# Beyond-Planarity: Turán-Type Results for Non-Planar Bipartite Graphs

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## Abstract

*Beyond-planarity* focuses on the study of geometric and topological graphs that are in some sense nearly planar. Here, planarity is relaxed by allowing edge crossings, but only with respect to some local forbidden crossing configurations. Early research dates back to the 1960s (e.g., Avital and Hanani 1966) for extremal problems on geometric graphs, but is also related to graph drawing problems where visual clutter due to edge crossings should be minimized (e.g., Huang et al. 2018).

Most of the literature focuses on Turán-type problems, which ask for the maximum number of edges a beyond-planar graph can have. Here, we study this problem for bipartite topological graphs, considering several types of beyond-planar graphs, i.e. 1-planar, 2-planar, fan-planar, and RAC graphs. We prove bounds on the number of edges that are tight up to additive constants; some of them are surprising and not along the lines of the known results for non-bipartite graphs. Our findings lead to an improvement of the leading constant of the well-known Crossing Lemma for bipartite graphs, as well as to a number of interesting questions on topological graphs.

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## 1 Introduction

Planarity has been a central concept in the areas of graph algorithms, computational geometry, and graph theory since the beginning of the previous century. While planar graphs were originally defined in terms of their geometric representation, they exhibit a number of combinatorial properties that only depend on their abstract representations. To mention some of the most important landmarks, we refer to the characterization of planar graphs in

**(a)** 1-planar **(b)** 3-quasiplanar **(c)** fan-planar **(d)** RAC

**Figure 1** Different forbidden crossing configurations.

terms of forbidden minors, to the existence of linear-time algorithms to test planarity, to the Four-Color theorem, and to the Euler's polyhedron formula, which implies that $n$-vertex planar graphs have at most $3n - 6$ edges.

For the applicative purpose of visualizing real-world networks, however, the concept of planarity turns out to be overly restrictive. Graphs representing such networks are too dense to be planar, even though one can often confine non-planarity in some local structures. Also, cognitive experiments [28] show that this does not affect the readability of the drawing too much, if these local structures satisfy specific properties. In other words, these experiments indicate that even non-planar drawings may be effective for human understanding, as long as the crossing configurations satisfy certain properties. Different requirements on the crossing configurations naturally give rise to different classes of *beyond-planar* graphs. Beyond-planarity is then defined as a generalization of planarity, which encompasses all these classes. Early works date back to 60's [12] in the field of extremal graph theory, and continued over the years [3, 7, 33]; also due to the aforementioned experiments, a strong attention on the topic was recently raised (e.g., [21]), which led to many results, described below.

Some of the most studied beyond-planar graphs include:

**(i)** *$k$-planar graphs*, in which each edge is crossed at most $k$ times [2, 32, 33], see Fig. 1a;

**(ii)** *$k$-quasiplanar graphs*, which disallow sets of $k$ pairwise crossing edges [1, 3, 24], see Fig. 1b;

**(iii)** *fan-planar* graphs, in which no edge is crossed by two independent edges or by two adjacent edges from different directions [14, 30], see Fig. 1c;

**(iv)** *RAC graphs*, in which crossings happen at right angles [20, 22]; see Fig. 1d.

Two notable sub-families of 1-planar graphs are the *IC-planar graphs* [38], where crossings are *independent* (i.e., no two crossed edges share an endpoint), and the *NIC-planar graphs* [37], where crossings are *nearly independent* (i.e., no two pairs of crossed edges share two endpoints). For a survey providing an overview on beyond-planarity see [21].

From the combinatorial point of view, the main extremal graph theory question, also called Turán-type [15], concerns the maximum number of edges for graphs in a certain class. Tight density bounds are known for several classes [20, 30, 33, 37, 38]; a main open question is to determine the density of $k$-quasiplanar graphs, which is conjectured to be linear in $n$ for any fixed $k$ [1, 3, 7, 24]. The new bounds for 1-, 2-, 3- and 4-planar graphs have led to progressive improvements on the leading constant of the lower bound on the number of crossings of a graph, provided by the well-known Crossing Lemma, from $\frac{1}{100} = 0.01$ [5, 31] to $\frac{1}{64} \approx 0.0156$ [4], to $\frac{1}{33.75} \approx 0.0296$ [33], to $\frac{1}{31.1} \approx 0.0322$ [32], to $\frac{1}{29} \approx 0.0345$ [2]. Related combinatorial problems concern inclusion relationships between classes [8, 14, 17, 22, 25].

From the complexity side, in contrast to efficient planarity testing algorithms [27], recognizing a beyond-planar graph has often been proven to be NP-hard [10, 14]. Polynomial-time testing algorithms can be found when posing additional restrictions on the produced drawings, namely, that the vertices are required to lie either on two parallel lines (see, e.g., [14, 19]) or on the outer face of the drawing (see, e.g., [11, 26]).

