Continued Radicals

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October 25, 2004

Extended Abstract

We will write

$$\sqrt{a_1+\sqrt{a_2+\sqrt{a_3+\cdots+\sqrt{a_n}}}} = S_n = \sqrt{a_1, a_2, a_3, \ldots, a_n},$$

and consider *continued radicals* of form $\lim_{n\to\infty} S_n = \sqrt{a_1, a_2, a_3, \ldots}$.

Convergence criteria for continued radicals are given in [2], and [3]. We consider the sets $S(M)$ of real numbers which are representable as a continued radical whose terms $a_1, a_2, \ldots$ are all from a finite set $M = \{m_1, m_2, \ldots, m_p\} \subseteq \mathbb{N}$ where $0 < m_1 < m_2 < \cdots < m_p$.

For any nonnegative number n, $\sqrt{n, n, n, \ldots}$ converges to $\varphi_n \equiv \frac{1+\sqrt{4n+1}}{2}$. It is easy to see that $\varphi_n = k \in \mathbb{N}$ if and only if $n = k(k-1)$ for some integer $k \geq 2$.

If $S(M)$ is to be an interval, then to insure that no gaps occur in $S(M)$, it is necessary that the largest value representable with $a_1 = m_i$ equal or exceed the smallest value representable with $a_1 = m_{i+1}$ (for $i = 1, \ldots, p-1$). That is, it is necessary that

$$\sqrt{m_i, m_p, m_p, m_p, \ldots} \geq \sqrt{m_{i+1}, m_1, m_1, m_1, \ldots} \quad \forall i \in \{1, \ldots, p-1\},$$

In fact, this condition will be necessary and sufficient: A greedy algorithm proves the following result.

Dagstuhl Seminar Proceedings 04351
Spatial Representation: Discrete vs. Continuous Computational Models
http://drops.dagstuhl.de/opus/volltexte/2005/128

Theorem 0.1 *Suppose $M = \{m_1, m_2, \dots, m_p\} \subseteq \mathbb{N}$ where $0 < m_1 < m_2 < \cdots < m_p$ and*

$$\sqrt{m_i + \varphi_{m_p}} \geq \sqrt{m_{i+1} + \varphi_{m_1}} \quad \forall i \in \{1, \dots, p-1\}.$$

Then the set of numbers representable as a continued radical $\sqrt{a_1, a_2, a_3, \dots}$ with terms $a_i \in M$ is the interval $[\varphi_{m_1}, \varphi_{m_p}]$.

To limit non-unique rerpesentation as much as possible, we should take the inequalities in Theorem 0.1 to be equalities. These equalities will not eliminate all duplication of representation, but will limit it to continued radicals of form

$$\sqrt{c_1, \dots, c_z, m_i, m_p, m_p, m_p, \dots} = \sqrt{c_1, \dots, c_z, m_{i+1}, m_1, m_1, m_1, \dots}$$

where repeating the largest value m_p is equal to raising the preceding term from m_i to m_{i+1} and repeating the smallest value m_1. (Compare to the decimal equation $1.3\overline{999} = 1.4\overline{000}$.)

The inequalities of Theorem 0.1 are equalties if and only if $\varphi_{m_1} = n+1 \in \mathbb{N}$ and $\varphi_{m_p} = j+1 \in \mathbb{N}$, so M contains $n+j+2$ equally spaced terms from $n(n+1)$ to $j(j+1)$.

Theorem 0.2 *If M is an "efficient" set of terms as described in the paragraph above and $x = \sqrt{a_1, a_2, a_3, \dots}$ where $(a_i)_{i=1}^{\infty}$ is a periodic sequence in M, then x is either an integer or irrational.*

Allowing zero as a term in our continued radicals introduces some complications. We now assume our terms a_n all come from a set $M = \{m_1, m_2, \dots, m_p\} \subseteq \mathbb{N} \cup \{0\}$ where $0 = m_1 < m_2 < \cdots < m_p$. To prevent gaps in the set $S(M)$ of numbers representable with these terms, the largest value of form $\sqrt{m_i, a_2, a_3, \dots}$ must equal or exceed the smallest value of form $\sqrt{m_{i+1}, b_2, b_3, \dots}$. That is, we must have

$$\begin{aligned} \sqrt{m_i, m_p, m_p, m_p, \dots} &\geq \sqrt{m_{i+1}, 0, 0, 0, \dots} \\ \sqrt{m_i + \varphi_{m_p}} &\geq \sqrt{m_{i+1}}. \end{aligned}$$

