, I)\).

For every \( k \geq 1 \), define \( R_k \) as the largest \( k \)-step time-abstract bisimulation over \( V_2 \). We define the relation \( \mathcal{R}_k \) over \( N \) as follows:

\[ n_1 \mathcal{R}_k n_2 \text{ iff } \exists y \in \gamma(n_1) \text{ such that } y_1 R_k y_2. \]

Let us prove, by induction on \( k \), that \( \mathcal{R}_k \) is a \((k\text{-step})\) \( \epsilon \)-bisimulation relation. The case \( k = 1 \) is immediate, hence we assume that \( k \geq 0 \) and that \( \mathcal{R}_{k-1} \) is a \((k-1)\)-step time-abstract bisimulation.

Let \( n_1 = (x_1, I_1) \) and \( n_2 = (x_2, I_2) \) be two nodes such that \( n_1 \mathcal{R}_{k-1} n_2 \), and let \( y_1 \in \gamma(n_1) \) and \( y_2 \in \gamma(n_2) \) such that \( y_1 R_k y_2 \). First, since \( I_1 \) is either \( x_1 \)-static or \( x_1 \)-suitable, we know that the function \( t \mapsto G_{\gamma(x_1, t)} \) is constant on \( I_1 \). Similarly, the function \( t \mapsto G_{\gamma(x_2, t)} \) is constant on \( I_2 \) and therefore \( L(n_1) = G_y = L(n_2) \).

Then, let \( n_1' = (x_1', I_1') \) be a node such that \( n_1 \sim n_1' \). By Lemma 24, there exists \( y_1' \in \gamma(n_1') \) such that \( y_1 \rightarrow y_1' \). Since \( y_1 R_k y_2 \), there also exists \( y_2' \) such that \( y_2 \rightarrow y_2' \) and \( y_1' R_{k-1} y_2' \). Let \( n_2' = (x_2', I_2') \) be a node such that \( y_2' \in \gamma(n_2') \). By construction, we have \( y_1' \mathcal{R}_{k-1} y_2' \). Since \( n_1 \) and \( n_2 \) play symmetric roles, \( \mathcal{R}_k \) is a \((k\text{-step})\) \( \epsilon \)-bisimulation relation.

Likewise, if \( R \) is the largest time-abstract bisimulation over \( V_2 \), the relation \( \mathcal{R} \) over \( N \) defined by \( n_1 \mathcal{R} n_2 \text{ iff } \exists y \in \gamma(n_1) \text{ such that } y_1 R y_2 \) is an \( \epsilon \)-bisimulation relation.

Consequently, if \( y_1 \) and \( y_2 \) are \((k\text{-step})\) time-abstract bisimilar, constructing the relation \( \mathcal{R}_k \) or \( \mathcal{R} \) as above proves that there exist \((k\text{-step})\) \( \epsilon \)-bisimilar nodes \( n_1 \) and \( n_2 \) of the bisimulation graph \( \mathcal{G} \) such that \( y_i \in \gamma(n_i) \).

Conversely, for every \( k \geq 1 \), define \( \mathcal{R}_k \) as the largest \( k \)-step \( \epsilon \)-bisimulation over \( N \). We define the relation \( R_k \) over \( V_2 \) as follows:

\[ y_1 R_k y_2 \text{ iff } \exists n \in N \text{ such that } y_1 \in \gamma(n_1) \text{ and } n_1 \mathcal{R}_k n_2 \].
Let us prove, by induction on $k$, that $R_k$ is a $k$-step $\varepsilon$-bisimulation relation. The case $k = -1$ is immediate, hence we assume that $k \geq 0$ and that $R_{k-1}$ is a $(k-1)$-step time-abstract bisimulation.