**Table 1** Summary of our results (from sparse to dense); the bound with asterisk (∗) is not tight.

| | General | | Bipartite | | | |
|---|---|---|---|---|---|---|
| Graph class | Bound (tight) | Ref. | Lower bound | Ref. | Upper bound | Ref. |
| IC-planar: | $3.5n - 7$ | [38] | $2.25n - 4$ | Thm.1 | $2.25n - 4$ | Thm.2 |
| NIC-planar: | $3.6n - 7.2$ | [37] | $2.5n - 5$ | Thm.1 | $2.5n - 5$ | Thm.3 |
| 1-planar: | $4n - 8$ | [34] | $3n - 8$ | [18] | $3n - 8$ | [18] |
| RAC: | $4n - 10$ | [20] | $3n - 9$ | Thm.4 | $3n - 7$ | Thm.5 |
| 2-planar: | $5n - 10$ | [33] | $3.5n - 12$ | Thm.13 | $3.5n - 7$ | Thm.15 |
| fan-planar: | $5n - 10$ | [30] | $4n - 16$ | Thm.6 | $4n - 12$ | Thm.11 |
| 3-planar: | $5.5n - 11$ | [32] | $4n - O(1)$ | Sec.6 | — | — |
| $k$-planar: | $3.81\sqrt{k}n$ * | [2] | — | — | $3.005\sqrt{k}n$ | Thm. 17 |

Another natural restriction, yet rarely explored in the literature, is to pose additional structural constraints on the graphs themselves, rather than on their drawings. For 3-connected 1-plane graphs, Alam et al. [6] presented a polynomial-time algorithm to construct 1-planar straight-line drawings. Further, Brandenburg [16] gave an efficient algorithm to recognize optimal 1-planar graphs, i.e., those with the maximum number of edges.

For the important class of bipartite graphs, very few results have been discovered so far. From the density point of view, the only result we are aware of is a tight bound of $3n - 8$ edges for bipartite 1-planar graphs [18, 29]. Didimo et al. [19] characterize the complete bipartite graphs that admit RAC drawings, but their result does not extend to non-complete graphs.

**Our contribution.** Along this direction, we study several classes of beyond-planar bipartite topological or geometric graphs, focusing on Turán-type problems. Table 1 shows our findings. The new bound on the edge density of bipartite 2-planar graphs leads to an improvement of the leading constant of the Crossing Lemma for bipartite graphs from $\frac{1}{29} \approx 0.0345$, which holds for general graphs [2], to $\frac{1}{18.1} \approx 0.0554$ (see Theorem 16). To the best of our knowledge, this is the first non-trivial adjustment of the Crossing Lemma that is specific for bipartite graphs, besides the Zarankiewicz conjecture [36], which however only concerns complete bipartite graphs. Our results also unveil an interesting tendency in the density of $k$-planar bipartite graphs with respect to the one of general $k$-planar graphs. At first sight, the differences seem to be around $n$, as it is in the planar and in the 1-planar cases (i.e., $n - 2$). This turns out to be true also for RAC and fan-planar graphs. However, for IC- and NIC-planar graphs, and in particular for 2-planar graphs, the differences are larger.

Another notable observation from our results is that, in the bipartite setting, fan-planar graphs can be denser than 2-planar graphs, while in the non-bipartite case these two classes have the same maximum density, even though none of them is contained in the other [14]. In Section 6 we discuss a number of open problems that are raised by our work.

**Methodology.** We focus on five classes of bipartite beyond-planar graphs; see Sections 2–5. To estimate the maximum edge density of each class we employ different counting techniques.

- For the class of bipartite IC-planar graphs, we apply a direct counting argument based on the number of crossings that are possible due to the restrictions posed by IC-planarity.
- Our approach is different for NIC-planarity. We show that a bipartite NIC-planar graph of maximum density contains a set of uncrossed edges forming a plane subgraph whose faces have length 6, and that each such face contains exactly one crossing pair of edges.

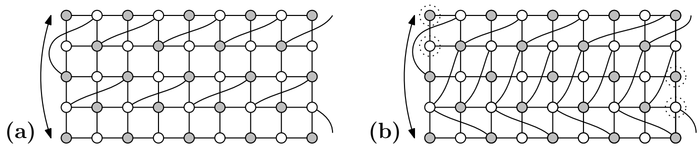



**Figure 2** Bipartite $n$-vertex IC- and NIC-planar graphs with (a) $2.25n - 4$ and (b) $2.5n - 5$ edges.

- To estimate the maximum number of edges of a bipartite RAC graph, we adjust a technique by Didimo et al. [20], who proved the corresponding bound for general RAC graphs.
- For fan-planarity, our technique is more involved. After examining structural properties of maximal bipartite fan-planar graphs, we show how to augment them so that they contain a planar quadrangulation as a subgraph. Then, we develop a charging scheme which charges edges involved in fan crossings to the corresponding vertices, to prove that there are at least as many edges in the quadrangulation as in the rest of the graph.
- For 2-planarity, we again show that maximal bipartite 2-planar graphs have a planar quadrangulation as a subgraph. We then use a counting scheme based on an auxiliary directed plane graph, defined by orienting the dual of the quadrangulation, describing dependencies of adjacent quadrangular faces posed by the edges not belonging to it.

**Preliminaries.** We consider connected *topological* graphs, i.e., drawn in the plane with vertices represented by points in $\mathbb{R}^2$ and edges by Jordan arcs connecting their endvertices, so that:
**(i)** no edge passes through a vertex different from its endpoints,
**(ii)** no two adjacent edges cross,
**(iii)** no edge crosses itself,
**(iv)** no two edges meet tangentially, and
**(v)** no two edges cross more than once.
A graph has no self-loops or multiedges. Otherwise, it is a topological *multigraph*, for which we assume that the two regions defined by self-loops or multiedges contain at least one vertex in their interiors, i.e., all edges are *non-homotopic.*

We refer to a beyond-planar graph $G$ with $n$ vertices and maximum possible number of edges as *optimal.* Consider all the plane spanning subgraphs of $G$ (i.e., in their drawings inherited from $G$ there exists no two crossing edges). Among those, we select one with the largest number of edges, which we denote by $G_p$ and call it *the planar structure* of $G$. Let $f = \langle u_0, u_1, \ldots, u_{k-1}\rangle$ be a face of $G_p$. We say that $f$ is *simple* if $u_i \neq u_j$ for each $i \neq j$; face $f$ is *connected* if edge $(u_i, u_{i+1})$ exists for each $i = 0, \ldots, k-1$ (indices modulo $k$).