However, note that besides the single value $\sqrt{m_{i+1}} = \sqrt{m_{i+1}, 0, 0, 0, \dots}$, every other value representable as $\sqrt{m_{i+1}, b_2, b_3, \dots}$ must be greater than

$$\sqrt{m_{i+1}, 1} = \sqrt{m_{i+1} + 1} = \lim_{k \to \infty} \sqrt{m_{i+1}, 0, 0, \dots, 0, 0, b_k, 0, 0, \dots} \text{ where } b_k \neq 0.$$

We now see that if $m_1 = 0$, then the numbers representable as $\sqrt{m_i, a_2, \dots}$ where $a_n \in M \ \ \forall n \in \mathbb{N}$ will be a subset of $\{\sqrt{m_i}\} \cup (\sqrt{m_i + 1}, \sqrt{m_i + \varphi_{m_p}}] = J_i$. To prevent any gaps in the set of numbers representable, it is necessary that $\bigcup\{J_i : i = 1, 2, \dots, p\}$ forms a solid interval. Consequently, it is necessary that

$$\sqrt{m_i + \varphi_{m_p}} \geq \sqrt{m_{i+1} + 1} \ \ \forall i = 1, 2, \dots, p - 1.$$

Again we find that choosing the values of $m_1, \dots, m_p$ so that the above inequalities are equalities will result in the most efficient representation of the largest possible interval using the smallest number of terms. For equality to hold, we must have that $\varphi_{m_p} \in \mathbb{N}$, and thus $m_p = (q+1)q$ for some $q \in \mathbb{N}$, so that $M = \{0, q, 2q, 3q, \dots, (q+1)q\}$.

Theorem 0.3 *A real number $b \in (\sqrt{q}, q + 1)$ has a unique representation as $\sqrt{a_1, a_2, a_3, \dots}$ where $a_i \in M_q = \{0, q, 2q, \dots, (q + 1)q\}$ with $q \geq 2$ if and only if it cannot be represented as a terminating continued radical $\sqrt{a_1, a_2, \dots, a_z, 0, 0, 0, \dots}$. A number $b \in (\sqrt{q}, q + 1)$ has a terminating continued radical representation $\sqrt{a_1, a_2, \dots, a_z, 0, 0, 0, \dots}$ if and only if it has a continued radical representation ending in repeating $(q-1)q$'s. Observe that*

$$\sqrt{q, 0, 0, 0, \dots} \quad \text{and} \quad \sqrt{(q + 1)q, (q + 1)q, (q + 1)q, \dots}$$

respectively are the unique representations of $\sqrt{q}$ and $\varphi_{(q+1)q} = q + 1$.

The result below was proved by Sizer [3].

Theorem 0.4 *Any number $b \in (1, 2)$ can be represented as a continued radical $\sqrt{a_1, a_2, \dots}$ where $a_i \in \{0, 1, 2\}$. This representation is unique unless b has such a representation ending in repeating 0s. A number $b \in (1, 2)$ has such a representation ending in repeating 0s if and only if it has such a representation ending in repeating 2s.*

Note that $\sqrt{1, 0, 0, 0, 0, \dots} = 1$ and $\sqrt{2, 2, 2, 2, 2, \dots} = 2$ are the unique representations of 1 and 2.

Finally, we consider continued radicals whose terms assume only two values. If $M = \{m_1, m_2\} \subseteq \mathbb{N}$, then $S(M)$ cannot be an interval, and the corresponding gaps in $S(M)$ recur regularly at at the i^{th} position of $\sqrt{a_1, a_2, a_3, \dots}$ for each $i \in \mathbb{N}$.

Theorem 0.5 *If m_1 and m_2 are natural numbers with $m_1 < m_2$, then the set $D = \{\sqrt{a_1, a_2, \dots} : a_i \in \{m_1, m_2\} \, \forall i \in \mathbb{N}\}$ is homeomorphic to the Cantor ternary set $\mathcal{C}$.*

References

[1] Jonathan M. Borwein and G. de Barra, Nested Radicals, *The American Mathematical Monthly*, 98 (8) Oct. 1991, 735–739.

[2] D. Laugwitz, Kettenwurzeln und Kettenoperationen, *Elemente der Mathematik*, 45 (4) 1990, 89–98.

[3] Walter S. Sizer, Continued Roots, *Mathematics Magazine* 59 (1) Feb. 1986, 23–27