Consider two states $y_1, y_2 \in V_2$ such that $y_1 R_k y_2$, and let $n_1 = (x_1, I_1)$ and $n_2 = (x_2, I_2)$ be two nodes such that $y_1 \in \gamma(n_1)$ and $n_1 \mathcal{R} n_2$. Once again, the function $t \mapsto G_{\gamma(x_{1,t})}$ is constant on $I_1$, and $t \mapsto G_{\gamma(x_{2,t})}$ is constant on $I_2$, hence $G_{y_1} = L(n_1) = L(n_2) = G_{y_2}$.

Then, let $y'_1$ be a state such that $y_1 \rightarrow y'_1$, and let $n'_1 = (x'_1, I'_1)$ be a node such that $y'_1 \in \gamma(n'_1)$. Lemma 24 proves that $n_1 \sim n'_1$, and since $\mathcal{R}_k$ is a $k$-step $\varepsilon$-bisimulation relation there exists a node $n'_2 = (x'_2, I'_2)$ such that $n_2 \sim n'_2$ and $n'_1 \mathcal{R}_k n'_2$. Lemma 24 proves that $y_2 \rightarrow y'_2$ for some $y'_2 \in \gamma(n'_2)$, and we have $y'_1 \mathcal{R}_k y'_2$ by construction. Since $y_1$ and $y_2$ play symmetric roles, $\mathcal{R}_k$ is a $k$-step $\varepsilon$-bisimulation relation.

Likewise, if $\mathcal{R}$ is the largest $\varepsilon$-bisimulation over $N$, the relation $R$ over $V_2$ defined by $y_1 R y_2$ iff $\exists n_1 \in N$ such that $y_1 \in \gamma(n_1)$ and $n_1 \mathcal{R} n_2$ is a time-abstract bisimulation relation. In particular, if $\mathcal{R}$ is an $\varepsilon$-bisimulation relation, then $R$ is a time-abstract bisimulation relation.

Consequently, if $n_1$ and $n_2$ are $(k$-step $\varepsilon$-bisimilar, constructing the relation $(R_k$ or $R$ as above proves that, for all states $y_1 \in \gamma(n_1)$, $y_1$ and $y_2$ are $(k$-step $\varepsilon$-time abstract) bisimilar, which completes the proof.

**Example 26.** The bisimulation graph for the dynamical system of Figure 1 is depicted on Figure 3. We infer that:

- all points of the interval $(-\infty, y_3) = \gamma(x_2, (3, +\infty)) = \gamma(x_3, (4, +\infty))$ are time-abstract bisimilar;
- the singleton $\{y_3\} = \gamma(x_1, [1, 2]) = \gamma(x_2, [1.5, 3]) = \gamma(x_3, \{4\})$ forms a class of the time-abstract bisimulation;
- all points of the interval $(y_3, +\infty) = \gamma(x_1, (-\infty, 1)) = \gamma(x_1, (2, +\infty)) = \gamma(x_2, (-\infty, 1.5)) = \gamma(x_3, (-\infty, 4))$ are time-abstract bisimilar.
4 Definability and decidability

In this section, we discuss definability and decidability issues. We say that a theory \( \mathcal{M} = (M, \lt, \ldots) \) is \textit{decidable} whenever for every first-order formula \( \varphi \), for every \( t \in M \), one can decide whether \( t \models \varphi \) holds.

So far we have not assumed any decidability of the structures, and, indeed, not all \( \omega \)-minimal structures are decidable. For instance, it is not known whether the \( \omega \)-minimal structure \( (\mathbb{R}, <, 0, 1, +, \cdot, \text{exp}) \) is decidable [27, 29]. Alternatively, if \( \omega \) is a non-computable real number, such as Chaitin’s constant [11], then the structure \( (\mathbb{R}, <, 0, 1, \omega, +) \) is \( \omega \)-minimal but not decidable.

In this section, we consider the relation \( \sim^* \), which is the (reflexive and) transitive closure of \( \sim \), with \( V_1^*(x) \overset{\text{def}}{=} \{ x' \in V_1 : x \sim^* x' \} \). We introduce the following assumption, called \textit{Finite Crossing}: every equivalence class of the relation \( \sim^* \) (i.e. every set \( V_1^*(x) \)) is finite. The stronger condition obtained when there is a uniform bound on the size of equivalence classes is called \textit{Uniform Crossing}.