## 2 Bipartite IC- and NIC-Planar Graphs

In this section, we give tight bounds on the density of bipartite IC- and NIC-planar graphs. For the proofs of the lower bounds, we refer to Fig. 2. Full proofs can be found in [9].

▶ **Theorem 1.** *For infinitely many values of $n$, there exists a bipartite $n$-vertex IC-planar graph with $2.25n - 4$ edges, and a bipartite $n$-vertex NIC-planar graph with $2.5n - 5$ edges.*

▶ **Theorem 2.** *A bipartite $n$-vertex IC-planar graph has at most $2.25n - 4$ edges*

**Proof.** Our proof is an adjustment of the one for general IC-planar graphs [38]. Let $G$ be a bipartite $n$-vertex optimal IC-planar graph. Let $cr(G)$ be the number of crossings of $G$. Since every vertex of $G$ is incident to at most one crossing, $cr(G) \leq \frac{n}{4}$. By removing one edge

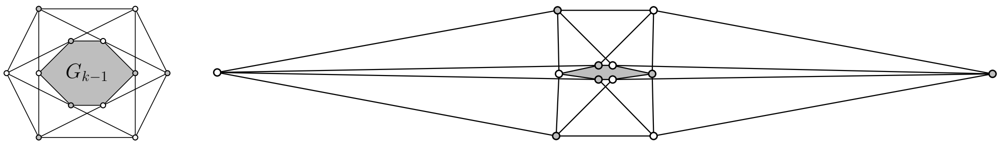



**Figure 3** Construction for a bipartite $n$-vertex RAC graph with $3n-9$ edges.

from every pair of crossing edges of $G$, we obtain a plane bipartite graph, which has at most $2n-4$ edges. Hence, the number of edges of $G$ is at most $2n-4+cr(G) = 2.25n-4$. ◀

▶ **Theorem 3.** *A bipartite $n$-vertex NIC-planar graph has at most $2.5n-5$ edges.*

**Proof.** Among all bipartite optimal NIC-planar graphs with $n$ vertices, let $G$ be the one with the maximum number of uncrossed edges, i.e., $G$ is such that the plane (bipartite) subgraph $H$ obtained by removing every crossed edge in $G$ has maximum density. It is not difficult to show that each face of $H$ containing two crossing edges in $G$ is connected and has length 6 (for details see [9]). Thus, every face of $H$ has length either 6, if it contains two edges crossing in $G$, or 4 otherwise (due to bipartiteness and maximality).

Let $\nu$ and $\mu$ be the number of vertices and edges of $H$, respectively. Clearly, $n=\nu$. Let also $\phi_4$ and $\phi_6$ be the number of faces of length 4 and 6 in $H$, respectively. We have that $2\phi_4+3\phi_6=\mu$. By Euler's formula, we also have that $\mu+2=\nu+\phi_4+\phi_6$. Combining these two equations, we obtain: $\phi_4+2\phi_6=\nu-2$. So, in total the number of edges of $G$ is $\mu+2\phi_6=2\phi_4+5\phi_6=2n-4+\phi_6$. By Euler's formula, the number of faces of length 6 of a planar graph is at most $(n-2)/2$, which implies that $G$ has at most $2.5n-5$ edges. ◀

## 3 Bipartite RAC Graphs

We continue our study on bipartite beyond-planarity with the class of geometric RAC graphs. We prove an upper bound on their density that is optimal up to a constant of 2.

▶ **Theorem 4.** *For infinitely many values of $n$, there exists a bipartite $n$-vertex RAC graph with $3n-9$ edges.*

**Proof.** For any $k>1$, we recursively define a graph $G_k$ by attaching six vertices and 18 edges to $G_{k-1}$; see the left part of Fig. 3. The base graph $G_1$ is a hexagon containing two crossing edges. So, $G_k$ has $6k$ vertices and $18k-10$ edges. The right part of Fig. 3 shows that $G_k$ is RAC: if $G_{k-1}$ is drawn so that its outerface is a parallelogram, then it can be augmented to a RAC drawing of $G_k$ whose outerface is a parallelogram with sides parallel to the ones of $G_{k-1}$. The bound follows by adding an edge in the outerface of $G_k$ by slightly "adjusting" its drawing; ses [9]. ◀

▶ **Theorem 5.** *A bipartite $n$-vertex RAC graph has at most $3n-7$ edges.*

**Proof.** Let $G$ be a (possibly non-bipartite) RAC graph with $n$ vertices. Since $G$ does not contain three mutually crossing edges, as in [20] we can color its edges with three colors (r, b, g) so that the crossing-free edges are the r-edges, while b-edges cross only g-edges, and vice-versa. Thus, the subgraphs $G_{rb}$, consisting of only r- and b-edges, and $G_{rg}$, consisting of only r- and g-edges, are both planar. Didimo et al. [20, Lemma 4] showed that each face of $G_{rb}$ has at least two r-edges, by observing that if this property did not hold, then the drawing could be augmented by adding r-edges. Thus, the number $m_b$ of b-edges is at most

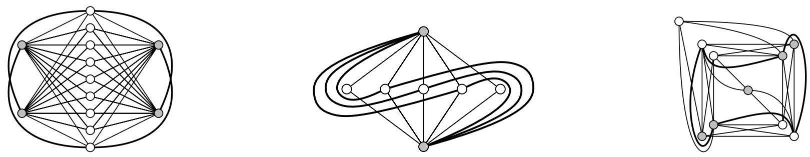



**Figure 4** (a) $K_{4,n-4}$ with four additional multiedges (thick), (b) $K_{2,n-2}$ with $2n-8$ additional multiedges (thick), and (c) $K_{5,5}-e$ with four additional multiedges (thick).

$n-1-\lceil\lambda/2\rceil$, where $\lambda \geq 3$ is the number of edges in the outer face of $G$. Suppose now that $G$ is additionally bipartite. We still have $m_b \leq n-1-\lceil\lambda/2\rceil$, but in this case $\lambda \geq 4$ holds (by bipartiteness). Hence, $m_b \leq n-3$. Since $G_{rg}$ is bipartite and planar, it has at most $2n-4$ edges (i.e., $m_r + m_g \leq 2n-4$). Hence, $G$ has at most $3n-7$ edges. ◀

## 4 Bipartite Fan-Planar Graphs

We continue our study with the class of fan-planar graphs. We begin as usual with the lower bound (Theorem 6), which we suspect to be best-possible both for graphs and multigraphs. For fan-planar bipartite graphs, we prove an almost tight upper bound (Theorem 11).

▶ **Theorem 6.** *For infinitely many values of $n$, there exists a bipartite $n$-vertex fan-planar*
**(i)** *graph with $4n-16$ edges, and*
**(ii)** *multigraph with $4n-12$ edges.*

**Proof sketch.** Figs. 4a, 4b, and 4c show constructions that yield bipartite $n$-vertex fan-planar multigraphs with $4n-12$ edges. Removing the thick edges in Figs. 4a and 4c gives bipartite $n$-vertex fan-planar graphs with $4n-16$ edges. ◀

To prove the upper bound, consider a bipartite fan-planar graph $G$ with a fixed fan-planar drawing. W.l.o.g. assume that $G$ is edge-maximal and connected, and $A$, $B$ are the two bipartitions of $G$. We shall denote vertices in $A$ by $a$, $a'$, or $a_i$ for some index $i$, and similarly vertices in $B$ by $b$, $b'$, or $b_i$. By fan-planarity, for each crossed edge $e$ of $G$ all edges crossing $e$ have a common endpoint $v$. We call $e$ an *$A$-edge* (respectively, *$B$-edge*) if this vertex $v$ lies in $A$ (respectively, $B$). If $e$ is crossed exactly once, it is $A$-edge and $B$-edge.

A *cell* of some subgraph $H$ of $G$ is a connected component $c$ of the plane after removing all vertices and edges in $H$; see also [30]. The *size* of $c$, denoted by $||c||$ is the total number of vertices and edge segments on the *boundary* $\partial c$ of $c$, counted with multiplicities.

▶ **Lemma 7** ([30]). *Each fan-planar graph $G$ admits a fan-planar drawing such that if $c$ is a cell of any subgraph of $G$, and $||c|| = 4$, then $c$ contains no vertex of $G$ in its interior.*

We choose a fan-planar drawing of $G$ with the property given in Lemma 7.

▶ **Corollary 8.** *If $e = (a, b)$, with $a \in A$ and $b \in B$, is crossed at point $p$ by an $A$-edge $e'$, then each edge crossing $e$ between $a$ and $p$ is an $A$-edge crossed by each edge crossing $e'$.*

**Proof.** Let $x$ be the common endpoint of all edges crossing $e$ and $e' = (x, y)$ be the $A$-edge crossing $e$ in $p$. Let $e'' = (x, y')$ be an edge that crosses $e$ between $p$ and $a$. If $e''$ is not an $A$-edge, it is crossed by an edge $e_1 = (a', b)$ with $a' \neq a$. The $A$-edge $e'$ is not crossed by $e_1$. But then there is a cell $c_1$ bounded by vertex $b$ and segments of $e$, $e''$ and $e_1$, which contains vertex $x$ or $y$ in its interior (see Fig. 5a), contradicting Lemma 7. Symmetrically, if there is

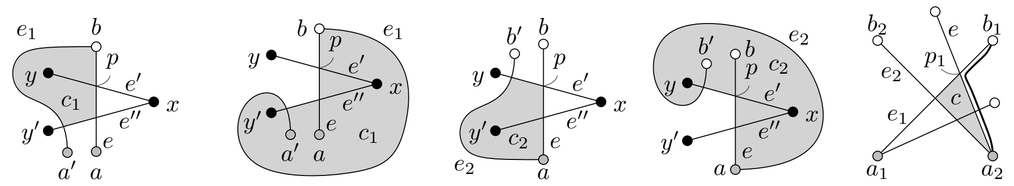



**Figure 5** Illustration of (a)-(b) the proof of Corollary 8, and (c) Lemma 10.

an edge $e_2 = (a, b')$ that crosses $e'$ but not $e''$, then there is a similar cell $c_2$ with $||c_2|| = 4$ containing vertex $x$ or $y'$ (see Fig. 5b), again contradicting Lemma 7. ◀

Kaufmann and Ueckerdt [30] derive Lemma 7 from the following lemma.

▶ **Lemma 9** ([30]). *Let $G$ be given with a fan-planar drawing. If two edges $(v, w)$ and $(u, x)$ cross in a point $p$, no edge at $v$ crosses $(u, x)$ between $p$ and $u$, and no edge at $x$ crosses $(v, w)$ between $p$ and $w$, then $u$ and $w$ are on the boundary of the same cell of $G$.*

By the maximality of $G$ we have in this case that $(u, w)$ is an edge of $G$, provided $u$ and $w$ lie in distinct bipartition classes. We can use this fact to derive the following lemma.