\textbf{Theorem 27.} Let \( (M, \gamma) \) be an \( \omega \)-minimal dynamical system. Under the Uniform Crossing assumption, the relation of time-abstract bisimulation is definable, and it contains finitely many equivalence classes.

\textbf{Proof.} Let \( y_1, y_2 \) be elements of \( V_2 \) and let \( x_1, x_2 \in V_1 \) be parameters such that \( y_1 \in \Gamma_{x_1} \). Let also \( P \) be a positive integer such that \( |V_1^*(x)| \leq P \) for all \( x \in V_1 \). Consider the sub-graph \( G' \) of the bisimulation graph \( G \) that consists in those nodes \((x', I)\) with \( x' \in V_1^*(x_1) \cup V_1^*(x_2) \). This sub-graph is finite, and Lemmas 18 and 20 prove that it contains at most \( k(L + M + M k) \) nodes, where \( k = |V_1^*(x_1) \cup V_1^*(x_2)| \). Since \( k \leq 2P \), it follows that \( G' \) contains at most \( N = 2P(L + M + 2M P) \) nodes.

It is well-known that, on \( G' \), the relations of \( \varepsilon \)-bisimulation and of \( N \)-step \( \varepsilon \)-bisimulation are equal to each other. Hence, it follows from Theorem 25 that \( y_1 \) and \( y_2 \) are time-abstract bisimilar if and only if they are \( N \)-step time-abstract bisimilar. In particular, the latter relation has finitely many equivalence classes, and is definable, which proves Theorem 27.

\textbf{Theorem 28.} Let \( (M, \gamma) \) be a decidable \( \omega \)-minimal dynamical system. Under the Finite Crossing assumption, the relation of time-abstract bisimulation is decidable: given \( y_1, y_2 \in V_2 \), one can decide whether \( y_1 \) and \( y_2 \) are time-abstract bisimilar.

\textbf{Proof.} For all \( k \geq 0 \) and \( x \in V_1 \), let \( V(k, x) = \{ x_k \in V_1 : \exists x_1, \ldots, x_k \in V_1 \text{ s.t. } x \sim x_1 \sim x_2 \ldots \sim x_{k-1} \sim x_k \} \), where we recall that the relation \( \sim \) is defined by: \( x \sim x' \) iff \( \Gamma_{x_1} \cap \Gamma_{x'} \neq \emptyset \). By construction, the set \( V(k, x) \) is definable and is a subset of \( V_1^*(x) \). Moreover, since \( V_1^*(x) \) is finite, there exists a minimal integer \( k \geq 0 \) such that \( V(k, x) = V(k + 1, x) \), and we have \( V_1^*(x) = V(k, x) \). Since the equality of definable sets is decidable, the set \( V_1^*(x) \) is therefore computable for every parameter \( x \in V_1 \).

Now, let \( y_1, y_2 \) be elements of \( V_2 \) and let \( x_1, x_2 \in V_1 \) be parameters such that \( y_1 \in \Gamma_{x_1} \). We just showed how to compute the set \( V_1^*(x_1) \cup V_1^*(x_2) \). Then, let \( R \) and \( R' \) be the respective time-abstract bisimulation relations in \( (M, \gamma) \) and of \( (M, \gamma') \), where \( \gamma' \) is the restriction of \( \gamma \) to the set \( V_1' \times V_1' \). Since \( R' \) coincides with the restriction of \( R \) to \( \{ y \in V_2 : \exists x \in V_1', y \in \Gamma_x \} \), it remains to compute the relation \( R' \).