▶ **Lemma 10.** *Assume that $e_1 = (a_1, b_1)$ and $e_2 = (a_2, b_2)$ cross. If $e_1$ and $e_2$ are both $A$- or $B$-edges, then $(a_2, b_1)$ belongs to $G$ and can be drawn so that each edge that crosses $(a_2, b_1)$ also crosses $e_2$. Otherwise, $(a_2, b_1)$ belongs to $G$ and can be drawn crossing-free.*

**Proof.** First assume that $e_1$ and $e_2$ are both $A$-edges; the case where $e_1$ and $e_2$ are both $B$-edges is analogous. Let $p_1$ be the crossing point on $e_1$ that is closest to $b_1$. Since $e_1$ is an $A$-edge crossing $(a_2, b_2)$, the edge $e$ crossing $e_1$ at $p_1$ (possibly $e = e_2$) is incident to $a_2$. Now either $e = e_2$ or the subgraph $H$ of $G$ consisting of $e$, $e_1$ and $e_2$ (and their vertices) has one bounded cell $c$ of size 4, which by Lemma 7 contains no vertex. In both cases, every edge of $G$ crossing $e$ between $a_2$ and $p_1$, also crosses $e_2$ and ends at $a_1$ (as $e_2$ is an $A$-edge crossing $(a_1, b_1)$). Thus, drawing an edge from $b_1$ along $e_1$ to $p_1$ and then along $e$ to $a_2$ does not violate fan-planarity; see Fig. 5c. By the maximality of $G$, $(a_2, b_1)$ belongs to $G$ .

Now assume that $e_1$ is an $A$-edge and $e_2$ is a $B$-edge. Let $p$ be the crossing point of $e_1$ and $e_2$. By Lemma 9, $a_2$ and $b_1$ lie on the same cell in $G$ and hence, by the maximality of $G$, we have that the edge $(a_2, b_1)$ is contained in $G$ and can be drawn crossing-free. ◀

We are now ready to prove the main theorem of this section (see also [9] for omitted parts).

▶ **Theorem 11.** *Any $n$-vertex bipartite fan-planar graph has at most $4n - 12$ edges.*

**Proof sketch.** We start by considering the planar structure $G_p$ of $G$, i.e., an inclusion-maximal subgraph of $G$ whose drawing inherited from $G$ is crossing-free. Let $E_A$ and $E_B$ be the set of all $A$-edges and $B$-edges, respectively, in $E[G] - E[G_p]$. Each $e \in E_A$ is crossed by a non-empty (by maximality of $G_p$) set of edges in $G$ with common endpoint $a \in A$, and we say that $e$ *charges* $a$. Similarly, every $e \in E_B$ charges a unique vertex $b \in B$.

For any vertex $v$ in $G$, let $\mathrm{ch}(v)$ denote the number of edges in $E_A \cup E_B$ charging $v$. Moreover, for a multigraph $H$ containing $v$, let $\deg_H(v)$ denote the degree of $v$ in $H$, i.e., the number of edges of $H$ incident to $v$. Our goal is to show that for every vertex $v$ of $G$ we have $\deg_{G_p}(v) - \mathrm{ch}(v) \geq 2$. However, this is not necessarily true when $G_p$ is not connected or has faces of length 6 or more. To overcome this issue, we shall add in a step-by-step procedure vertices and edges (possibly parallel but non-homotopic to existing edges in $G_p$) to the plane drawing of $G_p$ such that:

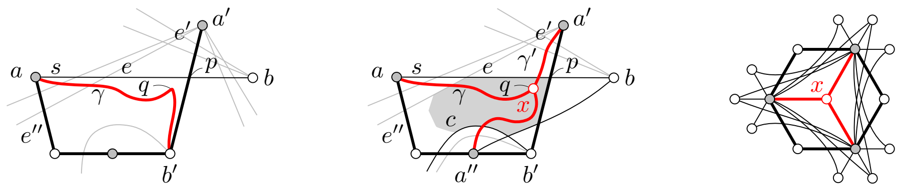



**Figure 6** Illustrations for Thm 11; edges in $G_p$ are thick, newly added vertices and edges are red.

**(P.1)** the obtained multigraph $\bar{G_p}$ is a planar quadrangulation,
**(P.2)** the drawing of the multigraph $\bar{G} := G \cup \bar{G_p}$ is again fan-planar, and
**(P.3)** each new vertex is added with three edges to other (possibly earlier added) vertices.

To find $\bar{G_p}$ (refer to [9] for a full proof), we first assume that $G_p$ is not connected. In this case there must be an edge $e$ with endpoints in different connected components of $G_p$, which is crossed by some edge $e'$ in $G_p$. Depending on which of $e, e'$ is an $A$-edge or $B$-edge, we either use Lemma 10 to add a new edge to $G_p$ or we carefully add a new vertex of degree three to $G_p$. Once we may assume that $G_p$ is connected but not a quadrangulation, there exists a face $f$ whose facial walk $W$ has length at least 6. For edges $e$ with one endpoint in $V[W]$ that run through face $f$ and leave $f$ by crossing an edge $e'$ of $G_p$, we define a *stick* to be the initial segment of $e$ that is contained in $f$. Such a stick $s$ splits $W$ into two parts, each going from the start vertex of $e$ to the crossing of $e$ and $e'$. As $G$ is bipartite, exactly one part, the *inner side* of $s$, contains an even number of vertices, and $s$ is called *short* if its inner side has only two vertices, and *long* otherwise. In case $f$ has a *long* stick, then again depending on which of $e, e'$ is an $A$-edge or $B$-edge, we either use Lemma 10 to add a new edge to $G_p$ (see Fig. 6a) or we carefully add a new vertex of degree three to $G_p$ (see Fig. 6b). Finally, if all sticks are short, we can add a crossing-free edge to $G_p$, or a new vertex with three crossing-free edges to $G_p$, as shown in Fig. 6c.

Adding to $G_p$ an edge or a vertex with three edges, strictly increases the average degree in $G_p$. Hence, we ultimately obtain supergraphs $\bar{G}$ of $G$ and $\bar{G_p}$ of $G_p$ satisfying P.1–P.3. Next, we show that the charge of every original vertex $v$ is at most its degree in $\bar{G_p}$ minus 2.

▶ **Claim 12.** *Every* $v \in V[G]$ *satisfies* $\deg_{\bar{G_p}}(v) - \mathrm{ch}(v) \geq 2$.

**Proof.** W.l.o.g. consider any $a \in A$ and let $k := \deg_{\bar{G_p}}(a)$ and $S \subseteq E_A$ be the set of edges charging $a$. Observe that no two edges of $S$ can cross. In fact, if $(a_1, b_1) \in E_A$ charges $a$ and $(a_2, b_2) \in E_A$ crosses $(a_1, b_1)$, then $(a_2, b_2)$ charges $a_1 \neq a$. Consider the face $f$ of $\bar{G_p} - \{a\}$ containing $a$, and the closed facial walk $W$ around $f$. Walk $W$ has length $2k$ (counting with repetitions) as $\bar{G_p}$ is a quadrangulation. Further, each edge in $S$ lies in $f$ and has both endpoints on $W$. Hence, the subgraph of $\bar{G}$ consisting of all edges in $W \cup S$ is crossing-free and has vertex set $V[W]$. Define graph $J$ by breaking the repetitions along $W$, i.e., $J$ consists of a cycle of length $2k$ and every edge in $S$ is an uncrossed chord of this cycle. $J$ has $\leq k-2$ chords, as it is bipartite outerplanar. Thus, $|S| = \mathrm{ch}(a) \leq k - 2 = \deg_{\bar{G_p}}(a) - 2$. ◀

Let $X = V[\bar{G}] - V[G]$ be the set of newly added vertices. For each $x \in X$, $\deg_{\bar{G_p}}(x) \geq 3$ and $\mathrm{ch}(x) = 0$ hold. Thus, $\deg_{\bar{G_p}}(x) - \mathrm{ch}(x) \geq 3$, and by Claim 12 we get $2|E[\bar{G_p}]| - (|E_A| + |E_B|) = \sum_{v \in V[\bar{G_p}]} \left(\deg_{\bar{G_p}}(v) - \mathrm{ch}(v)\right) \geq 2n + 3|X|$ which implies $|E_A| + |E_B| \leq 2|E[\bar{G_p}]| - 2n - 3|X|$. On the other hand, $|E[G_p]| + 3|X| \leq |E[\bar{G_p}]|$ by P.3 and $|E[\bar{G_p}]| = 2(n + |X|) - 4$ by P.1, which together give $|E[G]| = |E[G_p]| + |E_A| + |E_B| \leq 3|E[\bar{G_p}]| - 6|X| - 2n = 4n - 1$. ◀

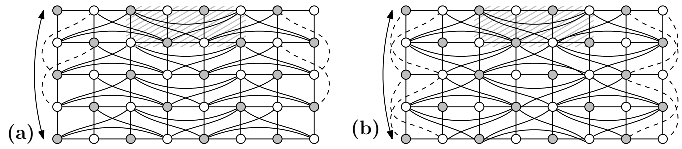
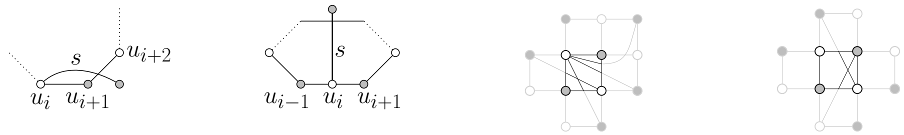



**Figure 7** Constructions for dense bipartite $n$-vertex 2-planar (a) graphs and (b) multigraphs.




**(a)** short stick **(b)** long stick **(c)** scissor+twin **(d)** pseudo-scissor

**Figure 8** Illustration of sticks, scissors and twins.

## 5 Bipartite 2-Planar Graphs

In this section, we overview our result for bipartite 2-planar graphs. For reasons of space, we sketch the proof; the full version is in [9]. We start with the lower bound; see Fig.7.

▶ **Theorem 13.** *For infinitely many values of $n$, there exists a bipartite $n$-vertex 2-planar*
**(i)** *graph with $3.5n - 12$ edges, and*
**(ii)** *multigraph with $3.5n - 8$ edges.*

For the upper bound, we study structural properties of the planar structure $G_p$ of an optimal bipartite 2-planar graph $G$. Let $(u, v)$ be an edge of $G$ that does not belong to $G_p$. By the maximality of $G_p$, $(u, v)$ has at least one crossing with an edge of $G_p$. As already mentioned, the part of $(u, v)$ that starts from $u$ ($v$) and ends at the first intersection point of $(u, v)$ with an edge of $G_p$ is a stick of $u$ ($v$). When $(u, v)$ has two crossings, there is a part that is not a stick, called *middle-part*. Each stick or middle-part lies in a face $f$ of $G_p$; we say that $f$ *contains* this part. Let $f = \langle u_0, u_1, \ldots, u_{k-1}\rangle$ be a face of $G_p$ with $k \geq 4$ and let $s$ be a stick of $u_i$ contained in $f$, $i \in \{0, 1, \ldots, k-1\}$. We call $s$ a *short* stick, if it ends either at $(u_{i+1}, u_{i+2})$ or at $(u_{i-1}, u_{i-2})$ of $f$; otherwise, $s$ is called a *long* stick; see Figs. 8a-8b.