Since \( V_1' \) is finite, we may apply Theorem 27 to \( (M, \gamma') \). We thereby prove that \( R' \) has finitely many equivalence classes, and therefore is equal to the \( N \)-step time-abstract bisimulation in \( (M, \gamma') \), for some integer \( N \). Consequently, the standard partition refinement procedure (see e.g. [7, p. 6]) will terminate, since there are finitely many classes, and we
will be able to detect termination, since the theory is decidable. The partition refinement procedure is therefore an effective algorithm which allows to compute the time-abstract bisimulation $R'$, which completes the proof. \hfill $\blacksquare$

Remark that all results still hold if we replace the conditions on the sets $V_1^*(x)$, $x \in V_1$ by a finer semantical definition:

$$V_1^*(x) = \{ x' \in V_1 : \exists y_1, \ldots, y_k \in V_2 \text{s.t. } y_1 \in \Gamma_x, y_k \in \Gamma_{x'} \text{ and } y_1 \to \cdots \to y_k \}.$$ 

Notice, however, that the assumption on the size of $V_1^*(x)$ could not be relaxed, due to the undecidability result of [4, Theorem 3.1].

**Recovering the main result of [25]**

The use of restricted dynamical systems also allows us to encompass the main result of [25].

**Theorem 29.** Let $\mathcal{M} = (\mathbb{R}, <, \ldots)$ be an expansion of the ordered set of the reals with an o-minimal theory, and let $V = \mathbb{R}$ and $V_1 = V_2 = \mathbb{R}^n$ for some integer $n \geq 1$. Assume that there exists a smooth, complete vector field $F$ over $\mathbb{R}^n$ such that the dynamics (called flow in [25]) $\gamma : (x, t) \to \gamma(x, t)$, which is defined by: $\gamma(x, 0) = x$ and $\frac{d}{dt} \gamma(x, t) = F(\gamma(x, t))$, is definable in $\mathcal{M}$. Then, the relation of time-abstract bisimulation is definable, and it contains finitely many equivalence classes.

**Proof.** By construction, if two trajectories $\Gamma_x$ and $\Gamma_{x'}$ have a non-empty intersection, then there exists a real number $t$ such that $x' = \gamma(x, t)$, and we have $\gamma(x', u) = \gamma(x, t + u)$ for all $u \in \mathbb{R}$, so that the trajectories $\Gamma_x$ and $\Gamma_{x'}$ are equal to each other. Hence, the relation $\sim$ is an equivalence relation.

Then, due to [28, Corollary 3.3.28], there exists a definable set $V_1'$ such that every equivalence class of $\sim$ contains a unique element in $V_1'$. Consider the restricted dynamical system $(\mathcal{M}, \gamma')$, where $\gamma'$ is the restriction of $\gamma$ to the set $V_1' \times V$. This restricted dynamical system satisfies the hypothesis of Theorem 27, and therefore there exists an integer $N \geq 0$ such that the time-abstract bisimulation relation and the $N$-step time-abstract bisimulation relation in $(\mathcal{M}, \gamma')$ are equal to each other. Since the transition systems associated with $(\mathcal{M}, \gamma)$ and $(\mathcal{M}, \gamma')$ are equal to each other, the result follows. \hfill $\blacksquare$

**5 Conclusion**

In this paper, we have proposed a new approach for the analysis of o-minimal dynamical systems. Our approach allows us to treat trajectories with overlapping portions, and with possibly rich intersections. There is however a restriction, which is that trajectory switches should remain within a finite family of trajectories, once the initial trajectory has been chosen. It is important to notice that, as mentioned in the end of Section 4, it would not be possible to arbitrarily relax that assumption, since the reachability problem is undecidable for dynamical systems allowing arbitrarily many switches, as proved in [4, Theorem 3.1].

Adding the standard decoupling hypothesis, where jumps between locations reinitialize trajectories, we obtain a decidable class of hybrid systems.

Our future work will consist in trying to adapt the idea of interrupt timed automata of [3], where no reinitialization is assumed, to systems with richer (o-minimal) dynamics.
References