W.l.o.g. we assume that among all optimal bipartite $n$-vertex 2-planar graphs, $G$ is such that its planar structure $G_p$ is the densest among the planar structures of all other optimal bipartite $n$-vertex 2-planar graphs; we call $G_p$ *maximally dense*. We first prove that $G_p$ is a spanning quadrangulation. For this, we first show that $G_p$ is connected, as otherwise it is always possible to augment it by adding an edge joining two connected components of it. Then, we show that all faces of $G_p$ are of length four. Our proof by contradiction is rather technical; assuming that there is a face $f$ with length greater than four in $G_p$, we consider two main cases:
**(i)** $f$ contains no sticks, but middle-parts, and
**(ii)** $f$ contains at least one stick.
With a careful case analysis, we lead to a contradiction either to the maximality of $G_p$ or to the fact that $G$ is optimal.

Since $G_p$ is a quadrangulation, it has exactly $2n - 4$ edges and $n - 2$ faces. Our goal is to prove that the average number of sticks for a face is at most 3. Since the number of edges of $G \setminus G_p$ equals half the number of sticks over all faces of $G_p$, this implies that $G$ cannot have more than $2n - 4 + \frac{3}{2}(n - 2) = 3.5n - 7$ edges, which gives the desired upper bound.

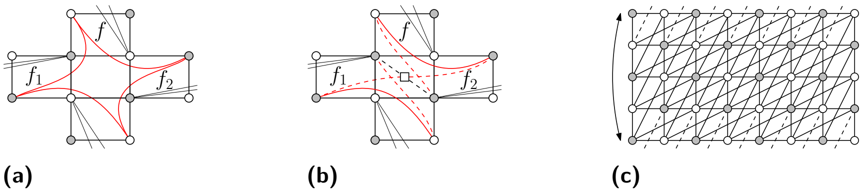



**Figure 9** Illustration of (a) the 8-sticks configuration, (b) its elimination, and (c) a bipartite 3-planar graph with $4n-O(1)$ edges; note that all vertices have degree 8, except for few boundary ones.

Let $f$ be a face of $G_p$. Denote by $h(f)$ the number of sticks contained in $f$. A *scissor* of $f$ is a pair of crossing sticks starting from non-adjacent vertices of $f$, while a *twin* of $f$ is a pair of sticks starting from the same vertex of $f$ crossing the same boundary edge of $f$; see Fig. 8c. We refer to a pair of crossing sticks starting from adjacent vertices of $f$ as a *pseudo-scissor*; see Fig. 8d. The following lemma shows that a face of $G_p$ contains a maximum number of sticks (that is, 4) only in the presence of scissors or twins, due to 2-planarity; see [9].

▶ **Lemma 14.** *Let $G$ be an optimal bipartite 2-planar graph, such that its planar structure $G_p$ is maximally dense. Then, for each face $f$ of $G_p$, it holds $h(f) \leq 4$. Further, if $h(f) = 4$, then $f$ contains one of the following: two scissors, or two twins, or a scissor and a twin.*

An immediate consequence of Lemma 14 is that $h(f) \leq 3$, for every face $f$ containing a pseudo-scissor. We now consider specific "neighboring" faces of a face $f$ of $G_p$ with four sticks and prove that they cannot contain so many sticks. Observe that each edge corresponding to a stick of $f$ starts from a vertex of $f$ and ends at a vertex of another face of $G_p$. We call this other face, a *neighbor* of this stick. The set of neighbors of the sticks forming a scissor (twin) of $f$ form the so-called *neighbors* of this scissor (twin).

By Lemma 14 and since $h(f) = 4$, face $f$ contains two sticks $s_1$ and $s_2$ forming a twin or a scissor, with neighbors $f_1$ and $f_2$. By 2-planarity and based on a technical case analysis, we show that $h(f_1) + h(f_2) \leq 7$ except for a single case, called 8-*sticks configuration* and illustrated in Fig. 9a, for which $h(f_1) + h(f_2) = 8$.

Assume first that $G$ does not contain any 8-sticks configuration. Let $H$ be an auxiliary graph, called *dependency graph*, having a vertex for each face of $G_p$. Then, for each face $f$ of $G_p$ containing a scissor or a twin with neighbors $f_1$ and $f_2$, s.t. $h(f_1) \leq h(f_2)$, there is an edge from $f$ to $f_1$ in $H$; $f_1 = f_2$ is possible. To prove that the average number of sticks for a face of $G_p$ is at most 3 (which implies the upper bound), it suffices to prove that the number of faces of $G_p$ that contain two sticks is at least as large as the number of faces that contain four sticks. This holds due to the following facts for every face $f$ of $G_p$:

**(i)** if $h(f) = 4$, then $f$ has two outgoing edges and no incoming edge in $H$,
**(ii)** if $h(f) = 3$, then the number of outgoing edges of $f$ in $H$ is at least as large as the number of its incoming edges, and
**(iii)** if $h(f) = 2$, then $f$ has at most two incoming edges in $H$.

So, $G$ has at most $3.5n - 7$ edges in the absence of 8-sticks configurations.

Finally, if $G$ contains 8-sticks configurations, we eliminate each of them (without creating new) by adding one vertex, and by replacing two edges of $G$ by six other edges violating neither bipartiteness nor 2-planarity, as in Fig. 9b. The derived graph $G'$ has a planar structure that is a spanning quadrangulation without 8-sticks configurations. Since $G'$ has one vertex and four edges more than $G$ for each 8-sticks configuration and since the vertices of $G'$ have degree at most 3.5 on average, by reversing the augmentation steps we conclude that $G$ cannot be denser than $G'$. We summarize our result in the following.

▶ **Theorem 15.** *A bipartite $n$-vertex 2-planar multigraph has at most $3.5n-7$ edges.*

**Implications of Theorem 15.** In the following, we adjust the well-known Crossing Lemma to bipartite graphs and use it to obtain a bound on the density of bipartite $k$-planar graphs, when $k>2$. Our proofs are inspired by the ones for general graphs; see, e.g., [4].

▶ **Theorem 16.** *Let $G$ be a bipartite topological graph with $n \geq 3$ vertices and $m \geq \frac{17}{4}n$ edges. Then, $cr(G) \geq \frac{16}{289} \cdot \frac{m^3}{n^2} \approx \frac{1}{18.1} \cdot \frac{m^3}{n^2}$, where $cr(G)$ is the crossing number of $G$.*

**Proof.** We first prove a weaker bound which holds for every $m$, that is, $cr(G) \geq 3m-\frac{17}{2}n+19$. This bound clearly holds when $m \leq 2n-4$. Hence, we may assume w.l.o.g. that $m > 2n-4$. It follows from [18] that if $m > 3n-8$, then $G$ has an edge that is crossed by at least two other edges. Also, by Theorem 15 we know that if $m > \frac{7}{2}n-7$, then $G$ has an edge that is crossed by at least three other edges. We obtain by induction on the number of edges of $G$ that $cr(G) \geq (m-(2n-4)) + (m-(3n-8)) + (m-(\frac{7}{2}n-7)) = 3m - \frac{17}{2}n+19$.

Assume that $G$ admits a drawing on the plane with $cr(G)$ crossings and let $p = \frac{17n}{4m} \leq 1$. Choose independently every vertex of $G$ with probability $p$, and denote by $H_p$ the graph induced by the chosen vertices. Let also $n_p$, $m_p$ and $c_p$ be the random variables corresponding to the number of vertices, of edges and of crossings of $H_p$. Taking expectations on the relationship $c_p \geq 3m_p - \frac{17}{2}n_p + 19$, which holds by our weaker bound, we obtain that $p^4 cr(G) \geq 3p^2 m - \frac{17}{2}np$, or equivalently that $cr(G) \geq \frac{3m}{p^2} - \frac{17n}{2p^3}$. The proof follows by plugging $p = \frac{17n}{4m}$ (which is at most 1 by our assumption) to the last inequality. ◀

▶ **Theorem 17.** *Let $G$ be a bipartite $k$-planar graph with $n \geq 3$ vertices and $m$ edges, for some $k \geq 1$. Then: $m \leq \frac{17}{8}\sqrt{2k}n \approx 3.005\sqrt{k}n$.*

**Proof.** For $k=1,2$, the bounds are weaker than the ones of [18] and of Theorem 15. So, we may assume w.l.o.g. that $k>2$. We may also assume that $m \geq \frac{17}{4}n$, as otherwise there is nothing to prove. Combining the fact that $G$ is $k$-planar with the bound of Theorem 16 we obtain that $\frac{16}{289} \cdot \frac{m^3}{n^2} \leq cr(G) \leq \frac{1}{2}mk$, which implies that $m \leq \frac{17}{8}\sqrt{2k}n \approx 3.005\sqrt{k}n$. ◀

## 6 Conclusions and Open Problems

We presented tight bounds for the density of bipartite beyond-planar graphs, yielding an improvement of the leading constant of the Crossing Lemma for bipartite graphs. We conclude with open problems.

- **(i)** What is the maximum density of bipartite $k$-planar graphs with $k>2$? Such bounds may further improve the leading constant of the Crossing Lemma for bipartite graphs; Fig. 9c shows a bipartite 3-planar graph with $4n-O(1)$ edges. Bounds for other classes of bipartite beyond-planar (e.g., quasi-planar) graphs are also interesting.
- **(ii)** The ratio of the maximum density of general over bipartite graphs for large $n$ approaches $\frac{3n}{2n} = 1.5$ for planar graphs, $\frac{4n}{3n} \approx 1.33$ for 1-planar graphs, $\frac{5n}{3.5n} \approx 1.43$ for 2-planar graphs and at most $\frac{5.5n}{4n} \approx 1.37$ for 3-planar graphs, leaving room for speculation on how it develops for $k$-planar graphs with $k>3$; note that for classes closed under subgraphs, it is at most 2 [23].
- **(iii)** Optimal 1-, 2- and 3-planar graphs allow for characterizations [13, 35], while recognizing general beyond-planar graphs is often NP-hard. Does the restriction of bipartiteness allow for characterizations or efficient recognition algorithms in some cases?
- **(iv)** Finally, one should study properties that not only hold for general beyond-planar graphs but also for bipartite ones, e.g., is every optimal bipartite RAC graph also 1-planar?

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